§7. Bimodules

The unital bimodules for an ordinary composition algebra $\mathbb{C}$ are semisimple, and the simple bimodules are homomorphic images of the regular and Cayley bimodules. The reason is that such bimodules are full of copies of the regular bimodule (generated by elements m which commute with $\mathbb{C}$, $am = ma$) and of the Cayley bimodule (generated by elements m which *-commute with $\mathbb{C}$, $am = ma^*$).

Suppose A is composition algebra over a field Φ, and consider the algebra $\mathbb{C}(A,\mu)$ obtained from A and the scalar μ by the Cayley-Dickson construction:

$$\mathbb{C}(A,\mu) = A \oplus A\ell.$$

Notice that both the space A and the space $A\ell$ are invariant under multiplication by A (recall $b(a\ell) = (ab)\ell$ and $(a\ell)b = (ab^*)\ell$). Thus the spaces A and $A\ell$ with the induced multiplications furnish us with unital A-bimodules; A is called the **regular bimodule** reg(A) and $A\ell$ the **Cayley-Dickson bimodule** cay(A). Note that A is always an alternative bimodule for A, but $A\ell$ will be such only when A is associative: $[a,b,c\ell] + [a,c\ell,b] = [a,b,c\ell] - [a,c\ell,b^*]$ $= \{c(ab) - (cb)a - (ca)b + (cb)a\}\ell = \{c(ab) - (ca)b\}\ell = -[c,a,b]\ell$. Thus when A is a Cayley algebra $A\ell$ is not an alternative bimodule just as $\mathbb{C}(A,\mu)$ is not an alternative algebra.

Another way to interpret the Cayley-Dickson bimodule is an follows. We can identify $A\ell$ with A as a Φ-module, and the birepresentation λ, ρ of A becomes $\lambda_a b = ba$ and $\rho_a b = ba^*$, i.e. $\lambda_a = R_a$ and $\rho_a = R_a{}^*$.

The regular bimodule M = A is generated by the element m = 1, and this element "commutes" with the elements of A: $a \cdot m = a = m \cdot a$. The Cayley-Dickson bimodule $M = A\ell$ is generated by the element $m = \ell$, which "*-commutes" with A: $a \cdot m = a\ell = \ell a^* = m \cdot a^*$ (or identifying $A\ell$ with A, $\lambda_a = R_a = \rho_a{}^*$).

Thus if we are looking for copies of the regular or Cayley-Dickson bimodule in a general bimodule M, it is reasonable to look for these distinguished generators. We say an element $m \in M$ is a **commuter** if am = ma for all $a \in A$, and a ***-commuter** if $am = ma^*$ for all $a \in A$. We begin with a technical lemma.

7.1 Lemma. If A is an ordinary composition subalgebra over a field Φ of a unital alternative algebra E and $m \in E$ is an element which commutes with A, $[A,m] = 0$, then it automatically associates with A, $[A,m,E] = 0$. If $m \in E$ *-commutes with A, then $a(mx) = m(a^*x)$, $(xm)a = (xa^*)m$ for all $a \in A$, $x \in E$.

Proof. Note that by left and right bumping (and commutativity) $a[a,m,x] = [a,ma,x] = [a,am,x] = [a,m,x]a$ for all $a \in A$, $x \in E$, so $0 = [-n(a)1,m,x] = [a^2 - t(a)a,m,x]$ (A is degree 2, and 1 is the unit for both A and E) $= a \circ [a,m,x] - t(a)[a,m,x]$ (middle bumping) $= \{a - t(a)\}[a,m,x] + [a,am,x]$ (right bumping) $= -a^*[a,m,x] + [a,ma,x]$ (commutativity) $= (a - a^*)[a,m,x]$ (left bumping). If A is an ordinary composition algebra either

$A = \Phi 1$ (whence $[A,m,E] = 0$ trivially), or else there are invertible $a - a^*$ by 4.13. We can cancel these from the relation $(a - a^*)[a,m,x] = 0$ to get $[a,m,x] = 0$ when $a - a^*$ is invertible; using a standard trick, when $b - b^*$ is not invertible we linearize to see $0 = (a - a^*)[b,m,x]$ $+ (b - b^*)[a,m,x] = (a - a^*)[b,m,x]$, so cancelling again gives $[b,m,x] = 0$ for any b and $[A,m,E] = 0$.

When m is a *-commuter we have $a(mx) = (am + ma)x - m(ax)$ (linearized left alternativity) $= \{m(a + a^*)\}x - m(ax)$ (*-commutativity) $= m\{(a + a^*)x - ax\}$ $(a + a^* = t(a)1)$ $= m(a^*x)$, and duality. ∎

Now we are ready to prove that a commuter gives rise to an image of the regular bimodule, and a *-commuter an image of the Cayley-Dickson bimodule.

7.2 (Commuter Lemma) If m is a commuter for A, $am = ma$ for all $a \in A$, then

$$a(bm) = (ab)m, \quad (bm)a = (ba)m$$

and the map $a \to am$ is a homomorphism of the regular bimodule A onto the cyclic submodule $\{m\}$ generated by m.

Proof. The formulas express the condition $F(a \cdot b) = aF(b), F(b \cdot a) = F(b)a$ that $F(a) = am$ be a homomorphism of bimodules. The first formula is just $[a,b,m] = 0$, and the latter is equivalent to $(mb)a = m(ba)$ or $[m,b,a] = 0$ since m commutes. But this associativity for a commuter follows from the first part of 7.1. ∎

7.3 (*-Commuter Lemma). If m is a *-commuter for A then

$$a(bm) = (ba)m, \quad (bm)a = (ba^*)m$$

for all $a,b \in A$ so that $a\ell \to am$ is a homomorphism of the Cayley-Dickson bimodule $A\ell$ onto $\{m\}$.

Proof. The formulas again express the homomorphism condition. By the second part of 7.1 we know $(bm)a = (ba^*)m$ and $a(mc) = m(a^*c)$ (therefore $a(bm) = a(mb^*) = m(a^*b^*) = m(ba)^* = (ba)m$ by *-commutativity). ∎

It is useful to observe that if m is a *-commuter, so is any am or ma: $b(am) = (ab)m = (am)b^*$. There is no analogous result for commuters. (When A is noncommutative, the only commuters in $M = A$ are the scalars $m = \alpha 1$, while all elements of $M = A\ell$ are *-commuters).

Having seen how to generate regular and Cayley-Dickson bimodules from commuting and *-commuting elements, it is natural to ask how we can build up such elements. The key induction step is a method of building commuters and *-commuters for $A = \mathbb{C}(B,\mu)$ starting from commuters or *-commuters for B. Our construction strategy is B-commuter $\to$ B *-commuter $\to$ $\mathbb{C}$-commuter and $\mathbb{C}^*$ commuter.

7.4 (Commuter Construction Lemma) Let B be an ordinary composition algebra, M a unital $\mathbb{C}(B,\mu)$-bimodule.

If n_o is a commuter for B then $m_o = \ell n_o$ is a *-commuter for B, with $n_o = \ell^{-1} m_o \in \mathbb{C} m_o$.

If m_o is a *-commuter for B then any $n = (b\ell) \circ m_o$ is a commutor for $\mathbb{C}(B,\mu)$ and $m = m_o - \mu^{-1} \Sigma\, d_i' n_i$ is a *-commuter for $\mathbb{C}(B,\mu)$, with $m_o = m + \mu^{-1} \Sigma d_i' n_i \in \mathbb{C}m + \Sigma \mathbb{C} n_i$, where $d_i' = b_i' \ell$, $d_i = b_i \ell$, $n_i = d_i \circ m_o$ relative to dual bases $\{b_i\}$, $\{b_i'\}$ for B relative to the bilinear form $n(x,y)$.

Proof. Our first step is to transform ordinary commuters for B into *-commuters for B: given $bn_o = n_o b$ for all $b \in B$, <u>any</u> element $m_o = (a\ell)n_o$ is a *-commuter for B since by associativity in Lemma 7.1 $b\{(a\ell)n_o\} = \{b(a\ell)\}n_o = \{(a\ell)b^*\}n_o = (a\ell)\{b^* n_o\} = (a\ell)\{n_o b^*\} = \{(a\ell)n_o\}b^*$. In brief, the product of a commuter $n_o \in M$ with a *-commuter $a\ell \in B\ell$ is a *-commuter $(a\ell)n_o \in M$.

Now that we have a *-commuter m_o for B rather than just a commuter, we show $n = (b\ell)\circ m_o$ is an ordinary commuter for all of $\mathbb{C}(B,\mu)$. First, since $b\ell \circ m_o = b(\ell m_o) + (m_o b)\ell$ (linearized Middle Moufang) $= \ell(b^* m_o) + (b^* m_o)\ell$ (Lemma 7.1 applied to ℓ) $= \ell \circ m_o'$ where we observed after 7.3 $m_o' = b^* m_o$ is another *-commuter for B, it suffices to consider the case $b = 1$, $m_o' = m_o$, $n = \ell \circ m_o$. Such n commutes with all $c \in B$: $cn = c(\ell m_o) + c(m_o \ell) = \ell(c^* m_o) + m_o(c^* \ell)$ (7.1 applied to ℓ and m_o) $= U_{\ell, m_o} c^* = (\ell c^*)m_o + (m_o c^*)\ell = (\ell m_o)c + (m_o \ell)c$ (7.1 again) $= nc$. For any $c\ell \in B\ell$ we have $[n, c\ell] = [n,c]\ell + c[n,\ell] - [n,c,\ell]$ by the Commutator Derivation Formula, where we just saw n commutes with c, and where $[n,\ell] = [\ell \circ m_o, \ell] = -[\ell^2, m_o] = 0$ and $[n,c,\ell] = [\ell \circ m_o, c, \ell] = -[\ell^2, c, m_o] = 0$ (linearizing $[x^2,x] = [x^2,c,x] = 0$) because $\ell^2 = \mu 1 \in \Phi 1$. Thus n commutes with anything from B or $B\ell$, hence with $\mathbb{C}(B,\mu)$. In brief, the circle product of a *-commuter $m_o \in M$ and a *-commuter $b\ell \in B\ell$ is an ordinary commuter $b\ell \circ m_o \in M$ for $\mathbb{C}$.

The final step is to show $m = m_o - \mu^{-1} \Sigma d_i' n_i$ is a *-commuter for $\mathbb{C}(B,\mu)$. It certainly *-commutes with all elements of B: m_o is a *-commuter for B to begin with, and we have just seen the $n_i = d_i \circ m_o$ are commuters for $\mathbb{C}$ (as circle products of *-commuters m_o and $b_i\ell$), so the $d_i' n_i$ are *-commuters for B (as products of commuters n_i with *-commuters $b_i'\ell$). To show m *-commutes with elements of $B\ell$ we need consider only basis elements $d_j = b_j\ell$ (the b_j's span B); here $d_j m - m d_j^* = d_j m + m d_j = d_j \circ m = d_j \circ m_o - \mu^{-1} \Sigma d_j \circ (d_i' n_i) = n_j - \mu^{-1} \Sigma (d_j \circ d_i') n_i$ (the n_i commute and hence associate with $\mathbb{C}$ by 7.1) $= n_j + \mu^{-1} \Sigma n(d_j, d_i') n_i$ (since $t(d_j) = t(d_i') = 0$) $= n_j - \Sigma n(b_j, b_i') n_i$ (recall $n(b\ell, b'\ell) = n(b,b')n(\ell) = -\mu n(b,b')$) $= n_j - n_j = 0$ by the hypothesis that the $\{b_i\}$, $\{b_i'\}$ are dual bases relative to n. Therefore m *-commutes with B and $B\ell$, so is a *-commuter for $\mathbb{C}(B,\mu)$. ∎

So far we have discovered ways of building certain kinds of elements, and from those elements copies of certain kinds of bimodules. What are the submodules of these two basic kinds of bimodules? Let's begin with the regular bimodules. The sub-bimodules are the subspaces of B invariant under all $\lambda_b = L_b$ and $\rho_b = R_b$, that is, precisely the two-sided ideals. Now by the Simplicity Theorem 4.19 we know the composition algebras are, with one exception, simple - all the Cayley algebras, all the quaternion algebras, and all the fields (Φ or inseparable Ω) are simple, but the split quadratic extensions $\Phi e_1 \boxplus \Phi e_2$ are only *-simple, being the direct sum of two ideals. The proper ideals are Φe_1 and Φe_2, and these are non-isomorphic as $B = \Phi e_1 \boxplus \Phi e_2$ bimodules.

Turning to the Cayley-Dickson bimodule, the second way of representing the representation $\lambda_b = R_b$, $\rho_b = R_b{}^*$ shows the sub-bimodules are precisely the right ideals of B. Division algebras have no right ideals, and neither do Cayley algebras (not even the split ones), so the only composition algebras with proper right ideals are the split quadratic extensions (with right ideals (= ideals) being Φe_1, Φe_2) and the split quaternion algebras (with right ideals of the form eQ for suitable idempotents e). In either of these cases ($\Phi e_1 + \Phi e_2$ or $eQ + (1-e)Q$) the bimodule splits into a direct sum of two simple bimodules.

Thus in all cases the regular and Cayley-Dickson bimodules are semisimple (completely reducible), and most of the time they are actually simple (irreducible). For the ordinary composition algebras this extends to all bimodules.

7.5 (Bimodule Theorem) Every unital bimodule for one of the ordinary composition algebras $\mathfrak{C}$ over a field Φ is semisimple. The simple bimodules for $\mathfrak{C}$ are isomorphic to the simple sub-bimodules of the regular and Cayley-Dickson bimodules, leading to the following list of simple bimodules:

I. $\mathfrak{C} = \Phi$: reg $\mathfrak{C}$

IIa. $\mathfrak{C} = \mathfrak{C}(\Phi;\mu_1)$ division algebra: reg $\mathfrak{C}$, cay $\mathfrak{C}$

IIb. $\mathfrak{C} = \mathfrak{C}(\Phi;1) = \Phi e_1 \boxplus \Phi e_2$ split: reg(Φe_1), reg(Φe_2), cay(Φe_1), cay(Φe_2)

IIIa. $\mathfrak{C} = \mathfrak{C}(\Phi;\mu_1,\mu_2)$ division algebra: reg $\mathfrak{C}$, cay $\mathfrak{C}$

IIIb. $\mathfrak{C} = \mathfrak{C}(\Phi;1,1) = e_1\mathfrak{C} \oplus e_2\mathfrak{C}$ split: reg $\mathfrak{C}$, cay$(e_1\mathfrak{C})$

IV. $\mathfrak{C} = \mathfrak{C}(\Phi;\mu_1,\mu_2,\mu_3)$: reg $\mathfrak{C}$.

Proof. We first verify all simple sub-bimodules of the regular and Cayley-Dickson bimodules $\mathbb{C}$ and $\mathbb{C}\ell$ have the above forms. In case I, $\mathbb{C} = \Phi$ and $\mathbb{C}\ell = \Phi\ell$ are isomorphic as bimodules and we can delete the second. In split IIIb, every right ideal is isomorphic (as right Q-module) to $e_1 Q = \Phi e_1 + \Phi e_1 j$. In case IV the Cayley-Dickson bimodule is not an alternative bimodule.

To verify that all bimodules are semisimple with the above types of simple bimodules, it suffices to prove every bimodule M is a sum of homomorphic images of regular and Cayley-Dickson bimodules (and therefore a sum of isomorphic copies of the simple constituents of these bimodules). By 7.2 and 7.3, in order to fill M up with homomorphic images of regular and Cayley-Dickson bimodules we need to be able to fill it up with commuters and *-commuters. That is, we must show M is **full** in the sense that it is generated by commuters and *-commuters.

Certainly any unital M is chock full of commuters when regarded as a $\Phi 1$-bimodule: <u>every</u> m is a commuter, $(\alpha 1)m = m(\alpha 1) = \alpha m$. From this humble beginning we build up elements which commute or *-commute with more and more of $\mathbb{C}$. The inductive step is the Commuter Construction Lemma, which says that if M is full as a B-bimodule (where $n(x,y)$ is nondegenerate on B) then it is also full as a $\mathbb{C}(B,\mu)$ - bimodule. Indeed, if M is generated by B-commuters n_o and *-commuters m_o it is generated by *-commuters alone (recall $n_o \in \mathbb{C}m_o$ for $m_o = \ell n_o$), in which case it is also generated by $\mathbb{C}$-commuters n_i and *-commuters m (since $m_o \in \mathbb{C}m + \Sigma\mathbb{C}n_i$).

We have seen M is full as a Φ-bimodule; if the characteristic $\neq 2$ then $n(x,y)$ is nondegenerate on $\Phi 1$ and we can apply the induction step to deduce M is full as a $\mathbb{C}(\Phi,\mu_1)$ bimodule. In characteristic 2 the algebra $\mathbb{C}(\Phi,\mu_1)$ is $\Phi 1 + \Phi u$ for $u^2 = u + \mu_1 1$. In this case for any $m \in M$ the element $n_o = m - u\circ m$ is a $\mathbb{C}$-commuter and $m_o = u\circ m$ a $\mathbb{C}$-*-commuter: we need only check $un_o = n_o u$, $um_o = m_o u^*$, and these follow from $[u,n_o] = [u,m] - [u^2,m] = -[\mu_1 1,m] = 0$ and $um_o - m_o u^* = u^2 m + umu + umu^* + \mu_1 m$ (characteristic 2!!) $= \{u^2 + u + \mu_1\}m = 0$. In either case we get $\mathbb{C}(\Phi,\mu_1)$ full.

The only ordinary composition algebra for which $n(x,y)$ could possibly be degenerate is $\mathbb{C} = \Phi 1$ of characteristic 2, so once M is full as a $\mathbb{C}(\Phi,\mu_1)$ bimodule it is also full as a $\mathbb{C}(\Phi,\mu_1,\mu_2)$ and $\mathbb{C}(\Phi,\mu_1,\mu_2,\mu_3)$ bimodule and therefore as a $\mathbb{C}$-bimodule.

Thus all $\mathbb{C}$-bimodules are full, and the proof is complete. ∎

VII.7 Exercises

7.1 If m is a $*$-commuter for B, is $(b\ell)m$ a commuter for B (M a $\mathbb{C}(B,\mu)$ bimodule)?

7.2 Prove the first partof 7.1 by passing to an infinite scalar extension, showing that the b for which $b-b^*$ form a Zariski-dense set on which $[b,m,a]$ vanishes.

7.3 In the proof of 7.5, show $d^\circ m = b\ell^\circ m$ is a commuter for $\mathbb{C}(B,\mu)$ if m is a $*$-commuter for B by showing it commutes with all $cd \in Bd = B(b\ell)$, then linearizing to show it commutes with all $c\ell \in B\ell$. Rather than linearizing, one can assume b is invertible, by passing to an infinite extension so that the invertible b are a Zariski dense set on which $[b\ell^\circ m,\mathbb{C}]$ vanishes. Alternately, passing to a scalar extension Ω with more than 2 elements, use 3.18 to show $[(b_i\ell)^\circ m,\mathbb{C}] = 0$ for a basis $\{b_i\}$ of B_Ω, hence $[B\ell^\circ m,\mathbb{C}] = 0$.

7.4 What can you say about Ω-bimodules for the extraordinary composition algebra Ω (purely inseparable of characteristic 2)?

7.5 If a,m are elements of a unital alternative algebra E such that $am = ma^*$, $ma = a^*m$ and $a + a^*$ associates with E, show $a(mx) = m(a^*x)$ and $(xm)a = (xa^*)m$ for all $x \in E$.