§3 Associativity theorems

The Basic Associativity Theorem has many useful consequences. In a typical application we are given an alternative algebra A and a certain collection of elements a,b,... whose basic associators N(a,b,...) all vanish, and we will conclude that the subalgebra Φ[a,b,...] they generate is associative. For example, there are no basic associators of degree 1 or 2 and only one of degree 3, so

3.1 (Power Associativity Theorem). Any element $a \in A$ generates an associative subalgebra Φ[a]. ∎

3.2 (Artin's Theorem). Any two elements a,b of an alternative algebra generate an associative subalgebra Φ[a,b]. ∎

3.3 (Generalized Artin Theorem). If the elements $a,b,c \in A$ associate, $[a,b,c] = 0$, then they generate an associative subalgebra Φ[a,b,c]. ∎

It is not true that three elements necessary generate an associative subalgebra - indeed, if $[a,b,c] = 0$ always held then all alternative algebras would be associative, to which the Cayley algebra would strenuously object.

It should be noted that Artin's theorem provides an elegant characterization of alternative algebras:

3.4 (Artin's Theorem). An algebra is alternative iff every two elements generate an associative subalgebra.

Proof. If each a,b generate an associative subalgebra then in particular the associators [a,a,b] and [b,a,a] vanish, so the algebra is left and right alternative. ∎

Already at this stage we can prove Wedderburn's Theorem on finite division algebras by reduction to the associative case.

3.5 (Wedderburn's Theorem on Finite Division Algebras). A finite alternative division algebra is a finite (commutative, associative) field.

This follows from the (formally) more general

3.6 (Wedderburn's Theorem on Finite Domains). A finite alternative domain is a finite (commutative, associative) field.

Proof. We induct on the number n of generators, n = 0 being vacuous. If the result is true for n then any subalgebra $\Phi[x_1,\ldots,x_n]$ (using a subdomain) is a finite field, hence by field theory generated by a single generator x (in fact, consists of zero and the powers $x, x^2, \cdots, x^{p^\ell-1} = 1$). But then for n+1 generators we have $\Phi[x_1,\ldots,x_n, x_{n+1}] = \Phi[x, x_{n+1}]$ generated by two elements, hence associative by Artin, and still a finite domain, so a field by the associative Wedderburn Theorem. ∎∎

Artin's Theorem makes it trivial to check whether an identity in 2 variables holds - just see whether it holds in all associative algebras, i.e. write it down without parentheses and see if it is formally zero. For example, any associator involving 2 variables, like $[(xy)^2, x, yxy]$, is guaranteed to vanish. We state this as

3.7 (Artin's Principle). If an identity $f(x,y) = 0$ in two variables holds for all associative algebras, it holds for all alternative algebras. ∎

3.8 (Throw-in-a-Nucleizer-Theorem). If B is an associative subalgebra of A and $n \in N(B)$, the nucleizer of B, then $\Phi[n, B]$ is an associative subalgebra.

Proof. We apply the Basic Associator Theorem to the ordered generating set $\{n\} \cup B$. The basic associators are the $[n, b_{i_1}, b_{i_2}(\cdots b_{i_n})]$ and $[b_{i_1}, b_{i_2}, b_{i_3}(\cdots b_{i_n})]$. The first vanishes since $n \in N(B)$ and the second because B is associative by hypothesis. ∎

3.9 Corollary. A maximal associative subalgebra B is nucleizer-closed, $N(B) = B$. ∎

It is essential in this result that we throw in nucleizing elements one at a time (so the elements we throw in must nucleize larger and larger subalgebras). We cannot throw in $N(B)$ all at once - the subalgebra generated by B and $N(B)$ need not be associative, as is easily seen from the case $B = 0$, $N(B) = A$ where

A is not associative. However, we can throw in elements of the nucleus since they nucleize anything.

3.10 (Throw-in-the-Nucleus Theorem). If B is an associative subalgebra of A, so is the subalgebra generated by B and the nucleus N(A).

Proof. We could use transfinite induction on the Throw-in-a-Nucleizer Theorem. More elegantly, let C be a maximal associative subalgebra containing B (Zornify). If $n \in N(A)$ then $n \in N(C)$, so by the Corollary 3.11 $n \in C$, and $N(A) \subset C$. Thus B and N(A) are contained in an associative subalgebra C. ∎

We can extend our previous associativity results to the case of inverses.

3.11 (Throw-in-an-Inverse Lemma). If B is an associative subalgebra of a unital alternative algebra A and $b \in B$ is invertible, then B and b^{-1} generate an associative subalgebra.

PROOF. By the Throw-in-a-nucleizer Lemma it suffices if $b^{-1} \in N(B)$; i.e. $[b^{-1},c,d] = 0$ for all $c,d \in B$. But $(b^{-1}c)d = (b^{-1}c)\{(db)b^{-1}\}$ (Inverse Formula) $= b^{-1}\{c(db)\}b^{-1}$ (middle Moufang) $= b^{-1}\{(cd)b\}b^{-1}$ (since B is associative) $= \{b^{-1}(cd)\}(bb^{-1})$ (Moufang again) $= b^{-1}(cd)$. ∎

We say a subalgebra C is <u>inverse-closed</u> in A if whenever an element of C has an inverse in A that inverse belongs to C. We can extend 3.11 by throwing in all inverses:

3.12 (Inverse Closure Theorem). Every maximal associative subalgebra is inverse-closed. Consequently, every associative subalgebra is contained in an inverse-closed associative subalgebra.

PROOF. Let C be a maximal associative subalgebra containing the given subalgebra B. If $c \in C$ is invertible then by 3.11 C and c^{-1} generate an associative subalgebra; by maximality this must be C, so $c^{-1} \in C$ and C is inverse-closed. ∎

3.13 REMARK. Actually we only need $bb^{-1} = 1$ since this implies $R_{b^{-1}}R_b = I$ by the One-Sided Inverse Theorem I.4.1, so it is enough if b^{-1} is only a right inverse for b. (Similarly if it is a left inverse). Thus we can throw in one-sided inverses.

If b is right invertible but not invertible, then $bz = 0$ for some z. If $bz = 0$ then $b(b^{-1}+z) = 1$ so $b^{-1}+z$ is also a right inverse, so $(b^{-1}+z) - b^{-1} = z$ associates with B. Thus we can throw in any right zero divisor of a right-invertible element. ∎

3.14 (Artin's Theorem with Inverses). If a,b are invertible elements of a unital alternative algebra, then a, b, a^{-1}, b^{-1} generate an associative algebra.

Proof. a,b are contained in an associative subalgebra $\Phi[a,b]$ by the ordinary Artin's Theorem 3.2. If C is a maximal associative subalgebra containing $\Phi[a,b]$ then C is inverse-closed by the Inverse Closure Theorem 3.12. Therefore a^{-1}, b^{-1} belong to C, and $\Phi[a,b,a^{-1},b^{-1}]$ is associative because it is a subalgebra of the associative algebra C. ∎

3.15 The same argument gives a one-inverse form of the result, namely if a is invertible then $\Phi[a,a^{-1},b]$ is associative. ∎

3.16 (Artin's Principle). If a rational expression f(x,y) vanishes in all associative algebras in which it makes sense (i.e. where all requisite inverses exist), then it vanishes in all alternative algebras in which it makes sense.

Proof. Any maximal associative subalgebra C containing $\Phi[a,b]$ is inverse-closed, hence contains all rational expressions in a,b which exist in A. If f(a,b) makes sense in A it makes sense in C, hence vanishes. ∎

3.17 (Throw-in-a-Quasi-Inverse Lemma). If B is an associative subalgebra of an alternative algebra A, and b is quasi-invertible with quasi-inverse c, then B and c generate an associative subalgebra.

Proof. Passing to the unital hull $\hat{A} = \Phi 1 + A$ if necessary, $\hat{B} = \Phi 1 + B$ stays associative; in $\hat{B}$ the element 1-b is invertible with inverse 1-c, so by the Throw-in-an-Inverse Lemma $\hat{B}$ and 1-c generate an associative algebra. Then its subalgebra generated by B and c is also associative. ∎

3.18 (Quasi-Inverse Closure Theorem). Every associative subalgebra of an alternative algebra is contained in a quasi-inverse closed associative subalgebra.

Proof. If C is a maximal associative subalgebra containing B, then C is quasi-inverse closed by the above. ∎

3.19 (Artin's Theorem with Quasi-Inverses). If a,b are quasi-invertible elements of an alternative algebra with quasi-inverses c,d then a,b,c,d generate an associative subalgebra. ∎

Further associativity results will be left to the exercise.

Exercise

3.1 Show $[c,b^{-1},d] = 0$ if $[c,b,d] = 0$ by showing $[(cb)b^{-1},b^{-1},b^{-1}(bd)] = b^{-1}[cb,b^{-1},bd]b^{-1} = -\ b^{-1}[c,b,d]b^{-1}$.

3.2 Prove that if $a,b,c,d \in A$ satisfy $[a,b,c] = [a,b,d] = [a,c,d] = [b,c,d] = [a,b,cd] = 0$ they generate an associative subalgebra.

3.3 If a subset S of A satisfies $[S,S,A] = 0$ prove $[B,B,A] = 0$ for $B = \Phi[S]$.

3.4 If $[S,S,A] = [T,T,A] = 0$ for subsets $S,T \subseteq A$ show $\Phi[S,T]$ is associative. More generally, if $[S,S,A] = [T,T,A] = [P,P,A] = [S,T,P] = 0$ show $\Phi[S,T,P]$ is associative.

3.5 If B is an associative subalgebra of A and $[S,B,B] = [S,S,A] = 0$ prove $\Phi[B,S]$ is associative.

3.6 If $a,b,c \in A$ are invertible and $[a,b,c] = 0$, show $\Phi[a,b,c,a^{-1},b^{-1},c^{-1}]$ is associative.

3.7 Is a maximal associative algebra "root-closed": if B is associative and $a \in A$ invertible with $a^n = b \in B$, is $\Phi[B,a]$ associative? What if a is not required to be invertible?