§6 Quadratic ideals

We have seen that a proper quadratic ideal cannot contain invertible elements, and elements in a composition algebra over a field are invertible if they have nonzero norm, therefore a proper quadratic ideal B cannot contain elements of nonzero norm: $n(B) = 0$, i.e. B is totally isotropic relative to the quadratic form n. In fact, the isotropic subspaces comprise all quadratic Φ-ideals; even more they are all the quadratic $\mathbb{Z}$-ideals ($\mathbb{Z}$-subspaces B satisfying $U_B A \subset B$).

6.1 **Proposition.** The quadratic $\mathbb{Z}$-ideals of a composition algebra A over a field Φ are precisely A together with all totally isotropic Φ-subspaces. In particular, A has dcc on quadratic ideals.

Proof. Any totally isotropic Φ-subspace B is a quadratic ideal by the U-formula (1.11): $U_B A = n(B,A^*)B - n(B)A^* = n(B,A^*)B \subset \Phi B \subset B$.

Now suppose B is a proper quadratic $\mathbb{Z}$-ideal of A. Then B contains no invertible elements, so as we remarked above $n(B) = 0$ and B is totally isotropic. This rules out $A = \Omega$ a purely inseparable field extension, so we can assume $n(x,y)$ is a nondegenerate bilinear form. Our task will be to see that B is actually a Φ-subspace, not merely a $\mathbb{Z}$-subspace.

If b is any nonzero element of B then $n(b,a) \neq 0$ for some $a \in A$ by nondegeneracy, so $\Phi b = \Phi n(b,a)b = \Phi\{n(b,a)b - n(b)a\} = U_b(\Phi a^*) \subset B$. Thus B is closed under scalar multiplication by elements of Φ.

Since any chain $A = B_o > B_1 > \ldots > B_{r+1} = 0$ must have $B_1 > \ldots > B_r$ proper isotropic Φ-subspaces, and since the dimension of a maximal isotropic subspace is at most $\frac{1}{2}$ dim A, we have $r \leq \frac{1}{2}$ dim A. Since dim A = 2, 4, 8 (Types 0 and 1 have no proper isotropic subspaces) we see A has dcc. ∎

We will investigate these totally isotropic subspaces in more detail for the case of a Cayley algebra. First off, we restrict ourselves to a <u>split</u> Cayley algebra $\mathbb{C}$, since a Cayley division algebra has no isotropic vectors at all except zero.

It is easy to build totally isotropic spaces from isotropic vectors: if $n(a) = 0$ then $n(a\mathbb{C}) = n(a)n(\mathbb{C}) = 0$ shows $a\mathbb{C}$ is totally isotropic. By nondegeneracy of $n(x,y)$, if $a \neq 0$ we can find an isotropic element b with $n(b) = 0$ but $n(a,b) = 1$. In this case

$$a\mathbb{C} + b\mathbb{C} = \mathbb{C} \quad \text{if } n(a,b) \neq 0$$

because $n(a,b)x = a(\bar{b}x) + b(\bar{a}x) \in a\mathbb{C} + b\mathbb{C}$ for any x. Since $a\mathbb{C}$, $b\mathbb{C}$ have dimension at most 4 by Witt's Theorem (being totally isotropic), they both must have dimension exactly 4. We will call such an $a\mathbb{C}$ a <u>left 4-space</u>. Similarly, if $a \neq 0$ is isotropic we have a <u>right 4-space</u> $\mathbb{C}a$ which is totally isotropic of dimension 4.

The ways in which these 4-spaces can intersect is given by

6.2 (Intersection Theorem) Two left 4-spaces determined by isotropic vectors $a,b \neq 0$ have intersection

(i) $a\mathbb{C} \cap b\mathbb{C} = a\mathbb{C} = b\mathbb{C}$ of dimension 4 if a, b are linearly dependent

(ii) $a\mathbb{C} \cap b\mathbb{C} = a(\bar{b}\mathbb{C}) = b(\bar{a}\mathbb{C})$ of dimension 2 if a, b are independent but $n(a,b) = 0$

(iii) $a\mathbb{C} \cap b\mathbb{C} = 0$ of dimension 0 if a, b are independent and $n(a,b) \neq 0$.

A right and left 4-space have intersection

(iv) $a\mathbb{C} \cap \mathbb{C}b = \Phi ab$ of dimension 1 if $ab \neq 0$

(v) $a\mathbb{C} \cap \mathbb{C}b = ab^{\perp} = a^{\perp}b$ of dimension 3 if $ab = 0$ (where $x^{\perp} = \{y \in A / n(x,y) = 0\}$).

Two left 4-spaces coincide, $a\mathbb{C} = b\mathbb{C}$, iff a and b are linearly dependent. No left 4-space ever coincides with a right 4-space, $a\mathbb{C} \neq \mathbb{C}b$.

Proof. Our basic tool is the formula

$$n(a,b)x = a(\bar{b}x) + b(\bar{a}x) \tag{6.3}$$

obtained by linearizing $n(a)x = (a\bar{a})x = a(\bar{a}x)$. From this

$$x \in a\mathbb{C} \Leftrightarrow \bar{a}x = 0 \qquad (n(a) = 0,\ a \neq 0) \tag{6.4}$$

since $\bar{a}(a\mathbb{C}) = n(a)\mathbb{C} = 0$ and whenever $\bar{a}x = 0$ we can choose b with $n(a,b) = 1$ so that $x = a(\bar{b}x) \in a\mathbb{C}$ by (6.3).

Clearly if $b = \lambda a$ then $a\mathbb{C} = b\mathbb{C}$ as in (i).

Suppose now a, b are independent but $n(a,b) = 0$. By (6.3) we have $a(\bar{b}x) = -b(\bar{a}x)$, so $a(\bar{b}\mathbb{C}) = b(\bar{a}\mathbb{C}) \subset a\mathbb{C} \cap b\mathbb{C}$. But conversely, if $x \in a\mathbb{C} \cap b\mathbb{C}$ (so by (6.4) $\bar{a}x = \bar{b}x = 0$) then by independence of a and b we can choose an element c with $n(a,c) = 1$ but $n(b,c) = 0$, so as in (6.3) $0 = n(b,c)x = b(\bar{c}x) + c(\bar{b}x) = b(\bar{c}x)$, therefore by (6.4) $\bar{c}x \in \bar{b}\mathbb{C}$, and so $x = n(a,c)x = a(\bar{c}x) + c(\bar{a}x) = a(\bar{c}x) \in a(\bar{b}\mathbb{C})$. Thus $a\mathbb{C} \cap b\mathbb{C} \subset a(\bar{b}\mathbb{C})$. This shows $a(\bar{b}\mathbb{C}) = b(\bar{a}\mathbb{C}) = a\mathbb{C} \cap b\mathbb{C}$.

To finish (ii) we must find the dimension of $a(\bar{b}\mathbb{C})$. We first show $a(\bar{b}\mathbb{C}) \neq 0$. Using (6.3) and its dual

$$x(\bar{a}b) = n(a,x)b - a(\bar{x}b)$$

$$= n(a,x)b - n(b,x)a + a(\bar{b}x)$$

so that if $a(\bar{b}\mathbb{C}) = 0$ we would have $\mathbb{C}(\bar{a}b)$ contained in the two dimensional subspace $\Phi a + \Phi b$; then $\mathbb{C}(\bar{a}b)$ would not be a 4-space, so $\bar{a}b = 0$. But then the above would reduce even further to $n(a,x)b - n(b,x)a = 0$, which is impossible if a, b are independent and n nondegenerate. Thus $a(\bar{b}\mathbb{C}) \neq 0$. Dually $(\mathbb{C}\bar{b})a \neq 0$, so applying the involution gives $\bar{a}(b\mathbb{C}) \neq 0$. This in turn shows $a\mathbb{C} \not\supset b\mathbb{C}$ $(\bar{a}(a\mathbb{C}) = 0$ but $\bar{a}(b\mathbb{C}) \neq 0)$, so $a\mathbb{C} > a\mathbb{C} \cap b\mathbb{C} = a(\bar{b}\mathbb{C})$. Therefore $a\mathbb{C} > a(\bar{b}\mathbb{C}) > 0$ has dimension 1, 2 or 3. Now $f(x,y) = n(\bar{a}x,\bar{b}y)$ is an alternate bilinear form since $f(x,x) = n(\bar{a}x,\bar{b}x) = n(\bar{a},\bar{b})n(x) = n(a,b)n(x) = 0$, and it induces a nondegenerate alternate form on $\mathbb{C}/\text{Rad } f$, so $\mathbb{C}/\text{Rad } f$ has even dimension; but $\text{Rad } f = \{z \mid n(\bar{a}z,\bar{b}\mathbb{C}) = 0\} = \{z \mid n(z,a(\bar{b}\mathbb{C})) = 0\} = a(\bar{b}\mathbb{C})^{\perp}$, and $\mathbb{C}/a(\bar{b}\mathbb{C})^{\perp}$ has the same dimension as $a(\bar{b}\mathbb{C})$. Thus $a(\bar{b}\mathbb{C})$ has even dimension; the only possibility is dimension 2.

If a, b are independent and $n(a,b) \neq 0$ then (6.3) shows $x = n(a,b)^{-1}\{a(\bar{b}x) + b(\bar{a}x)\}$; if $x \in a\mathbb{C} \cap b\mathbb{C}$ then $\bar{a}x = \bar{b}x = 0$ shows $x = 0$. This establishes (iii).

Now consider a mixed intersection $a\mathbb{C} \cap \mathbb{C}b$, where $n(a) = n(b) = 0$ but $a, b \neq 0$. Any $x = ay = zb$ in the intersection has $\bar{a}(zb) = \bar{a}x = \bar{a}(ay) = 0$, so by (6.3)

Looked at the other way, we can get from our basic spaces to the given isotropic space by a proper orthogonal transformation T. Now our basic spaces (i) - (iv) have the desired form of n-spaces ($1 \leq n \leq 4$), so if T preserves n-spaces then all isotropic spaces will be n-spaces. Preservation will follow from the Principle of Triality: there are proper orthogonal T', T" with (T, T', T") a isotopy.

Certainly T preserves 1-spaces, $T(\Phi a) = \Phi T(a)$ for $n(Ta) = n(a) = 0$. For 4-spaces, $T(a\mathbb{C}) = T'(a)T''(\mathbb{C}) = a'\mathbb{C}$ for $n(a') = n(T'a) = n(a) = 0$, and dually $T(\mathbb{C}a) = \mathbb{C}a''$. Then for 2-spaces we have $T(a\mathbb{C} \cap b\mathbb{C}) = T(a\mathbb{C}) \cap T(b\mathbb{C}) = a'\mathbb{C} \cap b'\mathbb{C}$ for $n(a') = n(b') = n(a) = n(b) = 0$, $n(a',b') = n(a,b) = 0$ and $a' = Ta$, $b' = Tb$ independent (dually for $\mathbb{C}a \cap \mathbb{C}b$), and for 3-spaces $T(a\mathbb{C} \cap \mathbb{C}b) = a'\mathbb{C} \cap \mathbb{C}b''$ where $n(a') = n(b'') = 0$ and $a'b'' = T'(a)T''(b) = T(ab) = 0$.

To prove uniqueness of the representatives, because the basic types have different dimensions we need only show the two members of (iv) are <u>not</u> conjugate under a proper orthogonal T. But this follows by preservation: if $\mathbb{C}e_{11}$ were a conjugate of $e_{11}\mathbb{C}$ it would have the form $a'\mathbb{C}$, and $a'\mathbb{C} \neq \mathbb{C}e_{11}$ by 6.6. ∎

$$(6.5) \qquad n(a,z)b = \bar{a}(zb) + \bar{z}(ab) = \bar{z}(ab)$$

$$(6.6) \qquad n(a,z)x = n(a,z)zb = z\{\bar{z}(ab)\} = n(z)ab \ .$$

Thus if $x \notin \Phi ab$ we must have $n(a,z) = 0$.

In the case when $ab \neq 0$ certainly any $x \in \Phi ab$ belongs to $a\mathbb{C} \cap \mathbb{C}b$. When $x \notin \Phi ab$ we have $n(a,z) = 0$ and $(\overline{ab})z = \overline{\bar{z}(ab)} = 0$ by (6.5), so by (6.4) $z \in (ab)\mathbb{C}$. Therefore $x = zb \in \{(ab)\mathbb{C}\}b = a\{b\mathbb{C}b\}$ (right Moufang) $= a\{n(b,\bar{\mathbb{C}})b - n(b)\bar{\mathbb{C}}\}$ (U-formula (1.11)) $= n(b,\bar{\mathbb{C}})ab \subset \Phi ab$. Thus x belongs to Φab even if it doesn't, and $a\mathbb{C} \cap \mathbb{C}b = \Phi ab$.

In the case $ab = 0$ we have $n(a,z) = 0$ when $x \notin \Phi ab$, i.e. $x = zb \in a^{\perp}b$ when $x \neq 0$. Conversely, whenever $n(a,z) = 0$ we have $\bar{a}(zb) = 0$ by (6.5) so $x = zb \in a\mathbb{C}$ by (6.4), thus $x \in a\mathbb{C} \cap \mathbb{C}b$. This shows $a\mathbb{C} \cap \mathbb{C}b = a^{\perp}b$, dually it equals $ab^{\perp}$. To compute the dimension in this case, note $z \to zb$ is a linear map of the 7-dimensional space $a^{\perp}$ onto $a\mathbb{C} \cap \mathbb{C}b = a^{\perp}b$ with kernel $\{z \in a^{\perp} \mid zb = 0\} = \{z \in a^{\perp} \cap \mathbb{C}\bar{b}\}$ (by (6.4)) $= \mathbb{C}\bar{b}$ (since automatically $\mathbb{C}\bar{b} \subset a^{\perp}$, $n(a,\mathbb{C}\bar{b}) = n(ab,\mathbb{C}) = 0$ by (2.9) if $ab = 0$). Since this kernel has dimension 4, the image $a\mathbb{C} \cap \mathbb{C}b$ has dimension $7 - 4 = 3$.

From this, $a\mathbb{C} = b\mathbb{C}$ implies a, b are dependent, and never $a\mathbb{C} = \mathbb{C}b$ since $a\mathbb{C} \cap b\mathbb{C}$ has dimension 1 or 3. ∎

All isotropic subspaces can be represented as intersections of 4-spaces.

6.7 (Theorem on Isotropic Subspaces) The totally isotropic subspaces of a split Cayley algebra are zero and the

(1-spaces) Φa for $n(a) = 0$

(2-spaces) $a(\bar{b}\mathbb{C}) = b(\bar{a}\mathbb{C}) = a\mathbb{C} \cap b\mathbb{C}$ or

$(\mathbb{C}\bar{b})a = (\mathbb{C}\bar{a})b = \mathbb{C}a \cap \mathbb{C}b$ for

$n(a) = n(b) = n(a,b) = 0$, a and b independent

(3-spaces) $ab^{\perp} = a^{\perp}b = a\mathbb{C} \cap \mathbb{C}b$ for $n(a) = n(b) = 0$, $ab = 0$

(4-spaces) $a\mathbb{C}$ or $\mathbb{C}a$ for $n(a) = 0$.

Any totally isotropic subspace is conjugate under a proper orthogonal transformation to zero or precisely one of

(i) Φe_{11}

(ii) $\Phi e_{11} + \Phi e_{12}^{(1)} = e_{11}\mathbb{C} \cap e_{12}^{(1)}\mathbb{C} = \mathbb{C}e_{21}^{(2)} \cap \mathbb{C}e_{21}^{(3)}$

(iii) $\Phi e_{11} + \Phi e_{12}^{(1)} + \Phi e_{12}^{(2)} = e_{11}(e_{21}^{(3)})^{\perp} = (e_{11})^{\perp} e_{21}^{(3)} = e_{11}\mathbb{C} \cap \mathbb{C}e_{21}^{(3)}$

(iv) $\Phi e_{11} + \Phi e_{12}^{(1)} + \Phi e_{12}^{(2)} + \Phi e_{12}^{(3)} = e_{11}\mathbb{C}$ or

$\Phi e_{11} + \Phi e_{21}^{(1)} + \Phi e_{21}^{(2)} + \Phi e_{21}^{(3)} = \mathbb{C}e_{11}$.

Proof. By Witt's Theorem and the nondegeneracy of $n(x,y)$, any two totally isotropic subspaces of the same dimension are conjugate under an orthogonal transformation T. Thus every isotropic subspace of dimension 1, 2, 3, or 4 is conjugate to the first member of (i), (ii), (iii) or (iv). Composing T with the (improper) involution if necessary, we obtain a proper orthogonal T' taking the given subspace into the first <u>or</u> second member of (i) - (iv).