§7 Bimodules

Suppose B is composition algebra over a field Φ, and consider the algebra $C(B,\mu)$ obtained from B and the scalar μ by the Cayley-Dickson process:

$$C(B,\mu) = B \oplus B\ell .$$

Notice that both the space B and the space $B\ell$ are invariant under multiplication by B (recall $b(c\ell) = (cb)\ell$ and $(c\ell)b = (c\bar{b})\ell$). Thus the spaces B and $B\ell$ with the induced multiplications furnish us with B bimodules; B is called the <u>regular bimodule</u> reg (B) and $B\ell$ the <u>Cayley-Dickson bimodule</u> cay (B). (Note that B is always an alternative bimodule for B, but $B\ell$ will be such only when B is associative: $[a,b,c\ell] + [a,c\ell,b]$ $= \{c(ab) - (cb)a + (ca)\bar{b} - (c\bar{b})a\}\ell = \{c(ab) - t(b)ca + (ca)\bar{b}\}\ell =$ $\{c(ab) - (ca)b\}\ell = - [c,a,b]\ell$. Thus when B is a Cayley algebra $B\ell$ is not an alternative bimodule just as $C(B,\mu)$ is not an alternative algebra.

Another way to interpret the Cayley-Dickson bimodule is as follows. We can identify $B\ell$ with B as a Φ-module, and the birepresentation λ, ρ of B becomes $\lambda_b c = cb$ and $\rho_b c = c\bar{b}$, i.e. $\lambda_b = R_b$ and $\rho_b = R_{\bar{b}}$.

What are the submodules of these two basic kinds of bimodules? Let's begin with the regular bimodules. The sub-bimodules are the subspaces of B invariant under all $\lambda_b = L_b$ and $\rho_b = R_b$, that is, precisely the two-sided ideals. Now by 4.4 we know the composition algebras are, with one exception, simple - all the Cayley algebras, all the quaternion algebras, and all the fields (Φ or inseparable Ω) are simple, but the split quadratic extensions $\Phi e_1 \boxplus \Phi e_2$ are only $*$-simple, being the direct sum of two ideals. The proper ideals are Φe_1 and Φe_2, and these are non-isomorphic as $B = \Phi e_1 \boxplus \Phi e_2$ bimodules.

Turning to the Cayley-Dickson bimodule, the second way of representing the representation $\lambda_b = R_b$, $\rho_b = R_{\bar{b}}$ shows the sub-bimodules are precisely the right ideals of B. Division algebras have no right ideals, and neither do Cayley algebras (not even the split ones), so the only composition algebras with proper right ideals are the split quadratic extensions (with right ideals (= ideals) being Φe_1, Φe_2) and the split quaternion algebras (with right ideals of the form eQ for suitable idempotents e). In either of these cases ($\Phi e_1 + \Phi e_2$ or eQ + (1-e)Q) the bimodule splits into a direct sum of two irreducible bimodules.

Thus in all cases the regular and Cayley-Dickson bimodules are completely reducible, and most of the time they are actually irreducible. For the ordinary composition algebras this extends to all bimodules.

7.1 (Bimodule Theorem) Every unital bimodule for one of the ordinary composition algebras $\mathbb{C}$ over a field Φ is completely reducible. The irreducible bimodules for $\mathbb{C}$ are isomorphic to the irreducible sub-bimodules of the regular and Cayley-Dickson bimodules, leading to the following list of irreducible bimodules:

I. $\mathbb{C} = \Phi$: reg $\mathbb{C}$

IIa. $\mathbb{C} = \mathbb{C}(\Phi,\mu_1)$ division algebra: reg $\mathbb{C}$, cay $\mathbb{C}$

IIb. $\mathbb{C} = \mathbb{C}(\Phi,1)$ split: reg (Φe_1), reg (Φe_2), cay (Φe_1), cay (Φe_2)

IIIa. $\mathbb{C} = \mathbb{C}(\Phi;\mu_1,\mu_2)$ division algebra: reg $\mathbb{C}$, cay $\mathbb{C}$

IIIb. $\mathbb{C} = \mathbb{C}(\Phi;1,1)$ split: reg $\mathbb{C}$, cay $(e_1\mathbb{C})$

IV. $\mathbb{C} = \mathbb{C}(\Phi;\mu_1,\mu_2,\mu_3)$: reg $\mathbb{C}$.

Proof. We first verify all irreducible sub-bimodules of the regular and Cayley-Dickson bimodules $\mathbb{C}$ and $\mathbb{C}\ell$ have the above forms. In case I, $\mathbb{C} = \Phi$ and $\mathbb{C}\ell = \Phi\ell$ are isomorphic as bimodules and we can delete the second. In split IIIb, every right ideal is isomorphic (as right Q-module) to $e_1 Q = \Phi e_1 + \Phi e_1 j$. In case IV the Cayley-Dickson bimodule is not an alternative bimodule.

To verify that all bimodules are completely reducible with the above types of irreducibles, it suffices to prove every bimodule M is a sum of homomorphic images of regular and Cayley-Dickson bimodules (and therefore a sum of isomorphic copies of the irreducible constituents of these bimodules).

Certain types of elements give rise naturally to such submodules. We say an element $m \in M$ is a <u>commuter</u> if $am = ma$ for all $a \in \mathbb{C}$.

7.2 (Commuter Lemma) If m is a commuter for $\mathbb{C}$, $am = ma$ for all $a \in \mathbb{C}$, then

$$a(bm) = (ab)m, \quad (bm)a = (ba)m$$

and the map $a \to am$ is a homomorphism of the regular bimodule $\mathbb{C}$ onto the cyclic submodule $\{m\}$ generated by m.

Proof. The formulas express the condition $F(a\cdot b) = aF(b)$, $F(b\cdot a) = F(b)a$ that $F(a) = am$ be a homomorphism of bimodules. The first formula is just $[a,b,m] = 0$, and the latter is equivalent to $(mb)a = m(ba)$ or $[m,b,a] = 0$ since m commutes. We need only prove one of these. Why should commutativity imply associativity of m?

7.3 (Commuter Sublemma). If B is a degree 2 subalgebra over a field Φ of a unital alternative algebra A, and $m \in A$ is an element such that $bm = mb$ for all $b \in B$, then $(b-\bar{b})[b,m,a] = 0$ for all $a \in A$. If B is an ordinary composition algebra, $[B,m,A] = 0$.

Proof. $(b-\bar{b})[a,m,b] = b[a,m,b] + \bar{b}[a,m,\bar{b}]$ (since $b + \bar{b} = t(b)1$) $= [b,a,mb] + [\bar{b},a,m\bar{b}]$ (left bumping) $= [b,a,bm] - [b,a,m\bar{b}]$ (m commutes) $= [b,a,b\circ m - t(b)m] = -[m,a,b^2] + [m,a,t(b)b]$ (linearizing $[x,a,x^2] = 0$) $= [m,a,n(b)1] = 0$ (since B is degree 2).

If B is an ordinary composition algebra either $B = \Phi 1$ (whence $[B,m,A] = 0$ trivially), or else there are invertible $b-\bar{b}$ by 3.18. We can cancel these to get $[b,m,a] = 0$ when $b-\bar{b}$ is invertible; when $c-\bar{c}$ is not invertible we linearize to see $0 = (b-\bar{b})[c,m,a] + (c-\bar{c})[b,m,a] = (b-\bar{b})[c,m,a]$, so again $[c,m,a] = 0$, and $[B,m,A] = 0$. ∎

Thus commuters give rise to images of the regular bimodule. An element m is a **-commuter* if $am = m\bar{a}$ for all $a \in \mathbb{C}$. These give rise to images of the Cayley-Dickson bimodule.

7.4 (*-Commuter Lemma). If m is a *-commuter for $\mathbb{C}$ then

$$a(bm) = (ba)m \quad (bm)a = (b\bar{a})m$$

for all $a,b \in \mathbb{C}$ so that $a\ell \to am$ is a homomorphism of the Cayley-Dickson bimodule $\mathbb{C}\ell$ onto $\{m\}$.

Proof. The formulas again express the homomorphism condition. It would be enough if we knew $b(ma) = m(\bar{b}a)$ and $(am)b = (a\bar{b})m$, since then

$a(bm) = a(m\bar{b}) = m(\bar{a}\bar{b}) = m(\overline{ba}) = (ba)m$ and $(bm)a = (b\bar{a})m$. So we prove

7.5 (*-Commuter Sublemma). If b, m are elements of a unital alternative algebra A such that $bm = m\bar{b}$ and $b+\bar{b}$ associates with A, then $b(ma) = m(\bar{b}a)$ and $(am)b = (a\bar{b})m$ for all $a \in A$.

Proof. $b(ma) = (bm + mb)a - m(ba)$ (linearized left alternativity) $= \{m(b+\bar{b})\}a - m(ba) = m\{(b+\bar{b})a - ba\}$ (association) $= m(\bar{b}a)$ and dually. ∎

It is useful to observe that if m is a *-commuter, so is any bm or mb: $a(bm) = (ba)m = (bm)\bar{a}$. There is no analogous result for commuters.

Returning to the proof of the Theorem, in order to fill M up with homomorphic images of regular and Cayley-Dickson bimodules we need to be able to fill it up with commuters and *-commuters. That is, we must show M is <u>full</u> in the sense that it is generated by commuters and *-commuters.

Certainly any unital M is chock full of commuters when regarded as a $\Phi 1$-bimodule: <u>every</u> m is a commuter, $(\alpha 1)m = m(\alpha 1) = \alpha m$. From this humble beginning we build up elements which commute or *-commute with more and more of $\mathbb{C}$. The inductive step is

7.6 Lemma. Let B be a composition algebra over a field with nondegenerate norm bilinear form $n(x,y)$, M a unital $\mathbb{C}(B,\mu)$ bimodule. If m is a commuter for B then $n = \ell m$ is a *-commuter for B with $m = \ell^{-1}n \in \mathbb{C}n$. If m is a *-commuter for B then any $n = (b\ell)\circ m$ is a commuter for $\mathbb{C}(B,\mu)$. If $\{b_i\}$, $\{b_i'\}$ are dual bases for B relative to $n(x,y)$ then $n = m - \mu^{-1}\Sigma d_i' n_i$ is a *-commuter for $\mathbb{C}(B,\mu)$ for $d_i' = b_i'\ell$, $d_i = b_i\ell$, $n_i = d_i \circ m$, with $m = n + \mu \Sigma d_i' n_i \in \mathbb{C}n + \Sigma \mathbb{C}n_i$.

Thus if M is full as a B-bimodule it is also full as a $\mathbb{C}(B,\mu)$-bimodule.

Proof. (Fullness comes about because if M is generated by B-commuters and *-commuters m then it is also generated by the n and n_i, which are $\mathbb{C}(B)$-commuters and *-commuters.)

Our construction strategy is B-commuter → B-*-commuter → $\mathbb{C}$-commuter $\mathbb{C}$-*-commuter.

Our first step is to transform ordinary commuters for B into *-commuters for B: given $bm = mb$ for all $b \in B$, any element $n = (a\ell)m$ is a *-commuter for B since by associativity 6.3 $b\{(a\ell)m\} = \{b(a\ell)\}m = \{(a\ell)\bar{b}\}m = (a\ell)\{\bar{b}m\} = (a\ell)\{m\bar{b}\} = \{(a\ell)m\}\bar{b}$. In brief, the product of a commuter $m \in M$ with a *-commuter $a\ell \in B\ell$ is a *-commuter $(a\ell)m \in M$.

Now that we have a *-commuter m for B rather than just a commuter, we show $n = d\circ m$ $(d = b\ell)$ is an ordinary commuter for $\mathbb{C}(B,\mu)$. First, n commutes with $c \in B$: $cn = c(dm) + c(md) = c(dm) + m(\bar{c}d)$ (by 7.5) $= c(dm) + m(dc)$ (recall $d = b\ell$) $= U_{c,m}d = (cd)m + (md)c = (d\bar{c})m + (md)c = (dm)c + (md)c = nc$. Next, consider an element $cd \in B(b\ell)$: n commutes with such a cd since $[cd,n] = (cd)n - n(cd) = [c,d,n] + c(dn) + [n,c,d] - (nc)d = 2[c,d,n] + c(nd) - (cn)d$ (we just saw n commutes with c, and $dn = d^2m + dmd = md^2 + dmd = nd$ since $d^2 = \mu n(b)1 \in \Phi 1$) $= 3[c,d,n] = 3[c,d,d\circ m] = -3[c,m,d^2]$ (linearizing $[a,x,x^2] = [a,x,x]\circ x = 0$) $= 0$ as $d^2 \in \Phi 1$. Thus $[c(b\ell), b\ell\circ m] = 0$ for all b; in particular, for $b = 1$ any $[a\ell,\ell\circ m] = 0$. But by linearizing $b \to b,1$ we see $0 = [c(b\ell), \ell\circ m] + [c\ell,b\ell\circ m]$, and since $[c(b\ell), \ell\circ m] = [(bc)\ell, \ell\circ m] = 0$ we see $[c\ell, d\circ m] = 0$ for all $c\ell \in B\ell$. Thus $n = d\circ m$ commutes with anything from B or $B\ell$, hence with $\mathbb{C}(B,\mu)$. In brief, the circle product of a *-commuter $m \in M$ and a *-commuter $b\ell \in B\ell$ is an ordinary commuter $b\ell \circ m \in M$.

The final step is to show $n = m - \mu^{-1}\Sigma\, d_i'n_i$ is a *-commuter for $\mathbb{C}(B,\mu)$. It certainly *-commutes with all elements of B: m is a *-commuter for B to begin with, and we have just seen the $n_i = d_i \circ m$ are commuters for B as circle products *-commuters d_i and m so the $d_i'n_i$ are *-commuters as products of *-commuters d_i' with commuters n_i. To show n *-commutes with elements of $B\ell$ we may consider only basis elements $d_j = b_j\ell$ (the b_j's span B); here $d_jn - n\bar{d}_j = d_jn + nd_j = d_j \circ m - \mu^{-1}\Sigma\, d_j \circ (d_i'n_i) = n_j - \mu^{-1}\Sigma(d_j \circ d_i')n_i$ (as before, recalling the definition of n_j) $= n_j + \mu^{-1}\Sigma n(d_j,d_i')n_i = n_j - \Sigma\, n(b_j,b_i')n_i$ (recall $n(b\ell,b'\ell) = n(b,b')n(\ell) = -\mu n(b,b')$) $= n_j - n_j = 0$ by the hypothesis that the $\{b_i\}$, $\{b_i'\}$ are dual bases relative to n. Therefore n *-commutes with B and $B\ell$, so is a *-commuter for $\mathbb{C}(B,\mu)$. ∎

We have seen M is full as a Φ-bimodule; if the characteristic $\neq 2$ then $n(x,y)$ is nondegenerate on $\Phi 1$ and we can apply the Lemma to deduce M is full as a $\mathbb{C}(\Phi,\mu_1)$ bimodule. In characteristic 2 the algebra $\mathbb{C}(\Phi,\mu_1)$ is $\Phi 1 + \Phi u$ for $u^2 = u + \mu_1 1$. In this case for any $m \in M$ the element $n_0 = m - u \circ m$ is a $\mathbb{C}$-commuter and $n = u \circ m$ a $\mathbb{C}$-*-commuter: we need only check $un_0 = n_0u$, $un = n\bar{u}$, and these follow from $[u,n_0] = [u,m] - [u^2,m] = -[\mu_1 1,m] = 0$ and $un - n\bar{u} = u^2m + umu + um\bar{u} + \mu_1 m$ (characteristic 2!!) $= (u^2 + u + \mu_1)m = 0$. In either case we get $\mathbb{C}(\Phi,\mu_1)$ full.

The only ordinary composition algebra for which $n(x,y)$ could possibly be degenerate is $\mathbb{C} = \Phi 1$ of characteristic 2, so once M is full as a $C(\Phi,\mu_1)$ bimodule it is also full as a $C(\Phi,\mu_1,\mu_2)$ and $C(\Phi,\mu_1,\mu_2,\mu_3)$ bimodule and therefore as a $\mathbb{C}$-bimodule.

Thus all $\mathbb{C}$-bimodules are full, and the theorem is complete. ∎

Exercise

7.1 If m is a *-commuter for B, is $(b\ell)m$ a commuter for B (M a $\mathbb{C}(B,\mu)$ bimodule)?

7.2 Prove the formula in 7.3 by passing to an infinite scalar extension, showing that the b for which $b-\bar{b}$ form a Zariski-dense set on which $[b,m,a]$ vanishes.

7.3 In the proof of 7.6, show $b\ell\circ m$ is a commuter for $\mathbb{C}(B,\mu)$ if m is a *-commuter for B by passing to an infinite extension so that the invertible b are a Zariski dense set on which $[b\ell\circ m,\mathbb{C}]$ vanishes. Alternately, passing to a scalar extension Ω with more than 2 elements, use 3.18 to show $[(b_i\ell)\circ m,\mathbb{C}] = 0$ for a basis $\{b_i\}$ of B_Ω, hence $[B\ell\circ m,\mathbb{C}] = 0$.

7.4 What can you say about Φ-bimodules for the extraordinary composition algebra Ω (purely inseparable of charactersitic 2)?