§2 Composition algebras

Alternativity of a degree 2 algebra has definite consequences for the norm form and involution. Conversely, under suitable conditions the mere existence of a norm form guarantees alternativity.

Despite its name, the standard involution $x^* = t(x)1 - x$ of a degree 2 algebra is not really an involution in general. To measure how far it deviates from the involution condition $(xy)^* = y^*x^*$ we write $y = z^*$ and compute $(xz^*)^* - zx^* = t(xz^*)1 - n(x,z)1$. The condition that * be an involution is therefore

(2.1) $$t(xy^*)1 = n(x,y)1$$

(Involution Condition)

(2.2) $$n(x,y)1 = t(x)t(y)1 - t(xy)1 \ .$$

Notice that since $n(x,y)$ is symmetric, (2.2) implies

(2.3) $$t(xy)1 = t(yx)1 \ .$$

For an alternative algebra over a nice Φ these conditions are always met.

2.4 (Scalar Involution Criterion) If A is a flexible degree 2 algebra and Φ contains no nilpotent elements, then $x^* = t(x)1 - x$ defines a scalar involution on A. If A is an alternative algebra with scalar involution, it is of degree 2 over $\Phi 1$ and

(2.5) $$n(xy) = n(x)n(y) \ .$$

Proof. Assume A is flexible of degree 2. We know

$0 = x(x\circ y) - x\circ xy \qquad (L_x(L_x+R_x) = (L_x+R_x)L_x$ by flexibility)

$= x\{t(x)y+t(y)x-n(x,y)1\} - \{t(x)xy+t(xy)x-n(x,xy)1\}$ (by (1.2))

$= t(y)\{t(x)x-n(x)1\} - n(x,y)x - t(xy)x + n(x,xy)1$ (by (1.1) on x)

$= \{t(x)t(y) - t(xy) - n(x,y)\}x + \{n(x,xy)-t(y)n(x)\}1$

or

$$(2.6) \qquad f(x,y)x + g(x,y)1 = 0$$

for $f(x,y) = t(xy^*) - n(x,y)$, $g(x,y) = n(x,xy) - t(y)n(x)$. To show * is an involution we must show $f(x,y)1$ vanishes identically. If Φ is a field this is easy: on the Zariski-dense set where $x \notin \Phi 1$ we have $f(x,y) = g(x,y) = 0$ by independence.

In the general case, we know $f(1,y)1 = \{t(1)t(y) - t(y) - n(1,y)\}1 = 0$ (linearizing (1.8) shows $n(x,1)1 = x1^* + 1x^* = x + x^* = t(x)1$, and $t(1) = 2$). Applying $f(\cdot,y)$ to the relation (2.6) gives $f(x,y)^2 = -g(x,y)f(1,y) = 0$, so our hypothesis that Φ contains no nilpotents guarantees $f(x,y)1 = 0$, and * is a scalar involution.

If A is alternative, the norm condition can be derived directly from alternativity and the involution: $n(xy) = (xy)(xy)^* = (xy)(y^*x^*) = (xy)(y^*\{t(x) - x\}) = t(x)(xy)y^* - (xy)(y^*x) = t(x)x(yy^*) - x(yy^*)x$ (middle Moufang) $= \{t(x)x - x^2\}n(y) = n(x)n(y)$. ∎

We say the norm form <u>permits composition</u> if it is multiplicative as in (2.5), $n(xy) = n(x)n(y)$, so that the norm of a product equals the product of the norms.

Inverses are easily described when the norm permits composition.

2.7 (Inverse Criterion) An element x of an alternative algebra with scalar involution is invertible iff its norm is invertible, in which case

$$x^{-1} = n(x)^{-1} x^* .$$

Proof. If $xy = 1$ then $n(x)n(y) = n(xy) = n(1) = 1$ shows $n(x) \in \Phi 1$ is invertible with inverse $n(y)$. Conversely if $n(x)$ invertible then $xx^* = x^*x = n(x)$ shows $y = n(x)^{-1}x^*$ has $xy = yx = 1$. ∎

2.8 Corollary. An alternative algebra with scalar involution over a field is a division algebra iff its norm form does not represent zero:

$$n(x) = 0 \text{ only for } x = 0 . \quad ∎$$

Radicals too are easily described when the norm permits composition. First we need some useful consequences of the norm formula. Linearizing the quadratic relation $n(xy) = n(x)n(y)$ in x yields $n(xy,y) = n(x,1)n(y)$; linearizing this in y gives $n(xy,z) + n(xz,y) = n(x,1)n(y,z)$, or $n(xy,z) = n(y,n(x,1)z-xz) = n(y,\{n(x,1)1-x\}z)$. Now a degree 2 algebra need not always have $n(x,1) = t(x)$, but linearizing $x^2 - t(x)x + n(x)1 = 0$ shows $2x - t(1)x - t(x)1 + n(x,1)1 = 0$. Since $t(1) = 2$ by hypothesis, we have at least $n(x,1)1 = t(x)1$. Our relation (and its dual) may be succinctly stated as

$$n(xy,z) = n(y,x^*z)$$

(2.9)

$$n(yx,z) = n(y,zx^*)$$

Thus left multiplication L_x by x becomes left multiplication by x* when moved across the bilinear form, or in other words the adjoint relative to the bilinear form is $L_x^* = L_{x^*}$. Dually $R_x^* = R_{x^*}$.

Degeneracy of the norm form (or any quadratic form) is measured by its radical

(2.10) $\text{Rad } n = \{z \mid n(z+x) = n(x) \text{ for all } x\} = \{z \mid n(z) = n(z,x) = 0 \text{ for all } x\}$

(Clearly if $n(z) = n(z,x) = 0$ then $n(z+x) = n(z) + n(z,x) + n(x) = n(x)$, conversely if $n(z+x) = n(x)$ then for $x = 0$ we get $n(z) = 0$, then $n(x) = n(z+x) = n(z,x) + n(x)$ implies all $n(z,x) = 0$.) Rad n is always a linear subspace, since $n(\alpha z + \beta w,x) = 0$ if $n(z,x) = n(w,x) = 0$, and $n(\alpha z + \beta w) = \alpha^2 n(z) + \alpha\beta n(z,w) + \beta^2 n(w) = 0$ if $n(z) = n(w) = n(z,\cdot) = 0$.
The norm form is nondegenerate iff Rad n = 0.

Degeneracy of the norm form corresponds to a certain degeneracy of the algebra. The next proposition gives us a glimpse of the interrelation of the various radicals we will consider in Chapter VI.

2.11 (Radical Proposition) If A is an alternative algebra with scalar involution over Φ then Rad n is a nil ideal which consists entirely of elements z which are strictly trivial and generate trivial ideals $Z = \hat{A}z = z\hat{A}$. If Φ contains no nilpotent elements then Rad n is the maximal nilideal, and contains all the trivial elements of A.

Proof. Rad n is a left ideal since $n(xz) = n(x)n(z) = 0$ by (2.5) and $n(xz,y) = n(z,x^*y) = 0$ by (2.9), so that $xz \in$ Rad N when $z \in$ Rad n. Similarly it is a right ideal. Its elements are trivial by the U-formula

(1.9), $U_z y = n(z,y^*)x - n(z)y^* = 0$. In particular, $z^2 = U_z 1 = 0$ and Rad n is nil. We have $\hat{A}z = z\hat{A}$ since $t(z)1 = n(z,1)1 = 0$ implies $z^* = -z$, then $0 = n(z,x)1 = zx^* + xz^* = zx^* - xz$ implies

$$xz = zx^* .$$

$Z = \hat{A}z = z\hat{A}$ is (for example) a left ideal since $x(zy) = -x(z^*y) = z(x^*y) - n(x,z)y = z(x^*y) \subseteq Z$; it is trivial since $Z^2 = (z\hat{A})(\hat{A}z) = U_z\hat{A}^2$ (middle Moufang) = 0 by triviality of z.

If Φ contains no nilpotent elements and B is a nil ideal in A, we must have $n(B) = 0$ since if $b^n = 0$ then $n(b)^n = n(b^n) = 0$ forces $n(b) = 0$ by hypothesis on Φ. Then $b^2 = t(b)b$ implies $b^n = t(b)^{n-1}b$ for all n by induction; if $b^n = 0$ then taking traces of $t(b)^{n-1}b = 0$ gives $t(b)^n = 0$, again forcing $t(b) = 0$. Then $n(b,a)1 = t(ba^*)1$ (by (2.1)) $\subseteq t(B)1 = 0$. Taking norms and using $n(1) = 1$ we get $n(b,a)^2 = 0$, once more forcing $n(b,a) = 0$. This shows $n(B) = n(B,A) = 0$, and all nil ideals B are contained in Rad n for such a Φ.

Also for such Φ, if z is trivial then $0 = U_z z^* = n(z)z$ implies (taking norms) $n(z)^3 = 0$, therefore $n(z) = 0$, and $0 = U_z a^* = n(z,a)z - n(z)a = n(z,a)z$ implies (applying $n(a,\cdot)$) $n(z,a)^2 = 0$, so again $n(z,a) = 0$. Thus $n(z) = n(z,a) = 0$ for trivial z, and all such belong to Rad n. ∎

Recalling that when Φ has no nilpotents * is a scalar involution (see 2.4), we have

2.12 Corollary. The following are equivalent for an alternative algebra of degree 2 over a ring without nilpotent elements:

(i) A is strongly semiprime

(ii) A is semiprime

(iii) A has no nil ideals

(iv) the norm form is nondegenerate. ∎

If A is a unital alternative algebra of degree 2 and Φ has no nilpotent elements, then A has scalar involution $xx^* = n(x)1$. Whenever A has scalar involution, it is of degree 2 over $\Phi 1$ and $\tilde{n}(x) = xx^* \in \Phi 1$ permits composition. (Observe: the original $n(x) \in \Phi$ need not permit composition, although it does if Φ acts faithfully and we can identify $\Phi = \Phi 1$). Conversely, the presence of a well-behaved norm form permitting composition implies the algebra is alternative of degree 2. To see that just having any old norm form permitting composition is not enough, consider the following example: take any horrible algebra A_o and tack on a unit, $A = \Phi 1 \oplus A_o$; then the norm form $n(x) = \alpha^2$ $(x = \alpha 1 + a)$ permits composition, $n(xy) = \alpha^2\beta^2 = n(x)n(y)$, but tells us nothing about A_o (since there's nothing to tell).

Thus it is reasonable to require the quadratic form n to be nondegenerate, i.e. there is no element $z \neq 0$ with

$$n(z) = n(z,A) = 0 .$$

(We allow $n(z,A) = 0$ as long as $n(z) \neq 0$, so the bilinear form $n(\cdot,\cdot)$ can be degenerate.)

2.13 (Composition Criterion) If a unital alternative algebra A carries a nondegenerate quadratic form n permitting composition,

$$n(xy) = n(x)n(y),\ n(1) = 1$$

then A is alternative of degree 2 over Φ with norm n.

Proof. If we define $t(x) = n(x,1)$ and define $x^* = t(x)1 - x$ then just as before we obtain (2.9), $n(xy,z) = n(y,x^*z)$. From this we see $n(x^*(xy),z) = n(xy,xz) = n(x)n(y,z)$ (linearization of composition formula with respect to y) $= n(n(x)y,z)$ and also $n(x^*(xy)) = n(x^*)n(x)n(y) = n(x)^2n(y) = n(n(x)y)$ (noting $n(x^*) = n(t(x)1-x) = t(x)^2n(1) - t(x)n(1,x) + n(x) = t(x)^2 - t(x)t(x) + n(x) = n(x)$). Now anytime $n(a) = n(b)$ and $n(a,z) = n(b,z)$ for all z we have $n(a-b,z) = 0$ for all z and $n(a-b) = n(a) - n(a,b) + n(b) = n(a) - n(a,a) + n(a) = 0$, therefore by nondegeracy $a-b = 0$. In our case this says

$$x^*(xy) = n(x)y\ .$$

Setting $y = 1$ establishes the degree 2 nature of A

$$x^*x = n(x)1 \quad\text{or}\quad x^2 - t(x)x + n(x)1 = 0\ .$$

Note that $n(1) = 1$ by hypothesis, so $t(1) = n(1,1) = 2$:

$$n(1) = 1,\ t(1) = 2\ .$$

Now $x^*(xy) = n(x)y = \{n(x)1\}y = (x^*x)y$, or $[x^*,x,y] = 0$. Thus $0 = [t(x)1 - x,x,y] = -[x,x,y]$ and A is left alternative. Similarly A is right alternative. ∎

In view of this we will call an algebra which carries a nondegenerate quadratic form permitting composition a <u>composition</u> <u>algebra</u>; these are necessarily alternative of degree 2, with scalar involution.

Summarizing our results,

2.14 (Equivalence Theorem for Composition Algebras) The following are equivalent for a unital algebra A over a ring of scalars Φ without nilpotent elements:

(i) A is a composition algebra: it carries a nondegenerate quadratic form n which permits composition, $n(xy) = n(x)n(y)$ and $n(1) = 1$

(ii) A is an alternative algebra of degree 2, $x^2 - t(x)x + n(x)1 = 0$ where t is linear and n nondegenerate quadratic

(iii) A is a strongly semiprime (resp. semiprime) alternative algebra of degree 2 over Φ.

These always imply

(iv) A is a strongly semiprime (resp. semiprime) alternative algebra with scalar involution, $xx^* \in \Phi 1$.

And conversely (iv) implies the others when Φ acts faithfully on A, $\Phi = \Phi 1$. ∎

Exercise

2.1 Find an expression for $n(U_x y)$ in a composition algebra. What can you say about $n(x \circ y)$? Prove $t(x^*) = t(x)$, $n(x^*) = n(x)$, $t(xy^*) = t(yx^*)$, $t(x,y) = t(xy)$ is a symmetric associative bilinear form.

2.2 Suppose $B \triangleleft A$ where A is alternative with scalar involution over a field Φ, and all $1-b$ for $b \in B$ are invertible (all b are <u>quasi-invertible</u>). Show B is a nil ideal, hence $B \subset \text{Rad } n$.

2.3 If $t(e) = 1$, $n(e) = 0$ show e is an idempotent $\neq 0, 1$. If A is of degree 2 over a field Φ, show any idempotent $e \neq 0, 1$ has $t(e) = 1$, $n(e) = 0$.

2.4 If $x_1(x_2(\ldots x_n)) = 1$ in a composition algebra, show each x_i is invertible.

2.5 If n is nondegenerate show that $\alpha n(x) = 0$ for all x implies $\alpha = 0$. Conclude that if $n(x)n(1) = n(x)$ for all x then $n(1) = 1$.

§7. Problem Set on Quasi-Composition Algebras

An algebra A carrying a nondegenerate quadratic form satisfying $n(xy) = n(x)n(y)$ is called a quasi-composition algebra; if A is unital it is an ordinary composition algebra. Throughout let A be quasi-composition over a field Φ.

1. Show that there is some $u \in A$ with $n(u) = 1$; in this case show L_u, R_u are injective. If A is finite-dimensional or a division algebra show L_u, R_u are bijective.
2. Assume L_u, R_u are bijective (not necessarily that A is finite dimensional). Define an isotope $\tilde{A}$ by $x \cdot y = (R_u^{-1}x)(L_u^{-1}y)$. Show $\tilde{A}$ is unital, that n still permits composition on $\tilde{A}$, and is still nondegenerate. Conclude $\tilde{A}$ is an alternative composition algebra.
3. Show how to recover A from the algebra $\tilde{A}$. Is A necessarily unital or alternative?