§2 Basic associativity tests.

We want to show that an element w nucleizes a subalgebra B as soon as certain "basic" associators, involving w and the generators of B, vanish. From this we show that if the basic associators on a set of generators for B vanish, then B is associative. This leads to a host of associativity theorems in the next section.

We fix an element $w \in A$ and a set $\{x_i\}_{i \in I}$ of generators for the subalgebra $B \subseteq A$. An associator $[w,p,q]$ where p,q are monomials in the x_i of degrees ∂p, ∂q respectivily is said to be a <u>w-associator</u> of <u>degree</u> $n = \partial p + \partial q$. A w-associator

$$N_w(x_{i_1},\cdots,x_{i_n}) = [w,x_{i_1},x_{i_2}(x_{i_3}(\cdots x_{i_n}))]$$

is said to be in (left) <u>normal form</u>. If the index set I is linearly ordered, a <u>basic w-associator</u> of degree n is a normal $N_w(x_{i_1},\cdots,x_{i_n})$ where $i_1 < i_2 < \cdots < i_n$ (in particular, all the variables are distinct).

2.1 (Basic Nucleizing Lemma). The ideal $A_{w,n}$ in A generated by all w-associators of degree $\le n$ coincides with the ideal $B_{w,n}$ in A generated by all basic w-associators of degree $\le n$.

Proof. Since basic w-associators are a special kind of w-associator, we clearly have $B_{w,n} \subseteq A_{w,n}$, so we need only prove $B_{w,n} \supseteq A_{w,n}$. We do this by induction, the result being vacuous for $n = 0,1$. Assume the result for $n-1$, $B_{w,n-1} \supseteq A_{w,n-1}$. It suffices if $A_{w,n} \equiv B_{w,n} \pmod{A_{w,n-1}}$.

We can write $p = M(1)$, $q = N(1)$ for $M,N \in M_A(B)$ multiplication operators, even though 1 may only be a figment of our imagination (i.e. an element of $\hat{A}$). By Corollary I,7.8 to the Generation Lemma, $M_A(B)$ is generated by the L_{x_i} and R_{x_i} since the x_i generate B. Therefore we need only consider associators $[w,p,q]$ for $p = M(1)$, $q = N(1)$ with $M = M_{x_{i_r}} \cdots M_{x_{i_2}}$,

$N = M_{x_{i_{r+1}}} \cdots M_{x_{i_n}}$ and each M_{x_k} is either L_{x_k} or R_{x_k}.

We have $[w,xr,s] + [w,rx,s] = x\circ[w,r,s] + r\circ[w,x,s]$ by linearized middle bumping and $[w,xr,s] + [w,xs,r] = r[w,x,s] + s[w,x,r]$ by linearized left bumping, so

$$[w,R_x r,s] \equiv -[w,L_x r,s] \equiv [w,r,L_x s]$$

modulo associators of lower degree. Thus we can move an R_x or L_x from r to s (perhaps introducing a minus sign or changing on R to an L); peeling off one operator at a time we can move all the operators from q to p,

$[w,p,q] = [w,M_{x_{i_r}} \cdots M_{x_{i_1}} 1, M_{x_{i_{r+1}}} \cdots M_{x_{i_n}} 1] \equiv \pm[w,M'_{x_{i_{n-1}}} \cdots M'_{x_{i_1}} 1, x_{i_n}]$. Now we move them all back to x_{i_n}, except this time we convert all R_x's to L_x's (our process has already done that to the R_x's in q, but now we do it to those in p too):

$$[w,p,q] \equiv \pm[w,x_{i_1}, L_{x_{i_2}} \cdots L_{x_{i_{n-1}}} x_{i_n}] = \pm[w,x_{i_1}, x_{i_2}(\cdots x_{i_n})].$$

This is almost basic; it still needs to be straightened so that $i_1 < \cdots < i_n$. This will follow if $[w, x_{i_1}, x_{i_2}(\cdots x_{i_n})]$ is an alternating function of its arguments mod $A_{w,n-1}$. Clearly it is linear in each variable and falls into $A_{w,n-1}$ if $x_{i_1} = x_{i_2}$ by right bumping. If $x_{i_k} = x_{i_{k+1}}$ for $k \geq 2$ we move L's over to x_{i_1} to get $[w, x_{i_1}, ML_x L_x N1] = \pm[w, M' x_{i_1}, L_x^2 s]$ $= \pm[w, r, x^2 s]$, use linearized right bumping to get $\pm[w, x^2, rs]$ modulo associators of lower degree, then middle bumping to get $\pm x \circ [w, x, rs]$ congruent to zero modulo associators of lower degree. Thus $[w, x_{i_1}, x_{i_2} \cdots x_{i_n}] \equiv 0 \mod A_{w,n-1}$ if two adjacent variables are equal, and the function is alternating. Consequently $[w,p,q]$ is congruent to a multiple of a basic associator $[w, x_{i_1}, x_{i_2} \cdots x_{i_n}]$ for $i_1 < \cdots < i_n$ modulo $A_{w,n-1}$ and $A_{w,n} \subset B_{w,n} + A_{w,n-1}$. ▨

From this we get

2.2 (Basic Nucleizing Theorem). If a subalgebra $B \subset A$ is generated by elements $\{x_i\}_{i \in I}$, then an element $w \in A$ nucleizes B iff all basic w-associators vanish,

$$[w, x_{i_1}, x_{i_2}(\cdots x_{i_n})] = 0 \quad (i_1 < i_2 < \cdots < i_n) .$$

Proof. If $[w,B,B] = 0$ certainly the basic w-associators must vanish. Conversely, saying these vanish says all $B_{w,n} = 0$, and since $A_{w,n} = B_{w,n}$ by the Lemma we have $A_{w,n} = 0$ too. Thus $[w,p,q] = 0$ for all monomials p,q in the x_i ; since the x_i generate B, B is spanned by such monomials, and $[w,B,B] = 0$. ▨

In characteristic $\neq 3$ situations we can break an associator involving a product into smaller pieces.

2.3. Lemma. In any linear algebra we have the associator identity

$$(2.4) \quad [xy,z,w] - [x,yz,w] + [x,y,zw] = x[y,z,w] + [x,y,z]w$$

while in any alternative algebra we have an identity

$$(2.5) \quad 3[x,y,zw] = [x,[y,z,w]] - [y,[x,z,w]] + w[x,y,z] + [w,x,y]z + w\circ[x,y,z] + [w,x,y]\circ z .$$

Proof. (2.4) is just a matter of verification:

$\{(xy)z\}w - (xy)(zw) - \{x(yz)\}w + x\{(yz)w\} + (xy)(zw) - x\{y(zw)\}$
$= \{(xy)z - x(yz)\}w + x\{(yz)w - y(zw)\} = [x,y,z]w + x[y,z,w]$.

The basic idea of (2.5) is clear: by the bumping formulas $[x,y,zw]$ is an alternating function of its arguments modulo smaller associators, so $3[x,y,zw] \equiv [z,w,xy] + [x,w,yz] + [x,y,zw] = [xy,z,w] - [x,yz,w] + [x,y,zw] \equiv 0$ by (2.4) .

To change congruence to equality, we simply fill in the details

$$\begin{aligned}
3[x,y,zw] &= -[y,x,zw] + [zw,x,y] + [x,y,zw] \\
&= \{[w,x,zy] - y[x,z,w] - w[x,z,y]\} + \{-[yw,x,z] \\
&\qquad + [y,w,x]z + [z,w,x]y\} + [x,y,zw] \quad \text{(bumping)} \\
&= [zy,w,x] + [z,x,yw] + [x,y,zw] - [y,[x,z,w]] \\
&\qquad + w[x,y,z] + [w,x,y]z
\end{aligned}$$

$$= \{-[xy,w,z] + [z,y,w]x + [x,y,w]z\} + \{-[w,x,yz] + z[x,y,w] + w[x,y,z]\} + [x,y,wz] - [y,[x,z,w]] + w[x,y,z] + [w,x,y]z \quad \text{(bumping again)}$$

$$= \{[xy,z,w] - [x,yz,w] + [x,y,zw]\} - [y,z,w]x + z\circ[x,y,w] + w[x,y,z] - [y,[x,z,w]] + w[x,y,z] + [w,x,y]z$$

$$= \{x[y,z,w] + [x,y,z]w\} - [y,z,w]x - [y,[x,z,w]] + w[x,y,z] + [w,x,y]\circ z + w[x,y,z] + [w,x,y]z \quad \text{(by (2.4))}$$

$$= [x,[y,z,w]] - [y,[x,z,w]] + w\circ[x,y,z] + [w,x,y]\circ z + w[x,y,z] + [w,x,y]z \ . \ \square$$

Don't bother trying to remember formula (2.5), just remember it exists.

Now we want a test for associativity of the whole algebra A. Again we fix a generating set $\{x_i\}_{i\in I}$ with linearly-ordered index set I. If p,q,r are monomials in the x_i we say [p,q,r] is an associator of degree $n = \partial p + \partial q + \partial r$. A basic associator of degree n is one of the form

$$N(x_{i_1},\cdots, x_{i_n}) = [x_{i_1}, x_{i_2}, x_{i_3}(\cdots x_{i_n})] (i_1 < i_2 < \cdots < i_n) \ .$$

2.6 (Basic Associating Lemma). The ideal A_n in A generated by all associators of degree $\leq n$ coincides with the ideal B_n in A generated by all basic associators of degree $\leq n$. Furthermore, $3^{n-3} A_n \subset B_3$.

Proof. If $[p,q,r]$ is an associator of degree n, let x_{i_1} be the lowest index appearing in p,q, or r . Interchanging p,q,r we may assume it occurs in q . Then holding p fixed gives $[p,q,r]$ congruent modulo p-associators of lower degree to $[p, x_{i_1}, q]$ (even $q = x_{i_2}(x_{i_3}\cdots x_{i_n})$) . Now $[p, x_{i_1}, q] = [x_{i_1}, q,p]$; repeating this procedure with x_{i_1} fixed, $[x_{i_1}, q,p]$ is congruent modulo x_{i_1}-associators of lower degree to $[x_{i_1}, x_{i_2}, x_{i_3}(\cdots x_{i_n})]$ for $i_2 < i_3 < \cdots < i_n$. By choice of i_1 we have $i_1 \leq i_2$; if $i_1 = i_2$ the associator vanishes, otherwise $i_1 < i_2$ and $[p,q,r] = [x_{i_1}, x_{i_2}, x_{i_3}(\cdots x_{i_n})]$ is congruent to a basic associator modulo A_{n-1} . Once more we have $A_n \subset B_n + A_{n-1}$, so by induction $A_n = B_n$. Identity (2.5) shows $3A_n \subset A_{n-1}$; repeating this, with each factor 3 lowering degree by one, gives $3^{n-3} A_n \subset A_3 = B_3$. ∎

This leads to the following associativity test:

2.7 (Basic Associativity Theorem) If $\{x_i\}_{i \in I}$ is a set of generators for an alternative algebra A, then A will be associative if each basic associator vanishes

$$[x_{i_1}, x_{i_2}, x_{i_3}(\cdots x_{i_n})] = 0 \quad (i_1 < \cdots < i_n) .$$

If 3 is injective it suffices if the generators associate

$$[x_{i_1}, x_{i_2}, x_{i_3}] = 0 .$$

Proof. A is associative if all associators of arbitrary degree in the generators vanish, i.e. all $A_n = 0$. Our hypothesis guarantees all basic associators vanish, $B_n = 0$. By the Basic Associating Lemma $A_n = B_n$. Therefore $A_n = 0$ and A is associative.

If we only assume the basic associators of degree 3 vanish, $B_3 = 0$, but 3 is injective, then $3^{n-3} A_n \subset B_3 = 0$ implies $A_n = 0$ and again A is associative. ∎

We stress again that IN CHARACTERISTIC $\neq 3$ SITUATIONS IT SUFFICES IF THE GENERATORS ASSOCIATE, but otherwise one must check all basic associators. Once more characteristic 3 breaks up a lovely theorem.

Another case where associating with generators is enough is when the element also commutes with generators. We need three preliminary formulas

(2.8) $3[x,y,z] = [xy,z] - x[y,z] - [x,z]y$

(2.9) $3[x,y,z] = [xy,z] + [yz,x] + [zx,y]$

(2.10) $[x,y,zw] = [z,[x,y],w] + [z,x,y]w + z[x,y,w]$.

The first two formulas are just verifications: $3[x,y,z] = [x,y,z] + [z,x,y] - [x,z,y] = (xy)z - x(yz) + (zx)y - z(xy) - (xz)y + x(zy) = [xy,z] - x[y,z] - [x,z]y$ and $3[x,y,z] = [x,y,z] + [y,z,x] + [z,x,y] = (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy) = [xy,z] + [yz,x] + [zx,y]$. For the third, $[z,[x,y], w] + [z,x,y]w$

$+ z[x,y,w] = [z,xy,w] - [z,yx,w] + [zx,y,w] - [z,xy,w]$
$+ [z,x,yw]$ (by (2.4)) $= -\{[z,yx,w] + [z,yw,x]\} + \{[zx,y,w]$
$+ [zw,y,x]\} - [zw,y,x] = -\{x[z,y,w] + w[z,y,x]\} + \{x[z,y,x]$
$+ w[z,y,x]\} + [x,y,zw]$ (left bumping) $= [x,y,zw]$. ▨

2.11 Proposition. If an element c commutes and associates with generators x_i of a subalgebra B, then c centralizes B:

$$[c,x_i] = [c,x_i,x_j] = 0 \quad \text{implies} \quad c \in C_A(B) .$$

Proof. We first show c nucleizes B. By the Basic Nucleizing Theorem it suffices if basic c-associators $[c,x_{i_1}, x_{i_2}(\cdots x_{i_n})]$ vanish. We induct on n, the result being true for n = 2 by hypothesis. If true for degrees < n then by (2.10)

$[x_{i_1}, c, x_{i_2} x] = [x_{i_2}, [x_{i_1}, c], x] + [x_{i_2}, x_{i_1}, c]x + x_{i_2}[x_{i_1}, c, x] = 0$ since $[c,x_{i_1}] = [c, x_{i_2}, x_{i_1}] = 0$ by hypothesis and $[c, x_{i_1}, x] = 0$ by induction.

Once c nucleizes B, commutativity is easy: the set of $b \in B$ which commute with c is a subspace containing the generators x_i and closed under multiplication,

$$[c, bb'] = [c, b]b' + b[c, b'] - 3[b, c, b'] = 0$$

by (2.8) and nuclearity of c , so it coincides with B. ▨

Exercises

2.1 Prove directly that if c commutes and associates with a set of generators for a subalgebra B, then c centralizes B. (Prove $[c,p] = [c,q,r] = 0$ for monomials by induction on $n = \partial p = \partial q + \partial r$) .

2.2 Prove the Basic Associativity Theorem directly, showing that $A_n \subset B_n + A_{n-1}$ by using induction to rearrange parentheses in the monomials p,q,r making up an associator [p,q,r] of degree n.

2.3 Prove that there is no relation $3[w,x,yz] = \alpha_1[w,x,y]z + \beta_1 z[w,x,y] + \alpha_2[w,y,z]x + \beta_2 x[w,y,z] + \alpha_3[w,z,x]y + \beta_3 y[w,z,x]$ valied in all alternative algebras (i.e. there is none in the free algebra). Thus in (2.5) there is no way make all the lower degree associators be w-associators. Give an example where $[w,x_i,x_j] = 0$ for a set of generators for B but w doesn't nucleize B. (even over a field of characteristic $\neq 3$).