8. Bimodules with involution

The notion of bimodule-with-involution, or *-bimodule, is important in applications to Jordan algebras, since Jordan matrix algebras coordinatized by alternative algebras have Jordan bimodules coordinatized by alternative *-bimodules.

The regular and Cayley-Dickson bimodules for a composition algebra carry natural involutions induced from the enveloping Cayley-Dickson process algebra. In characteristic $\neq 2$ a general *-bimodule for a composition algebra is a direct sum of *-simple images of these bimodules with the standard involution or its negative. This *-semisimplicity fails in characteristic 2; instead, in this case *-bimodules are built out of images of regular or Cayley-Dickson exchange bimodules.

A **bimodule with involution** or ***-bimodule** for a *-algebra A is a bimodule M together with an endomorphism J on M of period 2 such that

$$a \oplus m \to a^* \oplus J(m)$$

defines an involution on the split null extension $E = A \oplus M$, turning it into a *-algebra. When the particular map J is not important we will denote it and the extension to the split null extension both by our generic symbol * for an involution and write

$$(a+m)^* = a^* + m^*.$$

Because $(xy)^* = y^*x^*$ already holds for $x,y \in A$ (as A is a *-algebra) or for $x,y \in M$ (as $M^2 = 0$), the condition that J induce an involution on E amounts to $(am)^* = m^*a^*$ and $(ma)^* = a^*m^*$, or

$$J(am) = J(m)a^*, \; J(ma) = a^*J(m), \; J^2 = -I.$$

This makes it clear that the negative of an involution on M is again an involution. Also, the regular bimodule $M = A$ with the natural involution becomes a *-bimodule.

We have obvious notions of ***-sub-bimodule**, ***-simple** (***-irreducible**), ***-semisimple** (***-completely reducible**), ***-homomorphism** etc. One convenient fact about *-homomorphisms: if $M \xrightarrow{F} N$ is left A-linear ($F(am) = aF(m)$) and preserves involution ($F(m^*) = F(m)^*$), it is necessarily right A-linear: $F(ma) = F((a^*m^*)^*) = F(a^*m^*)^* = (a^*F(m^*))^* = F(m^*)^*a = F(m)a$.

For a composition algebra A we thus obtain the **regular *-bimodules**

$$\mathrm{reg}_+(A) \text{ and } \mathrm{reg}_-(A)$$

obtained from the regular bimodule $M = A$ by taking the usual involution or its negative. The **Cayley-Dickson *-bimodules**

$$\mathrm{cay}_+(A) \text{ and } \mathrm{cay}_-(A)$$

are obtained from the Cayley-Dickson bimodule $M = A\ell$ by taking the usual involution (namely $-I$) or its negative (namely $+I$), induced from the involution on $\mathbb{C}(A) = A \oplus A\ell$.

As an example, we consider the possible involutions on the regular and Cayley-Dickson bimodules for the composition algebras.

where M^* as linear space is just a copy of M, with A-bimodule structure

$$a(m \oplus n^*) = am \oplus (na^*)^*$$

$$(m \oplus n^*)a = ma \oplus (a^*n)^*$$

and exchange involution

$$\{m \oplus n^*\}^* = n \oplus m^*.$$

This is indeed an involution of the module structure since it is clearly of period 2 and $\{a(m \oplus n^*)\}^* = \{am \oplus (na^*)^*\}^*$ $= na^* \oplus (am)^* = (n \oplus m^*)a^* = (m \oplus n^*)^*a^*$. To see ex(M) is indeed an alternative A-bimodule, notice that $M \subset ex(M)$ carries its given bimodule structure while the birepresentation on M^* is given by $\ell^*_a = r_{a^*}$, $r^*_a = \ell_{a^*}$ in terms of ℓ, r on M. Now we know we can give M the structure of an A^{op}-bimodule M^{op} by $\ell^{op}_a = r_a$, $r^{op}_a = \ell_a$ (the split null extension is then just $A^{op} \oplus M^{op} = (A \oplus M)^{op}$, which is alternative if $A \oplus M$ is). Composing this with the isomorphism $A \to A^{op}$ by $a \mapsto a^*$, we get a birepresentation ℓ^*: $a \to a^* \to \ell^{op}_{a^*} = r_{a^*}$ and r^*: $a \to a^* \to r^{op}_{a^*} = \ell_{a^*}$. Thus as bimodule ex(M) is just the direct sum of the two bimodules M and M^*.

Note that this construction is additive,

$$ex(\oplus_i M_i) \cong \oplus_i\ ex(M_i).$$

The usefulness of the exchange bimodule resides in its universal property.

8.2 **Proposition.** If A is an ordinary composition algebra of dimension 1, 4, 8 over a field Φ then the only involutions on reg(A) are $\pm$ the standard involution; if A has dimension 2 the involutions are of the form $J(a) = c_J a^*$ where $n(c_J) = 1$, and in this case A_J is *-isomorphic to $\mathrm{reg}_-(A)$. If $A = Ae_1 \oplus Ae_2$ is split of dimension 2 there are no involutions on the module Ae_i.

If A is a division algebra of dimension 1, 2, 4 then the only involutions on cay(A) are $\pm$ the standard involution (i.e. $\mp I$); if $A = e_1A + e_2A$ is split of dimension 2 or 4, the only involution on cay (e_iA) are $\pm$ the restrictions of the standard involutions (i.e. $\mp I$).

Proof. According to the Criterion 4.14, in the regular bimodule reg(A) of dimension 1, 4, 8 the only commuters are the elements of $\Phi 1$, so an involution J must have $J(1) = \alpha 1$; since $J^2(1) = 1$ we see $\alpha^2 = 1$, so $\alpha = \pm 1$ and by (8.1) $J(a) = J(a \cdot 1) = J(1)a^* = \pm a^*$. Thus J is $\pm$ the standard involution. In dimension 2, if $J(1) = c$ then $J(a) = J(a \cdot 1) = J(1)a^* = ca^*$ and $1 = J(J(1)) = J(c) = cc^* = n(c)$. The map $a \to c^*a$ is a *-isomorphism $A_J \xrightarrow{F} A_+$: it is a linear bijection with $F(a \cdot b) = c^*ab = ac^*b = aF(b)$ and $F(b \cdot a) = c^*ba = F(b)a$ and $F(J(a)) = F(ca^*) = c^*ca^* = a^*$. In the split case, since $J(e_i) = J(e_ie_i) = J(e_i)e_i^* = J(e_i)e_j \in \Phi e_ie_j = 0$, there are no involutions on Φe_i.

Cay(A) $= A\ell$ is not an alternative bimodule if A has dimension 8. Let A be a division algebra of dimension 1, 2, or 4. If $J(\ell) = c\ell$ then $J(a\ell) = J(\ell)a = (c\ell)a = (ca)\ell$, where

$\ell = J(J\ell) = J(c\ell) = c^2\ell$. But $c^2 = 1$ in a division algebra implies $c = \pm 1$ (as $(c-1)(c+1) = 0$), so $J(a\ell) = \pm a\ell$. In this case the only involutions are $J = \pm I$. If A is split of dimension 2 or 4, $A = e_1A \oplus e_2A$, then cay A is not irreducible and has lots of involutions $J(a\ell) = (ca)\ell$ for $c^2 = 1$. However, cay (e_iA) is irreducible. If J is an involution on $(e_iA)\ell$ we have $J(e_i\ell) = c\ell$ for some $c = e_i c \in e_i\mathbb{C}$ and $J((e_ia)\ell) = J(a(e_i\ell)) = J(e_i\ell)a^* = (c\ell)a^* = (ca)\ell$. In particular $c\ell = J(e_i\ell) = J((e_ie_i)\ell) = (ce_i)\ell$, so $c = ce_i$ and $c = e_ice_i$ belongs to the Peirce space $e_iAe_i = \Phi e_i$: $c = \gamma e_i$. From $J(J(e_i\ell)) = e_i\ell$ we see $c^2 = e_i$, $\gamma^2 = 1$, $\gamma = \pm 1$, $c = \pm e_i$. Thus $J((e_ia)\ell) = \pm(e_ia)\ell$, and J is $\pm I$. ∎

Strange things happen with involutions in characteristic 2 (since + and − are the same), so we first consider the characteristic $\neq 2$ case.

8.3 *-Bimodule Theorem. (First Version) Every *-bimodule for an ordinary composition algebra $\mathbb{C}$ over a field Φ of characteristic $\neq 2$ is *-semisimple (completely *-reducible), with *-simple (*-irreducible) sub-bimodules isomorphic to the *-sub-bimodules of the regular and Cayley-Dickson *-bimodules. We thus obtain the following list of *-simple bimodules:

I. $\mathbb{C} = \Phi 1$: $reg_+(\mathbb{C})$, $reg_-(\mathbb{C})$

IIa. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1)$ division algebra: $reg_+(\mathbb{C})$, $cay_+(\mathbb{C})$, $cay_-(\mathbb{C})$

IIb. $\mathbb{C} = \mathbb{C}(\Phi,1)$ split: $reg_+(\mathbb{C})$, $cay_+(\Phi e_1)$, $cay_-(\Phi e_1)$, $cay_+(\Phi e_2)$, $cay_-(\Phi e_2)$

IIIa. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1\mu_2)$ division algebra: $reg_+(\mathbb{C})$, $reg_-(\mathbb{C})$, $cay_+(\mathbb{C})$ $cay_-(\mathbb{C})$

IIIb. $\mathbb{C} = \mathbb{C}(\Phi,1,1)$ split: $reg_+(\mathbb{C})$, $reg_-(\mathbb{C})$, $cay_+((e_1\mathbb{C})\ell)$, $cay_-((e_1\mathbb{C})\ell)$

IV. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1,\mu_2,\mu_3)$: $reg_+(\mathbb{C})$, $reg_-(\mathbb{C})$.

Proof. Once more the essential part of the proof is that every *-bimodule is a sum of images of regular and Cayley-Dickson *-bimodules. To fill M up with regular or Cayley-Dickson *-bimodules we need only fill it up with symmetric and skew commuters and *-commuters. In characteristic $\neq 2$ it is easy to fill M up with such elements: every commuter or *-commuter is the sum $m = m_+ + m_-$ of its symetric and skew parts, which are again commuters or *-commuters.

In more detail, in characteristic $\neq 2$ every element m is the sum of its symmetric and skew parts $m_+ = \frac{1}{2}(m + m^*)$ and $m_- = \frac{1}{2}(m - m^*)$. Since $x \to x^*$ is an anti-automorphism on $E = A \oplus M$, if m is a commuter or *-commuter so is its image m^*: $am^* = (ma^*)^* = (a^*m)^* = m^*a$ or $am^* = (ma^*)^* = (am)^* = m^*a^*$. Therefore the averages m_+ and m_- are also commuters or *-commuters, and every commuter (resp. *-commuter) is a sum of a symmetric and a skew commuter (resp. *-commuter). Since the ordinary commuters and *-commuters generate M by the Bimodule Theorem 7.0, so do the symmetric and skew ones.

A symmetric (resp. skew) commuter m generates a -bimodule {m} which is an image of $reg_+(A)$ (resp. $reg_-(A)$) since the module homomorphism $a \to am$ of 7.1 is automatically a *-homomorphism: $a^* \to a^*m = + a^*m^* = (ma)^* = (am)^*$ (resp. $a^* \to -(am)^*$). Similarly a skew (resp. symmetric) *-commuter generates a *-bimodule {m} which is an image of $cay_+(A)$ (resp. cay(A)) since $a\ell \to am$ as in 7.3 is a *-homomorphism : $(a\ell)^* = -a\ell \to -am = -ma^* = m^*a^* = (am)^*$ (resp. $(a\ell)^* \to -(am)^*$).

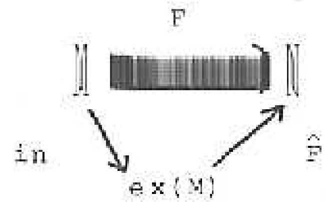
Thus M is generated by images of $reg_{\pm}(A)$ and $cay_{\pm}(A)$. These regular bimodules are *-semisimple with *-simple summands given by the above list. The difference between the present list and that in the Bimodule Theorem is due to the facts (I) again we don't need the Cayley-Dickson bimodules in dimension 1 since $cay_{\pm}(\Phi) \cong reg_{\mp}(\Phi)$ as *-bimodules, (II) we don't need both regular involutions in dimension 2 since $reg_{+}(\mathbb{C}) \overset{F}{\cong} reg_{-}(\mathbb{C})$ under the map $a \to ia$ for $\mathbb{C} = \Phi 1 + \Phi i$ $(i^2 = \mu 1,\ i^* = -i)$: $F(a^{*+}) = ia^* = -i^*a^* = -(ia)^* = F(a)^{*-}$, and moreover $reg_{+}(\mathbb{C})$ is simple <u>as a *-bimodule</u> even in the split case as Φe_1, Φe_2 alone don't form *-bimodules by 8.2; (III) again we don't need both pieces of the Cayley-Dickson bimodules in split dimension 4, since $cay_{\pm}((e_2\mathbb{C})\ell) \cong cay_{\pm}((e_1\mathbb{C})\ell)$ as *-bimodules. ∎

We now develop an alternate approach which works in all characteristics: instead of breaking things into symmetric and skew pieces $reg_{\pm}(A)$ and $cay_{\pm}(A)$, we keep the two glued together, filling a bimodule M up with homomorphic images of two bimodules reg(A) and cay(A) with exchange involution.

Suppose M is any bimodule (not necessarily with involution) for a *-algebra A. Then we can imbed M in the **exchange *-bimodule**

$$ex(M) = M \oplus M^*$$

8.4 (Universal Property of Exchange Bimodule) Any A-bimodule homomorphism $M \xrightarrow{F} N$ of a bimodule M into a *-bimodule N extends uniquely to a *-homomorphism $ex(M) \xrightarrow{\hat{F}} N$,



Proof. If we define $\hat{F}(m \oplus n^*) = F(m) + F(n)^*$ we have a *-homomorphism because $\hat{F}(a(m \oplus n^*)) = \hat{F}(am \oplus (na^*)^*) = F(am) + F(na^*)^* = aF(m) + aF(n)^* = a\hat{F}(m \oplus n^*)$ and $\hat{F}((m \oplus n^*)^*) = \hat{F}(n \oplus m^*) = F(n) + F(m)^* = \{F(n)^* + F(m)\}^* = \hat{F}(m \oplus n^*)^*$. This extension is unique since it is uniquely determined on the *-generating set M of ex(M). ∎

8.5 Example. If $\mathbb{C} = \Phi e_1 \boxplus \Phi e_2$ is split of dimension 2 then

$$ex(\Phi e_1) \cong ex(\Phi e_2) \cong reg_{\perp}(\mathbb{C})$$

since the imbedding $\Phi e_i \to reg_{\perp}(\mathbb{C})$ extends to a *-imbedding $ex(\Phi e_i) \to reg_{\perp}\,\mathbb{C}$ which is easily seen to be an isomorphism. ∎

8.6 Example. Just as we can represent $cay(\mathbb{C}) = \mathbb{C}\ell$ as $\mathbb{C}$ with operations $\ell_a = R_a$, $r_a = R_{a^*}$, the formulas $a\{b\ell \oplus (c\ell)^*\} = a(b\ell) \oplus \{(c\ell)a^*\}^* = (ba)\ell \oplus \{(ca)\ell\}^*$, $\{b\ell \oplus (c\ell)^*\}a = (b\ell)a \oplus \{a^*(c\ell)\}^* = (ba^*)\ell \oplus \{(ca^*)\ell\}^*$ show we can represent $ex(\mathbb{C}\ell)$ as $\mathbb{C} \oplus \mathbb{C}$ with action $a(b \oplus c^*) = ba \oplus (ca)^*$, $(b \oplus c^*)a = ba^* \oplus (ca^*)^*$, i.e. $\ell_a = R_a \oplus R_a$, $r_a = R_{a^*} \oplus R_{a^*}$. Thus $ex(cay(\mathbb{C})) = cay(\mathbb{C}) \oplus cay(\mathbb{C})$ as bimodule. ∎

Thus we have a universal way of building *-bimodules out of ordinary bimodules. What happens if M already carries an involution, i.e. an endomorphism J of period 2 satisfying $J(am) = J(m)a^*$ and $J(ma) = a^*J(m)$? By the universal property (with $N = M_J$, $F = I$) ex(M) must contain a copy of M_J as *-bimodule. In other words, ex(M) contains all possible involutions J on M.

Indeed, we can explicitly exhibit the copy of M_J inside ex(M): it consists of all

$$t_J(m) = m \oplus J(m)^*$$

of elements $m \in M$. The map $m \xrightarrow{F} t_J(m)$ is a *-isomorphism of M_J onto $t_J(M) \subset ex(M)$: clearly it is a linear bijection, and it is a homomorphism of *-bimodules since it is left A-linear $F(am) = am \oplus J(am)^* = am \oplus \{J(m)a^*\}^* = a\{m \oplus J(m)^*\} = aF(m)$ and preserves involutions $F(J(m)) = J(m) \oplus J(J(m))^* = J(m) \oplus m^* = \{m \oplus J(m)^*\}^* = F(m)^*$ by definition of operations in ex(M).

Thus if M is a *-bimodule it is *-imbedded in ex(M). What does the remaining part of ex(M) look like? We claim it looks like M, but relative to the involution $-J$:

$$ex(M)/t_J(M) \cong M_{-J}.$$

Indeed, by the Universal Property 8.4 the isomorphism $M \xrightarrow{I} M_{-J}$ induces an epimorphism $ex(M) \xrightarrow{\hat{F}} M_{-J}$ by $\hat{F}(m \oplus n^*) = F(m) + (-J)(F(n)) = m - J(n)$, with kernel $\{m \oplus n^* \mid m = J(n)\} = \{m \oplus n^* \mid J(m) = n\} = \{m \oplus J(m)^*\} = t_J(M)$. Thus $\hat{F}$ induces *-isomorphism $ex(M)/t_J(M) \to M_{-J}$.

In characteristic $\neq 2$ the exchange bimodule decomposes into the direct sum

$$ex(M) = t_J(M) \oplus t_{-J}(M) \cong M_J \oplus M_{-J} \qquad \text{(characteristic} \neq 2) \tag{8.7}$$

of one copy of M under its given involution, and one copy with the negative of this involution. One way to see it splits is to

J in *

observe that the bimodule isomorphism $M \to M \to ex(M) \to ex(M)$ (i.e. $F(m) = 0 \oplus J(m)^*$) extends to a *-automorphism $ex(M) \overset{\hat{F}}{\to} ex(M)$ of period 2 (i.e. $\hat{F}(m \oplus n^*) = J(n) \oplus J(m)^*$) by universality, so $ex(M)$ is the direct sum of the ± 1 eigenspaces of $\hat{F}$, namely the *-submodules $\{m \oplus n^* | J(n) \oplus J(m)^* = \pm m \oplus n^*\} = \{m \oplus n^* | J(m) = \pm n\} = \{m \oplus \pm J(m)^*\} = \{t_{\pm J}(m)\} = t_{\pm J}(M)$.

In characteristic 2, + 1 coincides with − 1 so $t_J(M) = t_{-J}(M)$ and $ex(M)$ does not break up into their direct sum. All we can say is

$$t_J(M) \cong M_J \cong ex(M)/t_J(M) \quad .$$

Next we investigate to what extent $ex(M)$ preserves irreducibility.

8.8 Proposition. Let M be an irreducible A-bimodule. Then the only proper *-submodules and proper *-homomorphic images of $ex(M)$ are the submodules $t_J(M) \cong M_J$ for all possible involutions J on M (if any).

Proof. We begin by recalling the basic fact from linear algebra that the only proper <u>submodules</u> of $M_1 \oplus M_2$ when the M_i are irreducible are

$$M_1, \ M_2, \ t_F(M_1) = \{m_1 \oplus F(m_1)\}$$

for all possible isomorphisms $M_1 \overset{F}{\to} M_2$. In our case, by definition of the bimodule structure on M^* an isomorphism $M \overset{F}{\to} M^*$ is determined by a map $M \overset{J}{\to} M$ which satisfies $J(am) = J(m)a^*$, $J(ma) = a^*J(m)$. If we demand proper <u>*-sub-bimodules</u>, M and M^* are

Proof. From the ordinary Bimodule Theorem 7.0 we know M is a (direct) sum of homomorphic images M_s of regular or Cayley-Dickson bimodules $\mathbb{C}1$ or $\mathbb{C}\ell$. By the Universal Property 8.4 of the exchange bimodule, the homomorphisms

$\mathbb{C}s \to M_s \xrightarrow{in} M$ extend to *-homomorphisms $ex(\mathbb{C}s) \to M$. Thus M is a sum of *-homomorphic images of $ex(\mathbb{C})$ and $ex(\mathbb{C}\ell)$.

It remains to list the images. The regular bimodule is simple when $\mathbb{C}$ is simple, i.e. in all cases but IIb; when $\mathbb{C}$ is simple we know by 8.8 the only nonzero images of $ex(\mathbb{C})$ are $ex(\mathbb{C})$ and $reg_{\pm}(\mathbb{C})$ (recall by 8.2 the only involutions on $\mathbb{C}$ are $\pm$ the standard involution in dimensions 1, 4, 8, and in dimension 2 are all equivalent to the standard involution). In IIb we have $\mathbb{C} = \Phi e_1 \boxplus \Phi e_2$ and $ex(\mathbb{C}) = ex(\Phi e_1) \oplus ex(\Phi e_2)$ for $ex(\Phi e_i) \cong reg_+(\mathbb{C})$ by Example 8.5, which by 8.8 are simple since there are no involutions J on Φe_i according to 8.2. Recall for I that $reg_{\pm}(\mathbb{C}) \cong cay_{\pm}(\mathbb{C})$ in dimension 1, for II that $reg_+(\mathbb{C}) \cong reg_-(\mathbb{C})$ in dimension 2 (this being trivial in characteristic 2), and for III that $(e_1\mathbb{C})\ell \cong (e_2\mathbb{C})\ell$ in dimension 4. This classifies the regular images.

Turning to the Cayley-Dickson images, we know $cay\ \mathbb{C} = \mathbb{C}\ell$ is simple when $\mathbb{C}$ has no right ideals, i.e. I, IIa, IIIa. When $\mathbb{C}\ell$ is simple, by 8.8 the only nonzero * images are $ex(\mathbb{C}\ell)$ and $\mathbb{C}\ell_{\pm} = cay_{\pm}(\mathbb{C})$ (by 8.2, the only involutions on $\mathbb{C}\ell$ are $\pm$ the standard involution in these cases). In the split cases IIb and IIIb we have $\mathbb{C}\ell = (e_1\mathbb{C})\ell \oplus (e_2\mathbb{C})\ell$ for $(e_1\mathbb{C})\ell \cong (e_2\mathbb{C})\ell$ simple, so $ex(\mathbb{C}\ell) = ex((e_1\mathbb{C})\ell) \oplus ex((e_2\mathbb{C})\ell)$. The * images of $ex(\mathbb{C}\ell)$ are thus sums of *-images of $ex((e_1\mathbb{C})\ell)$, which by 8.8 and 8.2 are again either $ex((e_1\mathbb{C})\ell)$ or $(e_1\mathbb{C})\ell_{\pm} = cay_{\pm}(e_1\mathbb{C})$. ∎

ruled out, and $t_J(M)$ is a *-bimodule only for those J of period 2: $t_J(m)^* = J(m) \oplus m^* \in t_J(M)$ implies $J(J(m)) = m$. Thus the only proper *-submodules are the $t_J(M)$ for the involutions J on M.

Since a proper *-homomorphic image of ex(M) is isomorphic to ex(M)/K for some proper *-submodule K, for $K = t_J(M)$, we get $t_{-J}(M)$, where $-J$ ranges over all possible involutions as J does. ■

For all characteristics, we can at least fill up a given *-bimodule M for a composition algebra $\mathbb{C}$ with images of

$$ex(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}^*, \; ex(\mathbb{C}\ell) = \mathbb{C}\ell \oplus (\mathbb{C}\ell)^*.$$

8.9 *-Bimodule Theorem (2nd Version) Every *-bimodule for an ordinary composition algebra $\mathbb{C}$ over a field Φ is a sum of *-homomorphic images of the regular and Cayley-Dickson exchange bimodules $ex(\mathbb{C})$ and $ex(\mathbb{C}\ell)$. The list of images is

I. $\mathbb{C} = \Phi 1$: $ex(\mathbb{C})$, $reg_-(\mathbb{C})$

IIa. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1)$ division: $ex(\mathbb{C})$, $reg_+(\mathbb{C})$; $ex(\mathbb{C}j)$, $cay_\pm(\mathbb{C})$

IIb. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1)$ split: $ex(\mathbb{C}e_1) \cong ex(\mathbb{C}e_2) \cong reg_+(\mathbb{C})$; $ex((e_1\mathbb{C})j)$, $ex((e_2\mathbb{C})j)$, $cay_\pm(e_1\mathbb{C})$, $cay_\pm(e_2\mathbb{C})$

IIIa. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1,\mu_2)$ division: $ex(\mathbb{C})$, $reg_+(\mathbb{C})$; $ex(\mathbb{C}\ell)$, $cay_\pm(\mathbb{C}$

IIIb. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1,\mu_2)$ split: $ex(\mathbb{C})$, $reg_\pm(\mathbb{C})$; $ex((e_1\mathbb{C})\ell)$, $cay_\pm(e_1\mathbb{C})$,

IV. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1,\mu_2,\mu_3)$: $ex(\mathbb{C})$, $reg_\pm(\mathbb{C})$.

The $ex(\mathbb{C})$ and $ex(\mathbb{C}\ell)$ are never *-simple, and are *-semisimple only in characteristic $\neq 2$.

VII.8 Exercises

8.1 Verify directly that I is an involution on cay($\mathbb{C}$); whenever $\pm$I is an involution on M, show ex(M) $\cong$ M $\oplus$ M as ordinary bimodule.

8.2 Let $t(m) = m + m^*$ in any *-bimodule. If m is a *-commuter, show t({m}) is a *-submodule, while if m is a commuter and A is a composition algebra with nontrivial involution then t({m}) generates {m,m*}.

8.3 Verify directly that if m is a commuter then $a \oplus b^* \to am + b^*m^*$ is a *-homomorphism ex($\mathbb{C}$) $\to$ {m,m*}, and if m is a *-commuter then $a \oplus b^* \to am + bm^*$ is a *-homomorphism ex($\mathbb{C}\ell$) $\to$ {m,m*}.

8.4 In Proposition 8.2 construct infinitely many involutions on the 2-dimensional module reg($\mathbb{C}$) when $\Phi = \mathbb{R}$. Similarly construct infinitely many involutions on cay($\mathbb{C}$) when $\mathbb{C}$ is split of dimension 2 or 4.

8.5 Verify $F(ma) = F(m)\cdot a$ directly in 8.4; in $ex(M)/t_J(M) \cong M_{-J}$; in $t_J(M) \cong M_J$.

8.6 Prove that the exchange bimodules ex($\mathbb{C}$) and ex($\mathbb{C}\ell$) are definitely not *-semisimple in characteristic 2.

8.7 Prove that the images listed in 8.8 are non-isomorphic; repeat for those in 8.9 (where of course $reg_+ = reg_-$, $cay_+ = cay_-$ in characteristic 2).