§6. Inner ideals

In a composition algebra the proper inner ideals are precisely the totally isotropic subspaces relative to the norm form. In a Cayley algebra they can be obtained as intersections of 4-dimensional maximal totally isotropic subspaces. These 4-spaces have the form of principal inner ideals $\mathbb{C}a$ or $a\mathbb{C}$ for some isotropic element $a \in \mathbb{C}$.

We have seen that a proper inner ideal cannot contain invertible elements, and elements in a composition algebra are invertible if they have nonzero norm, therefore a proper inner ideal B cannot contain elements of nonzero norm: $n(B) = 0$, i.e. B is **totally isotropic** relative to the quadratic form n. In fact, the totally isotropic subspaces comprise all inner Φ-ideals; even more they comprise all the inner $\mathbb{Z}$-ideals ($\mathbb{Z}$-subspaces B satisfying $U_B A \subset B$).

6.1 (Inner Ideal Theorem). The inner $\mathbb{Z}$-ideals of a composition algebra A over a field Φ are precisely A together with all totally isotropic Φ-subspaces. In particular, A has dcc on inner ideals.

Proof. Any totally isotropic Φ-subspace B is an inner ideal by the U-formula (1.19): $U_B A = n(B,A^*)B - n(B)A^* = n(B,A^*)B \subset \Phi B \subset B$.

Now suppose B is a proper inner $\mathbb{Z}$-ideal of A. Then B contains no invertible elements, so as we remarked above $n(B) = 0$ and B is totally isotropic. This rules out $A = \Omega$ a purely inseparable field extension, so we can assume $n(x,y)$ is a nonsingular bilinear form. Our task will be to see that B is actually a Φ-subspace, not merely a $\mathbb{Z}$-subspace.

If b is any nonzero element of B then $n(b) = 0$ but $n(b,a) \neq 0$ for some $a \in A$ by nonsingularity, so $\Phi b = \Phi n(b,a)b = \Phi\{n(b,a)b - n(b)a\} = U_b(\Phi a^*) \subset B$. Thus B is closed under scalar multiplication by elements of Φ.

Since any chain $A = B_o > B_1 > \ldots > B_{r+1} = 0$ must have $B_1 > \ldots > B_r$ proper totally isotropic Φ-subspaces, and since for any nonsingular quadratic form the dimension of a maximal totally isotropic subspace is at most $\frac{1}{2}$ dim A, (see Ex. 6.2), we have $r \leq \frac{1}{2}$ dim A. Since dim A = 2, 4, 8 (Types 0 and 1 have no proper isotropic subspaces) we see A has dcc. ∎

So far we have characterized inner ideals in terms of the norm form. We will investigate these totally isotropic subspaces in more detail for the case of a Cayley algebra and relate them to the algebra structure. First off, we restrict ourselves to a <u>split</u> Cayley algebra $\mathbb{C}$, since a Cayley division algebra has no isotropic vectors at all except zero.

It is easy to build totally isotropic spaces from isotropic vectors: if $n(a) = 0$ then $n(a\mathbb{C}) = n(a)n(\mathbb{C}) = 0$ shows $a\mathbb{C}$ is totally isotropic. We claim such an $a\mathbb{C}$ has dimension 4.

6.2 (4-Space Theorem) If a is a nonzero isotropic element of a split Cayley algebra $\mathbb{C}$ ($a \neq 0$ but $n(a) = 0$), then the principal inner ideal $a\mathbb{C}$ is a 4-dimensional totally isotropic subspace characterized by

$$x \in a\mathbb{C} \Longleftrightarrow a^*x = 0.$$

Proof. Certainly $a\mathbb{C}$ is totally isotropic. Further, it has a complement of the same form:

$$(*) \qquad \mathbb{C} = a\mathbb{C} \oplus b\mathbb{C} \qquad (\text{if } n(a) = n(b) = 0,\ n(a,b) = 1).$$

Indeed, it is a standard fact about nondegenerate quadratic forms that if $n(a) = 0$ we can find a vector b with $n(b) = 0$, $n(a,b) = 1$: by nondegeneracy $n(a,c) = 1$ for some c, so the vector $b = c - n(c)a$ has both $n(a,b) = n(a,c) - n(c)\ n(a,a) = n(a,c) = 1$ and $n(b) = n(c) - n(c)n(c,a) + n(c)^2n(a) = n(c) - n(c) = 0$ if $n(a) = 0$.

Once we have found b, the key to establishing (*) is the general formula

$$(6.3) \qquad n(a,b)x = a(b^*x) + b(a^*x) = a^*(bx) + b^*(ax)$$

obtained by linearizing $n(a)x = a(a^*x) = a^*(ax)$. When $n(a,b) = 1$ we see immediately all $x = a(b^*x) + b(a^*x)$ belong to $a\mathbb{C} + b\mathbb{C}$, so $a\mathbb{C} + b\mathbb{C} = \mathbb{C}$. Now $\mathbb{C}$ has dimension 8, and once more the totally isotropic subspaces $a\mathbb{C}$, $b\mathbb{C}$ have dimension at most $\frac{1}{2}\dim \mathbb{C} = 4$, so the only way they can add up to the whole 8-dimensional space is for them to have dimension exactly 4 (and for the sum to be direct). This establishes (*).

To derive the given characterization of $a\mathbb{C}$, note $x \in a\mathbb{C}$ $\Rightarrow a^*x \in a^*(a\mathbb{C}) = (a^*a)\mathbb{C} = n(a)\mathbb{C} = 0$, while conversely if $a^*x = 0$ we can choose b with $n(a,b) = 1$ so by (6.3) $x = n(a,b)x = a(b^*x) + b(a^*x) = a(b^*x) \in a\mathbb{C}$. ∎

Because of its resemblance to a right ideal, we will call such an $a\mathbb{C}$ a right 4-space (even though the element a appears on the <u>left</u>). Dually if $a \neq 0$ is isotropic we have a left 4-space $\mathbb{C}a$ which is totally isotropic of dimension 4 and characterized by $x \in \mathbb{C}a \Leftrightarrow xa^* = 0$. We will see that all inner ideals can be obtained as intersections of such principal inner ideals.

The ways in which these 4-spaces can intersect is given by

6.4 (Intersection Theorem) Two right 4-spaces determined by isotropic vectors $a, b \neq 0$ have intersection

(i) $a\mathbb{C} \cap b\mathbb{C} = a\mathbb{C} = b\mathbb{C}$ of dimension 4 if a, b are linearly dependent

(ii) $a\mathbb{C} \cap b\mathbb{C} = a(b^*\mathbb{C}) = b(a^*\mathbb{C})$ of dimension 2 if a, b are independent but $n(a,b) = 0$

(iii) $a\mathbb{C} \cap b\mathbb{C} = 0$ of dimension 0 if a, b are independent and $n(a,b) \neq 0$.

A right and left 4-space have intersection

(iv) $a\mathbb{C} \cap \mathbb{C}b = \Phi ab$ of dimension 1 if $ab \neq 0$

(v) $a\mathbb{C} \cap \mathbb{C}b = ab^{\perp} = a^{\perp}b$ of dimension 3 if $ab = 0$ (where $x^{\perp} = \{y \in \mathbb{C} \mid n(x,y) = 0\}$).

No left 4-space ever coincides with a right 4-space, $a\mathbb{C} \neq \mathbb{C}b$.

Proof. Clearly if $b = \lambda a$ then $a\mathbb{C} = b\mathbb{C}$ as in (i).

Suppose now a, b are independent but $n(a,b) = 0$. By (6.3) we have $a(b^*x) = -b(a^*x)$, so $a(b^*\mathbb{C}) = b(a^*\mathbb{C}) \subset a\mathbb{C} \cap b\mathbb{C}$. But conversely, if $x \in a\mathbb{C} \cap b\mathbb{C}$ (so by (6.2) $a^*x = b^*x = 0$) then by independence of a and b and nonsingularity of $n(x,y)$ we can choose an element c with $n(a,c) = 1$ but $n(b,c) = 0$; by (6.3) $0 = n(b,c)x = b(c^*x) + c(b^*x) = b(c^*x)$, therefore by (6.2) $c^*x \in b^*\mathbb{C}$, so by (6.3) again $x = n(a,c)x = a(c^*x) + c(a^*x) = a(c^*x) \in a(b^*\mathbb{C})$. Thus $a\mathbb{C} \cap b\mathbb{C} \subseteq a(b^*\mathbb{C})$. This shows $a(b^*\mathbb{C}) = b(a^*\mathbb{C}) = a\mathbb{C} \cap b\mathbb{C}$.

To finish (ii) we must find the dimension of $a\mathbb{C} \cap b\mathbb{C}$. We claim that for independent orthogonal isotropic a, b either

(ii') $\quad a\mathbb{C} \cap b\mathbb{C} = \Phi a \oplus \Phi b \qquad$ (if $a^*b = 0$)

(ii") $\quad a\mathbb{C} \cap b\mathbb{C} \oplus \Phi a \oplus \Phi b = \mathbb{C}(a^*b) \qquad$ (if $a^*b \neq 0$)

from which we see immediately $a\mathbb{C} \cap b\mathbb{C}$ has dimension 2. Using (6.3)

(**) $a(b^*x) = an(b,x) - a(x^*b) = n(b,x)a - n(a,x)b + x(a^*b)$.

As x ranges over $\mathbb{C}$, $a(b^*x)$ ranges over all $a(b^*\mathbb{C}) = a\mathbb{C} \cap b\mathbb{C}$ and by independence $n(b,x)a - n(a,x)b$ ranges over all $\Phi a \oplus \Phi b$ (we can find x with $n(b,x) = 1$, $n(a,x) = 0$ or $n(b,x) = 0$, $n(a,x) = 1$) Thus when $a^*b = 0$ we see $a\mathbb{C} \cap b\mathbb{C} = \Phi a \oplus \Phi b$ as in (ii'). When $a^*b \neq 0$ we have $a, b \in \mathbb{C}(a^*b)$ by the dual of (6.2) because $b(a^*b)^* = b(b^*a) = n(b)a = 0$ and $a(a^*b)^* = ab^*a = n(a,b)a - n(a)b = 0$ (by the U-formula (1.19)). Thus $a(b^*\mathbb{C}) + \Phi a + \Phi b \subset \mathbb{C}(a^*b)$ by (**), also $\mathbb{C}(a^*b) \subset a(b^*\mathbb{C}) + \Phi a + \Phi b$, so $\mathbb{C}(a^*b) = a(b^*\mathbb{C}) + \Phi a + \Phi b$. The latter sum is direct since if $a(b^*c) + \alpha a + \beta b = 0$ then left multiplying by a^* yields $\beta a^*b = 0$, hence $\beta = 0$ because $a^*b \neq 0$, dually $\alpha = 0$ (multiplying by b^* and using $a(b^*c) = -b(a^*c)$, $b^*a = (a^*b)^* \neq 0$) so $a(b^*c) = 0$. Thus $\mathbb{C}(a^*b) = a\mathbb{C} \cap b\mathbb{C} \oplus \Phi a \oplus \Phi b$ when $a^*b \neq 0$, as in (ii").

If a, b are independent and $n(a,b) \neq 0$ then (6.3) shows $x = n(a,b)^{-1}\{a(b^*x) + b(a^*x)\}$; if $x \in a\mathbb{C} \cap b\mathbb{C}$ then $a^*x = b^*x = 0$ shows $x = 0$. This establishes (iii).

Now consider a mixed intersection $a\mathbb{C} \cap \mathbb{C}b$, where $n(a) = n(b) = 0$ but $a, b \neq 0$. Any $x = ay = zb$ in the intersection has $a^*(zb) = a^*x = a^*(ay) = 0$, so by (6.3)

$$n(a,z)b = a^*(zb) + z^*(ab) = z^*(ab)$$

$$n(a,z)x = z\{n(a,z)b\} = z\{z^*(ab)\} = n(z)ab.$$

Thus if $n(a,z) \neq 0$ we have $x \in \Phi(ab)$.

First consider the case when $ab \neq 0$. Certainly any $x \in \Phi ab$ belongs to $a\mathbb{C} \cap \mathbb{C}b$. Conversely, assume $x \in a\mathbb{C} \cap \mathbb{C}b$. We saw above that if $n(a,z) \neq 0$ then $x \in \Phi ab$ and if $n(a,z) = 0$ then $z^*(ab) = 0$ (whence $(ab)^*z = \{z^*(ab)\}^* = 0$ implies $z \in (ab)\mathbb{C}$ by (6.2), therefore $x = zb \in \{(ab)\mathbb{C}\}b = a\{b\mathbb{C}b\}$ (right Moufang) $= a\{n(b,\mathbb{C}^*)b - n(b)\mathbb{C}^*\}$ (U-formula 1.19) $= n(b,\mathbb{C}^*)ab \subset \Phi ab$). Thus x belongs to Φab in either case, $a\mathbb{C} \cap \mathbb{C}b \subset \Phi ab$. Consequently $a\mathbb{C} \cap \mathbb{C}b = \Phi ab$ has dimension 1 as in (iv).

Now consider the case when $ab = 0$. If $n(a,z) \neq 0$ for $x = zb \in a\mathbb{C} \cap \mathbb{C}b$ we noticed above $x \in \Phi(ab) = 0$ (hence trivially $x \in a^{\perp}b$), while if $n(a,z) = 0$ then $z \in a^{\perp}$ and again $x = zb \in a^{\perp}b$. Thus $a\mathbb{C} \cap \mathbb{C}b \subset a^{\perp}b$. Conversely, whenever $n(a,z) = 0$ we have $0 = a^*(zb) + z^*(ab) = a^*(zb)$ by (6.3), so by (6.2) $x = zb$ belongs to $a\mathbb{C}$; $a^{\perp}b \subset a\mathbb{C} \cap \mathbb{C}b$. Thus $a\mathbb{C} \cap \mathbb{C}b = a^{\perp}b$ (and dually it equals $ab^{\perp}$). To compute the dimension in this case, note $z \mapsto zb$ is a linear map of the 7-dimensional space $a^{\perp}$ onto $a\mathbb{C} \cap \mathbb{C}b = a^{\perp}b$ with kernel $\{z \in a^{\perp} \mid zb = 0\} = \{z \in a^{\perp} \cap \mathbb{C}b^*\}$ (by (6.2)) $= \mathbb{C}b^*$ (since automatically $\mathbb{C}b^* \subset a^{\perp}$, $n(a, \mathbb{C}b^*) = n(ab, \mathbb{C}) = 0$ by (1.15) if $ab = 0$). Since this kernel has dimension 4, the image $a\mathbb{C} \cap \mathbb{C}b$ has dimension $7 - 4 = 3$.

From this, never $a\mathbb{C} = \mathbb{C}b$ since $a\mathbb{C} \cap \mathbb{C}b$ has dimension 1 or 3. ∎

All totally isotropic subspaces can be represented as intersections of 4-spaces.

6.5 (Theorem on Isotropic Subspaces) The totally isotropic subspaces of a split Cayley algebra are zero and the

(1-spaces) Φa for $n(a) = 0$

(2-spaces) $a(b^*\mathbb{C}) = b(a^*\mathbb{C}) = a\mathbb{C} \cap b\mathbb{C}$ or
$(\mathbb{C}b^*)a = (\mathbb{C}a^*)b = \mathbb{C}a \cap \mathbb{C}b$ for
$n(a) = n(b) = n(a,b) = 0$, a and b independent

(3-spaces) $ab^{\perp} = a^{\perp}b = a\mathbb{C} \cap \mathbb{C}b$ for $n(a) = n(b) = 0$, $ab = 0$

(4-spaces) $a\mathbb{C}$ or $\mathbb{C}a$ for $n(a) = 0$.

Any nonzero totally isotropic subspace is conjugate under a proper isometry to precisely one of

(i) Φe_{11}

(ii) $\Phi e_{11} + \Phi e_{12}^{(1)} = e_{11}\mathbb{C} \cap e_{12}^{(1)}\mathbb{C} = \mathbb{C}e_{21}^{(2)} \cap \mathbb{C}e_{21}^{(3)}$

(iii) $\Phi e_{11} + \Phi e_{12}^{(1)} + \Phi e_{12}^{(2)} = e_{11}\{e_{21}^{(3)}\}^{\perp} = \{e_{11}\}^{\perp} e_{21}^{(3)} = e_{11}\mathbb{C} \cap \mathbb{C}e_{21}^{(3)}$

(iv) $\Phi e_{11} + \Phi e_{12}^{(1)} + \Phi e_{12}^{(2)} + \Phi e_{12}^{(3)} = e_{11}\mathbb{C}$ or

$\Phi e_{11} + \Phi e_{21}^{(1)} + \Phi e_{21}^{(2)} + \Phi e_{21}^{(3)} = \mathbb{C}e_{11}$.

Proof. At this point let us refresh our memory of Witt's Theorem: if Q is a quadratic form on X with $Q(x,y)$ nonsingular, then any two subspaces which are isometric with each other are actually conjugate under an isometry of the total space X. In particular, two totally isotropic subspaces of the same dimension (which are rather trivially isometric) are conjugate under a global isometry.

In our present situation, by nonsingularity of $n(x,y)$ any two totally isotropic subspaces of the same dimension are conjugate under a norm isometry (orthogonal transformation) T. Thus every totally isotropic subspace of dimension 1, 2, 3, or 4 is conjugate to (i), (ii), (iii), or the <u>first</u> member of (iv). If T is proper, we are done. However, if T is improper we must modify it to obtain a proper isometry. (The reason we want a proper T is that later in the proof we will need to know T is an autotopy).

If T is improper, we modify it by the (improper) symmetry S_z: $x \longrightarrow x - n(z)^{-1}n(z,x)z = x + n(z,x)z$ where $z = e_{12}^{(3)} + e_{21}^{(3)}$ has $n(z) = -1$. The composite $T' = S_z \circ T$ will now be proper (if T is a product of an odd number of symmetries, T' will be a product of an even number), and it still sends the given subspace into (i), (ii), or (iii) if T does because S_z fixes $\Phi e_{11} + \Phi e_{12}^{(1)} + \Phi e_{12}^{(2)}$: $n(z,e_{11}) = n(z,e_{12}^{(1)}) = n(z,e_{12}^{(2)}) = 0$ and whenever $n(z,x) = 0$ we have $S_z(x) = x$. Thus in cases (i), (ii), (iii) we have found the proper conjugation.

In case (iv) T' does not send the given subspace into $e_{11}\mathbb{C}$ because S_z does not fix $e_{11}\mathbb{C}$: $n(z,e_{12}^{(3)}) = -1$ implies $S_z(e_{12}^{(3)}) = e_{12}^{(3)} - z = -e_{21}^{(3)}$. In fact, in this case we cannot get to $e_{11}\mathbb{C}$ by a proper isometry, and must settle for the second member of (iv), namely $\mathbb{C}e_{11}$. The easiest way to do this is to apply two more symmetries S_w and S_v (for $w = e_{12}^{(2)} + e_{21}^{(2)}$, $v = e_{12}^{(1)} + e_{21}^{(1)}$) to successively convert $e_{12}^{(2)}$ into $-e_{21}^{(2)}$ and then $e_{12}^{(1)}$ into $-e_{21}^{(1)}$ in the same way S_z converted $e_{12}^{(3)}$ into $-e_{21}^{(3)}$ leaving everything else alone (note S_w fixes e_{ii}, $e_{ij}^{(k)}$ for $k \neq 2$ and S_v fixes $e_{ii}, e_{ij}^{(\ell)}$ for $\ell \neq 1$ since $n(w,e_{ii}) = n(w,e_{ij}^{(k)}) = n(v,e_{ii}) = n(v,e_{ij}^{(\ell)}) = 0$); then $T'' = S_vS_wT' = S_vS_wS_zT$ is still proper (even number of symmetries), but sends the given subspace into $S_vS_wS_z(e_{11}\mathbb{C}) = \Phi e_{11} + \Phi e_{21}^{(1)} + \Phi e_{21}^{(2)} + \Phi e_{21}^{(3)} = \mathbb{C}e_{11}$.

Looked at the other way, we can get from our basic space to the given totally isotropic space by a proper isometry. Now our basic spaces (i)-(iv) all have the desired form of k-spaces ($1 \leq k \leq 4$), so to prove that each isotropic space actually is a k-space (rather than merely a conjugate of one) all we have to do is prove that a proper isometry takes k-spaces into k-spaces.

Preservation follows from the Principle of Triality 5.00: a proper similarity T of a Cayley algebra is an autotopy, so there exist similarities T', T'' with (T,T',T'') an isotopy. From this T preserves 4-spaces: $T(a\mathbb{C}) = T'(a)T''(\mathbb{C}) = a'\mathbb{C}$ where $a' = T'(a)$ is isotropic, $n(a') = n(T'a) = \tau'n(a) = 0$ (τ' the multiplier of the norm similarity T'), dually $T(\mathbb{C}a) = \mathbb{C}a''$ for isotropic $a'' = T''(a)$. By intersections, the other k-spaces follow. For 2-spaces: $T(a\mathbb{C} \cap b\mathbb{C}) = T(a\mathbb{C}) \cap T(b\mathbb{C})$ $= a'\mathbb{C} \cap b'\mathbb{C}$ for isotropic a', b' with $n(a',b') = n(T'a, T'b)$ $= T'n(a,b) = 0$ and $a' = T'(a)$, $b' = T'(b)$ independent (dually for $\mathbb{C}a \cap \mathbb{C}b$). For 3-spaces: $T(a\mathbb{C} \cap \mathbb{C}b) = a'\mathbb{C} \cap \mathbb{C}b''$ for isotropic a', b'' with $a'b'' = T'(a)T''(b) = T(ab) = T(0) = 0$. For 1-spaces, trivially $T(\Phi a) = \Phi T(a)$ where $n(ta) = Tn(a) = 0$. Thus an autotopy T takes one k-space into another, and therefore any given totally isotropic subspace is a k-space.

To prove uniqueness of the representatives, because the basic types have different dimensions we need only show the two members of (iv) are <u>not</u> conjugate under a proper isometry T. But this follows by preservation: if $\mathbb{C}e_{11}$ were a conjugate of $e_{11}\mathbb{C}$ it would again have the form $a'\mathbb{C}$, and $a'\mathbb{C} \neq \mathbb{C}e_{11}$ by 6.4. ∎

VII.6 Exercises

6.1 If A is a regular alternative algebra, show that any inner $\mathbb{Z}$-ideal $B \subset A$ is invariant under the centroid $\Gamma(A)$, $\Gamma B \subset B$. Show any degree 2 algebra with nondegenerate norm is regular.

6.2 If Q is a quadratic form on a finite-dimensional vector space X with $Q(x,y)$ nonsingular, show $\dim Y \leq \frac{1}{2} \dim X$ for any totally isotropic subspace Y; more generally, even if X is infinite show that there are subspaces Y', Z with $X = Y \oplus Y' \oplus Z$ with $\dim Y' = \dim Y$ and dual bases $\{y_\alpha\}$, $\{y'_\alpha\}$ for Y, Y' with $n(y_\alpha) = n(y'_\alpha) = 0$, $n(y_\alpha, y'_\alpha) = 1$, $n(y_\alpha, y'_\beta) = 0$.

6.3 If $a, b \in B \subset \mathbb{C}(B, \mu)$ have $aB \cap bB = ab^*B$, $Ba \cap Bb = Bb^*a$ show $a\mathbb{C} \cap b\mathbb{C} = ab^*B + (Bb^*a)\ell = a(b^*\mathbb{C})$. If B is quaternion show xB has dimension 2 if $n(x) = 0$, $x \neq 0$; if $n(x) = n(y) = 0$ show $xB \cap yB = 0$ if $n(x,y) \neq 0$ or if $xy^* = 0$, and $xB \cap yB = xB = yB = xy^*B$ if $n(x,y) = 0$ but $xy^* \neq 0$. If $n(x) = n(y) = n(x,y) = 0$ show $xy^* = 0 \Leftrightarrow y^*x \neq 0$. Conclude always $aB \cap bB = ab^*B$, $Ba \cap Bb = Bb^*a$ if $n(a) = n(b) = n(a,b) = 0$ and $a\mathbb{C} \cap b\mathbb{C} = ab^*B$ or $(Bb^*a)\ell$ always has dimension 2.

6.4 If $n(a) = n(b) = n(a,b) = 0$, $a^*b \neq 0$ in $\mathbb{C}$ show there exists $c \in \mathbb{C}$ with $n(c, a^*b) = 1$; show $\{ac, bc\}$ form a basis for $\mathbb{C}/a^\perp \cap b^\perp$. Show there are x, y forming a basis for $a^\perp \cap b^\perp/\mathbb{C}(a^*b)$. Show $x(a^*b) \in a\mathbb{C}$ iff $x \in a^\perp$, $x(b^*a) \in b\mathbb{C}$ iff $x \in b^\perp$. Then $\mathbb{C} = \Phi ac \oplus \Phi bc \oplus \Phi x \oplus \Phi y \oplus \mathbb{C}(a^*b)$, $\mathbb{C}(a^*b) = \Phi a \oplus \Phi b \oplus \Phi x(a^*b) \oplus \Phi y(a^*b)$ where $(ac)(a^*b) = -a$, $(bc)(a^*b) = -b$, $a\mathbb{C} \cap b\mathbb{C} = \Phi x(a^*b) \oplus \Phi y(a^*b)$.

6.5 If $n(a) = n(b) = 0$, $ab = 0$ show there are $\{x, y, z\}$ forming a basis for $a^\perp/\mathbb{C}b^*$; show $a^\perp b = \Phi xb \oplus \Phi yb \oplus \Phi zb$. Show $ub \in a\mathbb{C}$ iff $n(a,u) = 0$, so $a\mathbb{C} \cap \mathbb{C}b = a^\perp b$.

6.6 If $n(a) = n(b) = 0$ show $a\mathbb{C} \cap b\mathbb{C} = \{a\mathbb{C} + b\mathbb{C}\}^{\perp}$, $a\mathbb{C} \cap \mathbb{C}b = \{a\mathbb{C} + \mathbb{C}b\}^{\perp}$.

6.7 In a Cayley matrix algebra explicitly compute $e_{11}\mathbb{C}$, $e_{12}^{(1)}\mathbb{C}$, $e_{11}\mathbb{C} \cap e_{12}^{(1)}\mathbb{C}$; $\mathbb{C}e_{21}^{(2)}$, $\mathbb{C}e_{21}^{(3)}$, $\mathbb{C}e_{21}^{(2)} \cap \mathbb{C}e_{21}^{(3)}$; $\{e_{21}^{(3)}\}^{\perp}$, $\{e_{11}\}^{\perp}$, $e_{11}\{e_{21}^{(3)}\}^{\perp}$, $\{e_{11}\}^{\perp}e_{21}^{(3)}$, $e_{11}\mathbb{C} \cap \mathbb{C}e_{21}^{(3)}$.

6.8 Verify directly that in a Cayley matrix algebra

$$e_{11}\mathbb{C} \cap e_{21}^{(1)}\mathbb{C} = \mathbb{C}e_{22} \cap \mathbb{C}e_{21}^{(1)} = \Phi e_{12}^{(2)} \oplus \Phi e_{12}^{(3)} .$$

Find $e_{11}\mathbb{C} \cap e_{12}^{(1)}\mathbb{C}$ and show it also equals an intersection $\mathbb{C}a \cap \mathbb{C}b$.

6.9 Show in general $a\mathbb{C} \cap b\mathbb{C} = \mathbb{C}a \cap \mathbb{C}b$ if $a^*b = ba^* = 0$. If one merely assumes $a^*b = 0$, show $a\mathbb{C} \cap b\mathbb{C} = \mathbb{C}a \cap \mathbb{C}b$ iff $t(a)b = t(b)a$ (and iff $ab = ba$).

6.10 If $n(a) = n(b) = n(a,b) = a^*b = 0$ show $a\mathbb{C} \cap b\mathbb{C} = c\mathbb{C} \cap d\mathbb{C}$ iff $\Phi a \oplus \Phi b = \Phi c \oplus \Phi b$.