§5 Triality and Local Triality

Recall from Part 1 that if Q is any quadratic form on a vector space X over a field Φ, the generalized orthogonal group GO(Q) of Q, or group of similarities, consists of all bijective transformations T on X which preserve the quadratic form up to a scalar: $Q(Tx) = \tau Q(x)$ where $\tau \in \Phi$ is the multiplier of T. Those transformations with multiplier 1 comprise the orthogonal group O(Q). If the bilinear form Q(x,y) is non-degenerate and X is finite-dimensional the orthogonal group is generated by the symmetries

$$S_a(x) = x - Q(a)^{-1}Q(a,x)a$$

determined by non-isotropic vectors $a \in X$ $(Q(a) \neq 0)$. Those orthogonal transformations which are products of an even number of symmetries are called proper orthogonal transformations, and constitute a subgroup $O^+(Q)$ of index 2 in O(Q).

We wish to show that when the quadratic form Q is the norm form N of a composition algebra (in particular, a Cayley algebra) then "proper" norm similarities belong to the structure group. We do this in two steps, first considering those similarities which move the distinguished element 1 of X, and then those fixing this element.

5.1 Proposition. If N is the norm form of a composition algebra, every norm similarity T has a unique decomposition

$$T = L_x S$$

where $x = T(1)$ and S is orthogonal fixing 1, $1 = S(1)$.

Proof. Such a decomposition is clearly unique, for if $T = L_x S$ then $T(1) = L_x S(1) = L_x(1) = x$ determines x and $S = L_x^{-1} T$ determines S. Such a decomposition exists since $N(x) = N(T1) = \tau T(1) = \tau$ invertible guarantees x invertible (II.20), thence L_x invertible (Inverse Theorem I.42), so $S = L_x^{-1} T$ is also an invertible transformation, where now $N(Sy) = N(L_{x^{-1}} Ty) = N(x^{-1}) N(Ty) = N(x)^{-1} N(Ty) = \tau^{-1} \tau N(y) = N(y)$ shows S is orthogonal (not just a similarity) and $S(1) = L_{x^{-1}} T(1) = x^{-1} x = 1$.

Note that this result does not depend on Φ being a field, only on $\tau \in \Phi$ being invertible. ∎

Thus by a translation we can reduce any similarity to an orthogonal transformation fixing the unit element. For these we have

5.2 Proposition. If N is the norm form of a standard composition algebra over a field Φ, every proper orthogonal transformation $T \in O^+(N)$ belongs to the structure group.

$$T(xy) = T'(x)T''(y) \tag{5.3}$$

for similarities T', T''; every improper orthogonal $T \in O^-(N)$ satisfies

$$T(xy) = T''(y)T'(x) \tag{5.4}$$

for similarities T', T''.

Proof. The assumption that the composition algebra is standard is imposed to guarantee that the bilinear form $N(x,y)$ is nondegenerate and X is finite-dimensional, so that the symmetries S_a generate the orthogonal transformations.

First consider the case of a symmetry $T = S_a$. We claim

$$(5.5)\qquad S_a(xy) = -N(a)^{-1}(L_a\bar{y})(R_a\bar{x}) = T''(y)T'(x)$$

The reason for this is the intimate connection between symmetries and U-operators:

$$(5.6)\qquad S_a(x) = -N(a)^{-1}U_a\bar{x}$$

$$(5.7)\qquad U_a(x) = -N(a)S_a\bar{x}$$

recalling $S_a(x) = x - N(a)^{-1}N(a,x)a$ and $U_a(x) = N(a,\bar{x})a - N(a)\bar{x}$ or $U_a\bar{x} = N(a,x)a - N(a)x$ (II.2.0). Note also $\bar{x} = t(x)1 - x = N(1)^{-1}N(1,x)1 - x$

$$(5.8)\qquad \bar{x} = -S_1(x),$$

so we can also describe the U-operator by

$$(5.9)\qquad U_a = N(a)S_aS_1 .$$

These relation show once more how the algebraic structure (in this case the involution and U-operators) of a composition algebra are determined by the norm form N and its symmetries.

Once we have related the symmetry S_a to the U-operator U_a, the Moufang formula $U_a(xy) = (ax)(ya)$ shows $S_a(xy) = -N(a)^{-1}U_a(\overline{xy})$ (by (5.6)) $= -N(a)^{-1}U_a(\bar{y}\bar{x}) = -N(a)^{-1}(a\bar{y})(\bar{x}a)$ as required by (5.5), with T', T'' similarities.

For a product S_aS_b of two symmetries we get $S_aS_b(xy) = S_a\{T_b''(y)T_b'(x)\} = T_a''\{T_b'(x)\}T_a'\{T_b''(y)\} = T'(x)T''(y)$ repeating (5.5) twice. In the same way we see

$$T(xy) = T''(y)T'(x)$$

for all improper orthogonal T (odd number of symmetries) and

$$T(xy) = T'(x)T''(y)$$

for all proper orthogonal T (even number of symmetries). ∎

5.10 <u>Remark</u>: The T', T'' in (5.5) need not be orthogonal if T is. However, if Φ is closed under square roots (e.g. if it is algebraically closed) we can scale them up so they are orthogonal: if T', T'' have multipliers $\tau' = N(T'1)$, $\tau'' = N(T''1)$ then $\tau'\tau'' = N(T'1)N(T''1) = N(T'1 \cdot T''1) = N(T(1 \cdot 1)) = N(T1) = 1$ (T is orthogonal); if $\sigma^2 = \tau'$ then $T(xy) = S'(x)S''(y)$ for $S' = \sigma^{-1}T'$, $S'' = \sigma T''$ with multipliers $N(S'1) = \sigma^{-2}N(T'1) = \sigma^{-2}\tau' = 1$ and $N(S''1) = \sigma^2\tau'' = \sigma^2\tau'^{-1} = 1$, i.e. S' and S'' are both orthogonal. ∎

We say T is a <u>proper similarity</u>, and write $T \in GO^+(N)$, if $T(xy) = T'(x)T''(y)$, while if $T(xy) = T''(y)T'(x)$ we say $T \in GO^-(N)$ is an <u>improper similarity</u>.

5.11 (Triality Principal) Every norm similarity $T \in GO(N)$ of a standard composition algebra over a field is either a proper similarity,

$$T(xy) = T'(x)T''(y) \tag{5.3}$$

where T', T'' are also proper similarities, or it is an improper similarity

$$T(xy) = T''(y)T'(x) \tag{5.4}$$

where T', T'' are also improper.

Proof. Every invertible $T = L_x$ is proper

$$T(yz) = L_x(yz) = x\{y(xx^{-1}z)\} = (xyx)(x^{-1}z) = T'(y)T''(z)$$

for similarities $T' = U_x$, $T'' = L_{x^{-1}}$, and by the previous proposition every orthogonal T is proper or improper. Then an arbitrary similarity $T = L_x S$ is proper or improper according as its orthogonal part S is proper or improper.

A proper similarity T is an isotopy, which implies T', T'' are also isotopies. We go through the corresponding argument when T is improper, i.e. an anti-isotopy: (5.4) implies $T(x) = t''T'(x)$ and $T(y) = T''(y)t'$ for $t' = T'(1)$, $t'' = T''(1)$, so $T' = L_{t''}^{-1}T$ and $T'' = R_{t'}^{-1}T$ are improper if T is. ∎

We now see $GO^+(N)$ is a subgroup of $GO(N)$ of index 1 or 2; the natural representative for the complementary coset $GO^-(N)$ is the standard involution * ($(xy)^* = y^*x^*$). We have $GO^- = GO^+$ iff * is proper, which happens iff the composition algebra is commutative. Thus $GO = GO^+ = GO^-$ in dimensions 1 and 2, while GO^+ has index 2 in dimensions 4 and 8.

5.12 Corollary. The similarities T', T'' determined by (5.3), (5.4) are unique up to a multiple from the nucleus; in the case of a Cayley algebra, up to a scalar multiple.

Proof. In a Cayley algebra over Φ the nucleus is just $\Phi 1$; in general, $T'(x)T''(y) = S'(x)S''(y)$ implies $T' = L_n S'$, $T'' = L_n^{-1}S''$ for nuclear n (as in III.1.0). ∎

We can develop an analogous theory of local triality. A linear transformation W is <u>semi-alternating</u> relative to a quadratic form Q if $Q(Wx,x) = \omega Q(x)$ for all x, where $\omega \in \Phi$ is some fixed multiplier; W is <u>alternating</u> if $\omega = 0$, i.e. $Q(Wx,x) = 0$. This implies W is skew, but as usual in characteristic 2 skew does not imply alternating. We denote by $GL(Q)$ and $L(Q)$ the Lie algebras of semi-alternating and alternating transformations. If Q is nondegenerate and finite-dimensional over a field, the alternating transformations are spanned by the

$$S_{a,b}(x) = Q(x,a)b - Q(x,b)a .$$

We are interested in the case where $Q = N$ is the norm of a composition algebra, and begin by translating semi-alternating to alternating.

5.13 Proposition. Every semi-alternating $W \in GL(N)$ has a unique decomposition

$$W = L_z + Z$$

where $z = W1$ and Z is alternating with $Z1 = 0$.

Proof. Again such a decomposition is clearly unique since $z = W1$ and $Z = W - L_x$. It exists since $W - L_z$ is still semi-alternating (since L_z is: $N(zx,x) = T(z)N(x)$ with multiplier $T(z)$ by II.2.0), but now satisfies $Z1 = W1 - z = 0$. This forces Z to be alternating, since in general the multiplier is $\omega = \omega N(1) = N(W1,1)$. ∎

Since we are dealing with sums rather than products, and addition is always commutative, there is no propriety or impropriety in local triality.

5.14 Proposition. If N is the norm of a standard composition algebra over a field Φ, any alternating transformation $W \in L(N)$ is a local isotopy:

$$W(xy) = W'(x)y + xW''(y) \tag{5.15}$$

for semi-alternating W', W''.

Proof. Since W is spanned by the $S_{a,b}$ and everything is linear, it suffices to consider $W = S_{a,b}$. Recall $S_{a,b}(x) = N(x,a)b - N(x,b)a$. This is almost the same as $V_{a,\bar{b}}(x) = U_{a,x}\bar{b} = N(a,b)x + N(x,b)a - N(a,x)b$, indeed

$$S_{a,b} = N(a,b)I - V_{a,\bar{b}} . \tag{5.16}$$

Since $V_{a,\bar{b}}$ belongs to the structure algebra by IV.4.0, as does I, we see $S_{a,b}$ does too.

More directly, $S_{a,b}(xy) = N(xy,a)b - aN(xy,b) = \{(xy)\bar{a} + a(\overline{xy})\}b - \{a(\overline{xy})\}b - \{a\bar{b}\}(xy)$ (using $aN(z) = a(\bar{z}z) = (a\bar{z})z$) $= x \cdot U_{y,b}\bar{a} - \{(xb)\bar{a}\}y - \{a\bar{b} \circ x\}y + x\{(a\bar{b})y\} = x\{(y\bar{a})b + (b\bar{a})y + (a\bar{b})y\} - \{(xb)\bar{a} + (xa)\bar{b} + a(\bar{b}x)\}y = x[(y\bar{a})b\} + x\{N(a,b)y\} - \{xN(a,b)\}y - \{a(\bar{b}x)\}y = x\{(y\bar{a})b\} - \{a(\bar{b}x)\}y = xW''(y) + W'(x)y$. ∎

5.17 <u>Remark</u>: If Φ is closed under division by 2, i.e. has characteristic $\neq 2$ (the additive analogue of the multiplicative condition that Φ be closed under square roots), then W', W'' can be chosen alternating if W is. In fact since the multipliers ω', ω'' are negatives by $\omega' + \omega'' = N(W'1,1) + N(W''1,1) = N(W'(1)\cdot 1 + 1\cdot W''(1),1) = N(W1,1) = \omega = 0$ we have $W(xy) = S'(x)y + xS''(y)$ for $S' = W' - \frac{1}{2}\omega' I$, $S'' = W'' - \frac{1}{2}\omega'' I$ where S', S'' now have multipliers $\sigma' = N(W'1,1) - \frac{1}{2}\omega' N(1,1) = \omega' - \frac{1}{2}\omega'\{2N(1)\} = 0$ and $\sigma'' = N(W''1,1) = \omega'' - \frac{1}{2}\omega''\cdot 2 = 0$. ∎

For a general $W = L_z + Z$ we have (5.15) for the alternating transformation Z by the above, and also for L_z since $z(xy) = (z\circ x)y - x(zy)$, so (5.15) holds for W:

5.18 (Local Triality Principle) If W is semi-alternating relative to the norm form of a standard composition algebra then W is a local isotopy,

$$W(xy) = W'(x)y + xW''(y)$$

for semi-alternating W', W''. ∎

Again, W' and W'' are unique up to translation by a nuclear element ($W'(x)y + xW''(y) = S'(x)y + xS''(y)$ implies $S' = W' - L_z$, $S'' = W'' + L_z$ for nuclear z); if the composition algebra has dimension 8 its nucleus is just $\Phi 1$, so W', W'' are unique up to a scalar. When W is alternating in characteristic $\neq 2$, the alternating W', W'' are uniquely determined by W.

Exercises

5.1. Show $V_a = S_{1,a}$ when $T(a) = 0$; if $T(b) = 0$ too show $[V_a, V_b] = 2S_{a,b}$. In characteristic $\neq 2$ show the V_a with $T(a) = 0$ generate the $S_{a,b}$ as Lie algebra; conclude that since V_a lie in the structure algebra, so do all $S_{a,b}$.

5.2. In characteristic $\neq 2$ and dimension 8, where skew W', W'' are uniquely determined by a skew W, show $W \to W'$ and $W \to W''$ are automorphisms of $L(N)$.

5.3. Prove the multiplicative analogue of #2 when Φ is closed under square roots.

5.4. Prove that for arbitrary Φ (Not necessarily closed under $\frac{1}{2}$ or under square roots) the maps $W \to W'$, $W \to W''$ are automorphisms of $GL(N)/\Phi I$ and $T \to T'$, $T \to T''$ of $GO(N)/\Phi I$ for N the norm of a Cayley algebra (dimension 8).

5.5. Show that $GL(N) = \Phi I \oplus L(N)$ for a Cayley algebra in characteristic $\neq 2$, and $GO(N) = \Phi 1 \times O(N)$ when Φ is closed under square roots, so $GL(N)/\Phi I$ and $GO(N)/\Phi I$ can be canonically identified with $L(N)$ and $O(N)$.