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Chapter IV

Radicals

§1. Nilpotence

The associative concept of nilpotence splits into several different concepts for nonassociative algebras. In this section we study the concept of nonassociative nilpotence, and relate it via Etherington's Theorem to ordinary nilpotence of the associative multiplication algebra. We also relate nilpotence to the simpler notions of left and right nilpotence.

A nonassociative algebra A is **nilpotent** if for some n every product of n or more factors (no matter how associated) is zero. The smallest such integer n is the **index** of nilpotence. The nilpotency condition is that any monomial $f(x_1,\cdots,x_m)$ of degree $\geq n$ in the free nonassociative algebra F vanishes when evaluated at $a_1,\cdots,a_m \in A$, $f(a_1,\cdots,a_m) = 0$.

Actually, if all products with <u>exactly</u> n factors vanish so do those with more than n factors: in general, a product of $n+1$ factors $a_1\cdots(a_i a_{i+1})\cdots a_{n+1}$ can be written as a product $a_1\cdots a\cdots a_{n+1}$ of n factors by taking the first product $(a_i a_{i+1})$ as a single factor a.

We can define the **n^{th} power A^n** to be the space spanned by all monomials of degree $\geq n$ (by the above remark we could get by with those of degree exactly n). Thus we have a decreasing series of powers

$$A = A^1 \supset A^2 \supset A^3 \supset \cdots \supset A^n \supset \cdots$$

where each term is clearly an ideal in A. An algebra is nilpotent iff some power $A^n = 0$.

A subalgebra (in particular, an ideal) B is called **nilpotent** if it is nilpotent in its own right as an algebra: $B^n = 0$. If B is an ideal in A it is not true that the powers B^n are again ideals in a general nonassociative algebra. However, consider the following non-elementwise definition of the powers of B:

$$B^n = \sum_{1 \leq i \leq n-1} B^i B^{n-i} \qquad (n \geq 2). \tag{1.1}$$

This can be established by induction, the result $B^2 = BB$ for $n = 2$ being pretty clear. For general n, clearly everything on the right side is a sum of monomials with $\geq i + (n-i) = n$ factors from B. Conversely, every monomial with n factors from B belongs to some $B^i B^{n-i}$, since in any $b_1 \cdots b_n$ (with some distribution of parentheses) there is a last multiplication which takes place, $(b_1 \cdots b_i)(b_{i+1} \cdots b_n)$ for $1 \leq i \leq n$ (with some distribution of parentheses inside each factor), and by induction $b_1 \cdots b_i \in B^i$ and $b_{i+1} \cdots b_n \in B^{n-i}$.

This way of building the n^{th} power out of lower powers shows by induction that B^n is an ideal if B is whenever a product of ideals is again an ideal (sums always are). In particular, by the Product Theorem III.1.1. this holds in the alternative case.

1.2 (Power Proposition) In an alternative algebra the powers of an ideal are again ideals: $B \triangleleft A$ implies $B^n \triangleleft A$. ∎

The most important general result about nilpotence is that it is equivalent to nilpotence of the multiplication algebra; this reduces nilpotence of a nonassociative algebra A to nilpotence of an associative algebra M(A).

1.3 (Etherington's Theorem) A nonassociative algebra A is nilpotent iff its multiplication algebra M(A) is nilpotent.

Proof. Clearly if A is nilpotent so is its multiplication algebra M(A) since $M(A)^n A \subset A^{n+1}$ (recall from I.7 that M(A) is built out of L_x's and R_y's without throwing in the identity operator).

Conversely, nilpotence $A^{2^n} = 0$ of A will follow from nilpotence $M(A)^n = 0$ of M(A) if we can establish a general relation

$$A^{2^n} \subset M(A)^n A \qquad \text{(A arbitrary nonassociative)}. \tag{1.4}$$

This is certainly true for n = 1 since $A^2 = AA \subset L_A A$. If true for n then by (1.1) $A^{2^{n+1}} = \sum A^i A^j$ where $i+j = 2^{n+1}$, so either $i \geq 2^n$ or $j \geq 2^n$. But if $i \geq 2^n$ then $A^i A^j \subset R_A A^i \subset R_A A^{2^n} \subset R_A M(A)^n A$ by induction, and if $j \geq 2^n$ then $A^i A^j \subset L_A M(A)^n A$. In either case $A^i A^j \subset M(A) M(A)^n A = M(A)^{n+1} A$ and the induction is complete. ∎

1.5 Remark. The idea of the proof is that in any monomial $x_1 \cdots x_{2^{n+1}}$ (with some distribution of parentheses) there is a last multiplication which is performed, say $(x_1 \cdots x_i)(x_{i+1} \cdots x_{2^{n+1}})$ (with some distribution of parentheses inside each of the two factors). One of these two factors must have degree $\geq 2^n$ and hence by induction be analyzable into n multiplications acting on A, so the product of both factors can be analyzed into n+1 multiplications.

These results are best possible; in the free nonassociative algebra $(x_1x_2)(x_3x_4) = L_{x_1x_2} L_{x_3} x_4 \in A^{2^2}$ belongs to $M(A)^2 A$ but not to $M(A)^3 A$, $\{(x_1x_2)(x_3x_4)\}.\ \{(x_5x_6)(x_7x_8)\} = L_{(x_1x_2)(x_3x_4)} L_{x_5x_6} L_{x_7} x_8 \in A^{2^3}$ belongs to $M(A)^3 A$ but not to $M(A)^4 A$, etc. ∎

1.6 Remark. If we were willing to content ourselves with the alternative case, Etherington's Theorem could be proved much more simply. The Generation Theorem I.7.6 shows

$$M(A^n) \subset M(A)^n \qquad \text{(A alternative)} \tag{1.7}$$

for an alternative algebra, since any left or right multiplication $M_{p(x_1,\cdots,x_m)}$ by a monomial $p(x_1,\cdots,x_m)$ of degree n can be expressed as a sum of monomials of degree n in the generators $L_{x_1},\cdots,L_{x_m}, R_{x_1},\cdots,R_{x_m}$.

This leads immediately to the formulas (compare (1.4))

(1.8) $A^n = M(A)^n \hat{1},\ A^{n+m} = M(A)^n A^m$ (A alternative).

Indeed, the second follows from the first since $M(A)^{n+m}\hat{1} = M(A)^n\{M(A)^m \hat{1}\}$, and in the first $M(A)^n \hat{1} \subset A^n$ is clear by degree considerations, while by (1.7) $A^n = M(A^n)\hat{1} \subset M(A)^n \hat{1}$.

From this Etherington follows: $A^{n+1} = 0 \Leftrightarrow M(A)^n A = 0 \Leftrightarrow M(A)^n = 0$. ■

We could define other notions of powers and corresponding notions of nilpotence by choosing particular associations in (1.1). In the next section we will discuss solvability and the derived series. For another example, we can define the **left powers** $B^{n,L}$ inductively by

$$B^{1,L} = B,\ B^{n+1,L} = BB^{n,L}$$

and **left nilpotence** by $B^{n,L} = 0$ for some n. There is a completely analogous notion of **right nilpotence**. Again, whenever products of ideals in a non-associative algebra are ideals these $B^{n,L}$ and $B^{n,R}$ will be ideals when B is.

Because the association is fixed in left powers $A^{n,L}$ these are sometimes technically easier to deal with than the ordinary unrestricted powers A^n. For example, the left powers can be described explicitly.

(1.9) $A^{n,L} = L_A^n \hat{1} = L_A^{n-1} A$ (A nonassociative).

From this it is clear that A is left nilpotent iff the subspace L_A of left multiplications or the subalgebra L(A) it generates is nilpotent (in the associative algebra of linear transformations), a sort of "Left Etherington Theorem".

1.10 Proposition. A nonassociative algebra A is left nilpotent iff L(A) is nilpotent. ■

In fact, we could stick to left or right nilpotence if we wished, because of the following result.

1.11 (Left Nilpotence Theorem) An alternative algebra is nilpotent iff it is left nilpotent.

Proof. If A is nilpotent it is automatically left, right, and every-which-way nilpotent.

Conversely, assume A is left nilpotent: $A^{n,L} = 0$. By the Left Normal Form Theorem for Elements I.7.10 we know A^{n^2} is spanned by second-order monomials $y = y_1(y_2(\cdots y_r))$ of degree $\geq n^2$ where each y_i is a first-order monomial $x_1(x_2(\cdots x_s))$ (in some fixed set of generators $\{x_i\}$ for A). Such a first-order monomial $L_{x_1}\cdots L_{x_s}\hat{1}$ belongs to $A^{s,L}$ and hence vanishes if $s \geq n$. Therefore y vanishes unless all y_i have degrees $\partial y_i < n$, in which case $n^2 \leq \partial y = \sum_1^r \partial y_i < \sum_1^r n = nr$ implies $n < r$. But then $y = L_{y_1}\cdots L_{y_r} 1 \in A^{r,L} \subset A^{n,L} = 0$, so y vanishes in any event. This shows the monomials spanning A^{n^2} all vanish, so $A^{n^2} = 0$ and A is nilpotent. ∎

This type of combinatorial argument is very common. Roughly, it says that a large enough object is either composed of some very big pieces or else is composed of a very large number of small pieces. Typically the pieces vanish if they are of size $\geq n$, and a product of $\geq m$ pieces vanishes, therefore any object of size $\geq mn$ vanishes. (In the above case we have $n = m$).

We can also reduce the condition that an algebra act nilpotently on a bimodule to the condition that it act both left and right nilpotently on the bimodule. (Here the algebra itself need not be nilpotent).

1.12 (Left-Right Nilpotence Theorem) An alternative algebra A acts nilpotently on a bimodule M, $\{M(A)|_M\}^k = 0$, iff it acts left and right nilpotently on M, $\{L(A)|_M\}^n = \{R(A)|_M\}^m = 0$.

Proof. We really should write $L_E(A)|_M$, $R_E(A)|_M$, $M_E(A)|_M = M_E(A\ M)$ to indicate the multiplications are taking place in the split null extension $E = A \oplus M$, but we abbreviate by leaving out the E. Since $\{L(A)|_M\}^k$ and $\{R(A)|_M\}^k$ are contained in $\{M(A)|_M\}^k$, nilpotence implies one-sided nilpotence.

Conversely, if $L(A)^n = R(A)^m = 0$ on M we claim $M(A)^{n+m} = 0$ on M. In fact, we claim

$$(1.13) \qquad M_E(A)^{n+m} \subset \hat{M}_E(A)L_E(A)^n + \hat{M}_E(A)R_E(A)^m.$$

Since M is A-invariant, restriction $T \to T|_M$ is a homomorphism $M_E(A) \to M_E(A)|_M$, so restricting (1.13) to M and using $\{L(A)^n\}|_M = \{L(A)|_M\}^n = 0$, $\{R(A)^m\}|_M = \{R(A)|_M\}^m = 0$ we will have $\{M(A)|_M\}^{n+m} = 0$.

To establish (1.13), note that $M(A)^{n+m}$ is spanned by monomials $T = T_1 \cdots T_r$ for $r \geq n+m$ where each multiplication T_i is an L_x or an R_y for x or y in A. Since there are $\geq n+m$ factors there must be $\geq n$ L's or $\geq m$ R's. In case there are $\geq n$ factors L_x we can move them to the right across R_y's by means of $L_x R_y = R_y L_x + L_{xy} - L_x L_y$ (recall $-[x,m,y] = +[x,y,m]$); this does not decrease the number of L's, so by repeated application we can rewrite the original monomial as a combination of monomials of the form $S = S' L_{x_1} \cdots L_{x_n}$ with n L's lumped together at the right: $S \in \hat{M}(A)L(A)^n$. Similarly if T has $\geq m$ R's it belongs to $\hat{M}(A)R(A)^m$. This finishes (1.13) and the theorem. ∎

The example of right modules (where $L(A)|_M = 0$) shows left nilpotence on a bimodule does not imply two-sided nilpotence, though by (1.11) this does hold for the regular bimodule M = A.

As in the associative case, an infinite sum of nilpotent ideals need not be nilpotent, so nilpotence is not a radical property in the sense of Amitsur (see Section 10). But it is not even weak radical property: it is not recoverable ! That is, because of nonassociativity we cannot conclude from the fact that B and A/B are nilpotent that A itself is nilpotent. This leads us to look for something which (i) coincides with nilpotence in the associative case, (ii) the absence of which coincides with the absence of nilpotence, and (iii) which does form a true weak radical property. The notion we seek is that of solvability.

IV.1 Exercises

1.1 If B is a nilpotent ideal in A and $b \in B$, show $bz = z$ implies $z = 0$; if A is alternative, show $c(bz) = z$ implies $z = 0$.

1.2 If $B = \Phi x$, $C = \Phi y$ are trivial ideals show B+C is a nilpotent ideal.

1.3 If $I = \{T \in M(A) \mid TB^i \subset B^{i+1}$ for all $i\}$ show I contains $M_A(B)$ and $I^k B^i \subset B^{i+k}$ for all i,k. Show $I(C,D) = \{T \in M(A) \mid T(C) \subset D\}$ is an ideal in M(A) if C and D are ideals in A, and conclude $I = \bigcap I(B^i, B^{i+1}) \lhd M(A)$ contains M(B;A) if all $B^i \lhd A$. Deduce 1.9.

1.4 Generalize 1.11 to show that any nonassociative algebra, in which products of ideals are ideals, is nilpotent iff it is both left and right nilpotent.

1.5 Show that if B is a left-nilpotent ideal of index n in a nonassociative algebra where products of ideals are ideals, then $C = B^{n-1,L}$ is a (two-side) nilpotent ideal. Conclude that A is free of nilpotent ideals iff it is free of left-nilpotent ideals. Conclude that the smallest ideals R, R_L, R_R such that A/R, A/R_L, A/R_R are free respectively of nilpotent, left nilpotent, right nilpotent ideals all coincide.

1.6 Show that an alternative algebra is nilpotent iff it is left nilpotent by showing $A^{3^n} \subset A^{n,L}$ directly from (1.1).

1.7 The estimate $A^{3^n} \subset L_A^n 1$ for alternative algebras is unduly pessimistic. If $\frac{1}{2} \in \Phi$ show $A^{3n} \subset L_A^n 1$ (using the fact that $2\, x\{y(zw)\}$ and $2(xy)(zw)$ are sums of Jordan products $U_a \hat{b}$).

1.8 Prove that if an alternative algebra has no nilpotent ideals, it has no nilpotent one-sided ideals.

1.9 If $B \triangleleft A$ for alternative A, show $M_A(B^n) \subset M_A(B)^n$, $B^n = M_A(B)^n \hat{A}$, $B^{n+m} = M_A(B)^n B^m$. Repeat for $M(B;A)$.

1.10 Deduce 1.11 from 1.12 by showing $L(A)^n = 0 \Rightarrow R(A)^{f(n)} = 0$ for $f(n) = \frac{n(n+3)}{2}$. (Hint: show $R(A^k) \subset \hat{M}(A)L(A)\hat{M}(A) + R(A^{k,L})$ and $R(A)^{f(n)} A \subset L(A)^n A$ by induction, noting $f(n+1) = f(n)+n+2$).

1.11 If A is any nonassociative algebra in which $L_A R_A \subset \hat{M}(A) L_A$ show the Left-Right Nilpotence Theorem 1.12 still holds.

1.12 Prove <u>Etherington's</u> <u>Ideal</u> <u>Theorem</u>: An ideal B in a linear algebra A is nilpotent iff its multiplication algebra $M_A(B)$ acts nilpotently on A. If all powers B^n are ideals then B is nilpotent iff its multiplication ideal $M(B;A)$ is nilpotent. In particular, an ideal B in an alternative algebra A is nilpotent iff $M(B;A)$ is nilpotent.

1.13 Show that if A is nilpotent so is any subalgebra and any homomorphic image. If powers of ideals are ideals, show any finite sum of nilpotent ideals in A is again a nilpotent ideal. (Hint: use exercise 1.12).

1.14 If A is 4-dimensional over Φ with basis b,c,z,w such that $cz = w$, $bw = z$, and all other products of basis elements are zero, show $A = B+C$ where $B = \Phi b+\Phi z+\Phi w$ and $C = \Phi c+\Phi z+\Phi w$ are nilpotent ideals - yet A itself is not nilpotent. Show A/B and B are nilpotent yet A is not nilpotent.

This example is not alternative, and indeed we will see A/B and B nilpotent implies A nilpotent when A is alternative with finiteness conditions (because these guarantee nilpotence equals solvability).