§5 Isotopy

Another useful way of constructing new alternative algebras out of old ones is by means of homotopes. It will turn out that passage to a homotope provides a means of escaping from the nonassociativity of A.

Let A be an alternative algebra and u an element of A. If we attempt to define a homotope $A^{(u)}$ as in the associative case we have two possibilities for the product $x \cdot_u y$, namely $(xu)y$ and $x(uy)$. In general these will be distinct. We define the <u>left u-homotope</u> $A^{(u,L)}$ to have the same module structure as A but a new product

$$x \cdot_{u,L} y = (xu)y. \qquad (x,y \in A^{(u,L)})$$

Similarly the <u>right u-homotope</u> $A^{(R,u)}$ has product

$$x \cdot_{R,u} y = x(uy). \qquad (x,y \in A^{(R,u)})$$

These are special cases of (two-sided) <u>u,v-homotopes</u> $A^{(u,v)}$ defined by

(5.1) $$x \cdot_{u,v} y = (xu)(vy) \qquad (x,y \in A^{(u;v)}),$$

where we allow u,v to come from the unital hull $\hat{A}$ in case A is not unital. The multiplication operators in $A^{(u,v)}$ have the form

(5.2) $$L_x^{(u,v)} = L_{xu} L_v$$
$$R_y^{(u,v)} = R_{vy} R_u$$
$$U_z^{(u,v)} = U_z U_{uv} .$$

The first two can be read off immediately from (5.1). For the third we calculate $\{z\cdot_{u,v}y\}\cdot_{u,v}z = \{(zu)(vy)\}u \cdot vz = z\{u(vy)u\}\cdot vz$ (right Moufang) $= U_z(\{u(vy)u\}\cdot v)$ (middle Moufang) $= U_z(R_vU_uL_vy) = U_zU_{uv}y$ (right fundamental).

Note that if A is associative, or more generally if u,v associate with everything, the u,v-homotope reduces to the ordinary associative notion

$$x\cdot_{u,v} y = x(uv)y.$$

We will see many instances of properties of $A^{(u,v)}$ which depend only on the product uv and not on u,v individually. (Witness the formula for $U_z^{(u,v)}$.)

5.3 (Homotope Theorem) The u,v-homotope $A^{(u,v)}$ of an alternative algebra A is again alternative.

Proof. By symmetry we need only check left alternativity. We have $L_x^{(u,v)}\,L_x^{(u,v)} = L_{xu}\,L_v\,L_{xu}\,L_v = L_{(xu)v(xu)}\,L_v$ (left Moufang) $= L_{\{x(uv)x\}u}\,L_v$ (right fundamental formula $U_{xu}\,v = R_uU_xL_uv$) $= L_{\{(xu)(vx)\}u}\,L_v$ (middle Moufang) $= L_{(x\cdot_{u,v}x)u}\,L_v = L^{(u,v)}_{x\cdot_{u,v}x}$. ∎

If we iterate this procedure we get nothing new, since the homotope of a homotope is a homotope.

5.4 (Transitivity of Homotopy) Homotopy is a transitive relation,

$$\{A^{(u,v)}\}^{(u',v')} = A^{(u'',v'')} \quad \text{for } u'' = U_u(vu'),\ v'' = U_v(v'u).$$

Proof. Observe

$(x\cdot_{u,v}u')\cdot_{u,v}(v'\cdot_{u,v}y) = \{(xu)(vu')\}u\cdot v\{(v'u)(vy)\} = x\{u(vu')u\}\cdot\{v(v'u)v\}y$ (left & right Moufang) $= (xu'')(v''y) = x\cdot_{u'',v''}y$, so the multiplications in $A^{(u,v)(u',v')}$ and $A^{(u'',v'')}$ coincide. ∎

Inverses and units are further examples of properties of a homotope depending only on uv.

5.5 (Unit Criterion for Homotopes) A homotope $A^{(u,v)}$ has unit $1^{(u,v)}$ iff A has unit 1 and uv is invertible, in which case

$$1^{(u,v)} = (uv)^{-1}.$$

Proof. If w is a unit for $A^{(u,v)}$ then $I = U_w^{(u,v)} = U_wU_{uv}$ by (5.2), so U_w is surjective. By the U-Test 4.4 this implies A is unital and w invertible. Then $U_{uv} = U_w^{-1}$ is also invertible, so by the Inverse Theorem the element uv is invertible with inverse $(uv)^{-1} = U_{uv}^{-1}(uv) = U_w(uv) = (wu)(vw)$ (middle Moufang) $= w\cdot_{u,v}w = w$ since w is the unit for $A^{(u,v)}$.

Conversely, if uv is invertible in the unital algebra A with inverse $w = (uv)^{-1}$ then $(wu)v = u(vw) = 1$. Indeed, $U_{w^{-1}}\{(wu)v\} = \{w^{-1}(wu)\}\{vw^{-1}\}$ (middle Moufang) $= u(vw^{-1})$ (left inverse property $L_{w^{-1}} = L_w^{-1}$) $= u(v(uv)) = (uv)(uv) - [uv,uv] = w^{-1}w^{-1} - [u,v,v]u$ (bumping) $= U_{w^{-1}}\{1\}$, and we can cancel $U_{w^{-1}}$. Similarly $u(vw) = 1$. Thus by the One-Sided Inverse theorem $L_{wu}L_v = R_{vw}R_u = I$, which by (5.2) is just the condition $L_w^{(u,v)} = R_w^{(u,v)} = I$ that w be a left and right unit $1^{(u,v)}$ for $A^{(u,v)}$. ∎

5.6 (Invertibility Criterion for Homotopes) An element x is invertible in a unital homotope $A^{(u,v)}$ iff x is invertible in A, in which case the inverses are related by

$$x^{-1(u,v)} = U_{uv}^{-1} x^{-1} .$$

Proof. x invertible in $A^{(u,v)} \Longleftrightarrow U_x^{(u,v)} = U_x U_{uv}$ is invertible $\Longleftrightarrow$ U_x is invertible (since U_{uv} is automatically invertible if $A^{(u,v)}$ is unital, by the previous Criterion) $\Longleftrightarrow$ x is unital in A, and then $x^{-1(u,v)} = U_x^{(u,v)-1} x = (U_x U_{uv})^{-1} x = U_{uv}^{-1} U_x^{-1} x = U_{uv}^{-1} x^{-1}$, all due to the Inverse Theorem. ∎

In general much of the structure of A is lost in passing to a homotope. For example, if u or v is zero, multiplication in $A^{(u,v)}$ is completely trivial. Put another way, homotopy is not symmetric: we cannot recover A from $A^{(u,v)}$. We now investigate a class of homotopes where we can recover the structure.

If u,v are invertible we call $A^{(u,v)}$ the <u>u,v-isotope</u>. We reserve the term isotope for the case when both u and v are invertible, not merely uv. By the Unit Criterion such an isotope is again a unital alternative algebra with unit $1^{(u,v)} = (uv)^{-1} = v^{-1}u^{-1}$ (recall 4.2 (x)). Moreover, isotopy is an equivalence relation: it is reflexive since $A^{(1,1)} = A$, transitive since $\{A^{(u,v)}\}^{(u',v')} = A^{(u'',v'')}$ where $u'' = U_u(vu')$, $v'' = U_v(v'u)$ are invertible if u,v,u',v' are, and it is symmetric since if we take $u' = v^{-1}u^{-2}$, $v' = v^{-2}u^{-1}$ we get $u'' = u(vu')u = uu^{-2}u = 1, v'' = v(v'u)v = 1$.

$$\{A^{(u,v)}\}^{(u',v')} = A \qquad (u' = v^{-1}u^{-2},\ v' = v^{-2}u^{-1}). \tag{5.7}$$

For these results it is essential that u,v be individually invertible, not just that uv be invertible.

The form of the operators in (5.2), where left multiplications in $A^{(u,v)}$ are built out of left multiplications in A, and right multiplications out of right multiplications, shows that a left, right, or two-sided-ideal in A remains left, right, or two-sided in $A^{(u,v)}$. Since isotopy is symmetric, it is clear that left, right, or two-sided ideals in an isotope coincide with those in the original algebra. In particular, A is simple iff all its isotopes are. Similarly A has a chain condition on ideals iff all isotopes do. Further, since isotopy preserves invertibility by the Invertibility Criterion 5.6, an algebra is a division algebra iff all its isotopes are.

Any isotope is isomorphic to a left (and a right) isotope:

5.8 Proposition. The left, right, and two-sided isotopes $A^{(vuv,L)}$, $A^{(R,uvu)}$, $A^{(u,v)}$ are isomorphic under maps

$$L_u: A^{(R,uvu)} \to A^{(u,v)} \qquad R_v: A^{(vuv,L)} \to A^{(u,v)} .$$

Proof. The maps are linear bijections since u,v are invertible. To see $F = L_u$ (for example) is an algebra homomorphism, note $F(x \cdot_{R,uvu} y) = F(L_x^{(R,uvu)} y) = L_u L_x L_{uvu}\, y$ (by (5.2)) $= L_u L_x L_u L_v L_u y$ (left Moufang) $= L_{(ux)u} L_v L_u y$ (left Moufang again) $= L_{ux}^{(u,v)} L_u y = ux \cdot_{u,v} uy = F(x) \cdot_{u,v} F(y)$, and similarly for R_v. ∎

Here again both u,v must be invertible, for if only the product uv is invertible then $A^{(u,v)}$ will have a unit but $A^{(R,uvu)}$ and $A^{(vuv,L)}$ will not (uvu or vuv is invertible iff u and v both are, according to Corollary 4.3).

An *isotopy* $\tilde{A} \xrightarrow{F} A$ is an isomorphism from $\tilde{A}$ to an isotope of A. Two algebras are *isotopic* if there is an isotopy from one to the other. This is a symmetric relation, for if $\tilde{A} \xrightarrow{F} A^{(u,v)}$ is an isomorphism then the isomorphism $A^{(u,v)} \xrightarrow{F^{-1}} \tilde{A}$ is also an isomorphism $A = \{A^{(u,v)}\}^{(u',v')} \xrightarrow{F^{-1}} \tilde{A}^{(F^{-1}u',F^{-1}v')}$ $(u' = v^{-1}u^{-2},\ v' = v^{-2}u^{-1}$ by (5.7)). It is also transitive, since if $\tilde{A} \xrightarrow{F} A^{(u,v)}$ and $A \xrightarrow{G} A'^{(u',v')}$ are isomorphism so is the composite $\tilde{A} \xrightarrow{F} A^{(u,v)} \xrightarrow{G} \{A'^{(u',v')}\}^{(Gu,Gv)} = A'^{(u'',v'')}$. Thus isotopy is an equivalence relation between unital alternative algebras which is weaker than isomorphism.

There is another more general notion of isotopy which makes sense in any linear algebra. If we have a linear algebra structure A on a space X and three linear bijections ρ, σ, τ from X to another space $\tilde{X}$, then we can transfer the algebraic structure from X to $\tilde{X}$ by defining

$$(5.9) \qquad \sigma(x) \tilde{\cdot} \tau(y) = \rho(x \cdot y) \qquad (x,y \in X),$$

that is, $\tilde{x} \tilde{\cdot} \tilde{y} = \rho(\sigma^{-1}(\tilde{x})\cdot\tau^{-1}(\tilde{y}))$ for all $\tilde{x},\tilde{y} \in \tilde{X}$. The linearity of ρ,σ,τ guarantee that the resulting product $\tilde{\cdot}$ is bilinear. Since the three maps ρ,σ,τ are completely arbitrary, we have a method for creating many new linear algebras $\tilde{A}$ out of a given algebra A.

An *isotopy* from a linear algebra A to an algebra $\tilde{A}$ is a triple (ρ,σ,τ) of bijections satisfying the relation (5.9). Notice that if $\rho = \sigma = \tau$ all coincide, their common value F is just an isomorphism: $F(x) \tilde{\cdot} F(y) = F(x\cdot y)$. We say two algebras are *isotopic*, or that one is an *isotope* of the other, if there is an isotopy between them. (The above method therefore is a method for creating isotopes). If (ρ,σ,τ)

is an isotopy $A \to \tilde{A}$ then $(\rho^{-1},\sigma^{-1},\tau^{-1})$ is an isotopy in the reverse direction, $\rho^{-1}(\tilde{x} \,\tilde{\cdot}\, \tilde{y}) = \rho^{-1}(\sigma(\sigma^{-1}\tilde{x}) \,\tilde{\cdot}\, \tau(\tau^{-1}\tilde{y})) = \rho^{-1}(\rho(\sigma^{-1}\tilde{x} \cdot \tau^{-1}\tilde{y})) = \sigma^{-1}(\tilde{x}) \cdot \tau^{-1}(\tilde{y})$. If $A \xrightarrow{(\rho,\sigma,\tau)} \tilde{A}$ and $\tilde{A} \xrightarrow{(\tilde{\rho},\tilde{\sigma},\tilde{\tau})} \tilde{\tilde{A}}$ are isotopies, so is their composite $A \xrightarrow{(\tilde{\rho}\rho,\tilde{\sigma}\sigma,\tilde{\tau}\tau)} \tilde{\tilde{A}}$ since $\tilde{\rho}\rho(x \cdot y) = \tilde{\rho}(\sigma(x) \,\tilde{\cdot}\, \tau(y)) = \tilde{\sigma}(\sigma(x)) \,\tilde{\tilde{\cdot}}\, \tilde{\tau}(\tau(y)) = \tilde{\sigma}\sigma(x) \,\tilde{\tilde{\cdot}}\, \tilde{\tau}\tau(y)$. Consequently isotopy is an equivalence relation, weaker than isomorphism (but only slightly).

Luckily, just as the general notion of division algebra coincided for alternative algebras with our definition, so too the general notion of isotopy coincides for unital alternative algebras with our previous definition of isotope and isotopy.

Let us first see how our notion of isotopy fits into the general setup. Suppose $A \overset{\rho}{\to} \tilde{A}$ is an isotopy in our sense, i.e. an isomorphism $A \overset{\rho}{\to} \tilde{A}^{(\tilde{u},\tilde{v})}$. Set $\sigma(x) = \rho(x) \,\tilde{\cdot}\, \tilde{u}$, $\tau(x) = \tilde{v} \,\tilde{\cdot}\, \rho(x)$. Then ρ,σ,τ are all linear bijections from A to $\tilde{A}$ (since $\tilde{u}$, $\tilde{v}$ are invertible in $\tilde{A}$), and $\sigma(x) \,\tilde{\cdot}\, \tau(y) = \{\rho(x) \,\tilde{\cdot}\, \tilde{u}\} \,\tilde{\cdot}\, \{\tilde{v} \cdot \rho(y)\} = \rho(x) \cdot_{\tilde{u},\tilde{v}} \rho(y) = \rho(x \cdot y)$ by virtue of ρ being an isomorphism, so the triple (ρ,σ,τ) furnishes an isotopy in the general sense from A to $\tilde{A}$. In particular, $A^{(u,v)}$ is always an isotope of A in the general sense; such an isotope will be called the <u>principal isotope</u> determined by u and v. We will show that all isotopes in unital alternative algebras are principal, so the general notion reduces to our (principal) notion.

Isotopes of alternative (or even associative) algebras in this general sense need not be alternative. However, if both algebras are unital then alternativity is preserved, and a general isotopy (ρ,σ,τ) necessarily comes from one of our principal isotopies in the manner indicated above. This is the content of

5.10 (Schafer's Isotopy Theorem) Let $\tilde{A}$ be unital and A an alternative algebra. If (ρ,σ,τ) is an isotopy from $\tilde{A}$ to A then both $\tilde{A}$ and A are unital and alternative, ρ is an isomorphism of $\tilde{A}$ with the principal isotope $A^{(u,v)}$ for $u = \tau(\tilde{1})^{-1}$, $v = \sigma(\tilde{1})^{-1}$, and σ, τ have the form $\sigma(x) = \rho(x)\cdot u$, $\tau(y) = v\cdot\rho(y)$.

Proof. From $\rho(x \tilde{\cdot} y) = \sigma(x)\cdot\tau(y)$ we see $\rho(x) = \rho(x \tilde{\cdot} \tilde{1}) = \sigma(x)\cdot\tau(\tilde{1})$, $\rho(y) = \rho(\tilde{1} \tilde{\cdot} y) = \sigma(\tilde{1})\cdot\tau(y)$ since $\tilde{A}$ has unit $\tilde{1}$, and $\rho = R_{\tau(\tilde{1})}\circ\sigma = L_{\sigma(\tilde{1})}\circ\tau$. But then $R_{\tau(\tilde{1})}$, $L_{\sigma(\tilde{1})}$ are bijective since ρ,σ,τ are, and since A is assumed alternative we can apply the L,R-Test 4.5 to conclude A is unital and $\tau(\tilde{1}) = u^{-1}$, $\sigma(\tilde{1}) = v^{-1}$ are invertible. Then $\rho = R_{u^{-1}}\circ\sigma = L_{v^{-1}}\circ\tau$ implies $\sigma = R_u\circ\rho$, $\tau = L_v\circ\rho$, and $\rho(x \tilde{\cdot} y) = \sigma(x)\cdot\tau(y) = \{\rho(x)\cdot u\}\cdot\{v\cdot\rho(y)\} = \rho(x)\cdot_{u,v}\rho(y)$. This means ρ is an isomorphism of $\tilde{A}$ with $A^{(u,v)}$. In particular, $\tilde{A}$ itself must be alternative. ∎

5.11 Example. An associative algebra, even a unital one, can have isotopes which are not alternative (naturally these isotopes will not be unital).

For example, let A be any unital associative algebra which posses an invertible linear transformation T not of the form L_x for $x \in A$ (eg. if $T = R_y$ for some invertible y not in the center). Let $\tilde{A}$ be the algebra with multiplication $x \tilde{\cdot} y = x(T^{-1}y)$. Then $(\rho,\sigma,\tau) = (I,I,T^{-1})$ is an isotopy $\tilde{A} \to A$. $\tilde{A}$ doesn't even have a left unit, since if $\tilde{L}_e = L_e T^{-1} = I$ then $T = L_e$, contrary to hypothesis. Also, it is not alternative: we have $[1,1,Tx] = (1 \tilde{\cdot} 1) \tilde{\cdot} Tx - 1 \tilde{\cdot} (1 \tilde{\cdot} Tx) = (1\cdot T^{-1}1) \tilde{\cdot} Tx - 1\cdot T^{-1}(1\cdot x) = (T^{-1}1)x - T^{-1}x = (L_{T^{-1}(1)} - T^{-1})x$, and by the hypothesis $T \neq L_z$ this must be nonzero for some x. ∎

5.12 Example. Even if A, $\tilde{A}$ are both associative, an isotopy need not be an isomorphism (naturally neither algebra will be unital).

Let A be the associative algebra on two generators z, w with $z^2 = w^2 = zw + wz = 0$. Then $a^2 = 0$ for all a $\quad A = \Phi z \oplus \Phi w \oplus \Phi zw$. If T is defined on a bases by $T(z) = w$, $T(w) = z$, $T(zw) = zw$ then the isotope $\tilde{A}$ with product $x \tilde{\cdot} y = x(Ty)$ has $z^{\tilde{2}} = zT(z) = zw \neq 0$, so $\tilde{A}$ cannot be isomorphic to A, yet $\tilde{A}^3 = 0$ since $\tilde{A}^3 \quad A^3 = 0$ and $\tilde{A}$ is trivially associative. ∎

Isotopy too has a geometric meaning. The different coordinate systems for a projective plane give rise to isotopic coordinate rings. Therefore it is important to know if all isotopes are actually isomorphic. (So the coordinate ring is an invariant of the plane.) We know this is the case with associative algebras, and we will see this also for the (non-associative) Cayley algebras in the next section. Since these exhaust the alternative division algebras according to the structure theory, isotopy coincides with isomorphism for alternative division rings. No intrinsic proof of this is known. The best we can do is

5.13 Proposition. If uvu has a cube root z, $z^3 = uvu$, then the isotope $A^{(u,v)}$ is isomorphic to A.

Since $A^{(u,v)}$ is isomorphic to the right isotope $A^{(R,uvu)}$ by Proposition 5.8, it suffices to prove

5.14 Lemma. If w has a cube root z, $z^3 = w$, then $L_z^{-2} R_z^{-1}$ is an isomorphism of A with the isotope $A^{(R,w)}$.

Proof. $F = L_z^{-2} R_z^{-1}$ is a linear bijection, and $F(x) \cdot_{R,w} F(y) = \{z^{-2} x z^{-1}\} \cdot w \{z^{-2} y z^{-1}\} = \{z^{-2} x z^{-1}\} \cdot z^3 \{z^{-2} y z^{-1}\} = \{z^{-1}(z^{-1}x)z^{-1}\} \cdot \{zyz^{-1}\}$ (Moufang) $= z^{-1}\{z^{-1}xz^{-1} \cdot zy\}z^{-1}$ (middle Moufang) $= z^{-1}\{z^{-1}(xy)\}z^{-1}$ (left Moufang) $= z^{-2}(xy)z^{-1} = F(xy)$. ∎

We will see an example in Section III.4 of unital alternative algebras which are isotopic but not isomorphic. Of course, they are not division algebras.

Exercises

5.1 Define a new product on A by $x \cdot y = (xu)y + x(vy)$. Is the resulting algebra alternative? Flexible? Does $(x \cdot x) \cdot x = x \cdot (x \cdot x)$?

5.2 Try $x \cdot y = x(yu)$. Is this algebra alternative? If A is associative, what can you say?

5.3 If A is alternative with involution $*$ (so $x^{**} = x$, $(xy)^* = y^*x^*$) try defining $x \cdot y = xy^*$. What if A is associative? Try $x \cdot y = y^*x^*$.

5.4 Define a homotopy from A to $\tilde{A}$ to be a homomorphism $A \xrightarrow{F} \tilde{A}^{(\tilde{u},\tilde{v})}$ from A to a homotope of A. Show any homomorphism $B \xrightarrow{G} \tilde{B}$ is also a homomorphism $B^{(u,v)} \xrightarrow{G} \tilde{B}^{(Gu,Gv)}$. If $A \xrightarrow{F} \tilde{A} \xrightarrow{\tilde{F}} A'$ are homotopies, show $\tilde{F} \circ F$ is a homotopy.

5.5 Show $\{A^{(u,L)}\}^{(v,L)} = A^{(uvu,L)}$, $\{A^{(R,u)}\}^{(R,v)} = A^{(R,uvu)}$, $\{A^{(u,v)}\}^{(w,L)} = A^{(u[vw]u,L)}$, $\{A^{(u,v)}\}^{(R,w)} = A^{(u,v[wu]v)}$. Be careful with the last two.

5.6 Show any isotope may be obtained as the right homotope of a left homotope (and vice versa).

5.7 Expand upon the words "and similarly for R_v" at the end of Proposition 5.5.

5.8 If x is trivial in A, is it still trivial in $A^{(u,v)}$? If it is regular in A? (See Ex. 3.8, 3.9.)

5.9 Find expressions for the squares and cubes $x^{2(u,v)}$, $x^{3(u,v)}$ of x in $A^{(u,v)}$.

5.10 We know in the alternative case that an algebra A and its principal isotopes $A^{(u,v)}$ have the same ideals. This is no longer true for non-principal (i.e. non-unital) isotopes $\tilde{A}$. Let A be any n-dimensional unital algebra (associative or not, simple or not) over a field Φ, and $p(\lambda)$ an irreducible polynomial over Φ of degree n. We will construct a simple algebra $\tilde{A}$ isotopic to A. Take $\tilde{A}$ to be the field $\Omega = \Phi[\lambda]/(p(\lambda))$ as n-dimensional vector space over Φ, and define T on $\tilde{A}$ by $T(x) = \lambda x$. Show T is a bijective Φ-linear transformation which acts irreducibly on $\tilde{A}$. Let $\tau: A \to \tilde{A}$ be a bijective linear map (are there any?) and set $\sigma = \tau$, $\rho = T \circ \tau$. Use ρ, σ, τ to turn the space $\tilde{A}$ into an algebra isotopic to A. Find a formula for $\tilde{L}_{\tau(x)}$ and $\tilde{R}_{\tau(y)}$. Show A has no proper subspaces invariant under $\tilde{L}_{\tau(1)}$ or under $\tilde{R}_{\tau(1)}$. Conclude $\tilde{A}$ has no proper left or right ideals, hence is simple, no matter what A is like (as long as it is unital). In particular, we could have $A = \Phi 1 \oplus B$ for B an n-1 dimensional trivial algebra, so $\tilde{A}$ certainly doesn't reflect the ideal structure of A.

#3. Problem Set on Isotopy in Groupoids

A <u>groupoid</u> is a set G with a binary product (not necessarily associative - it is a "nonassociative semigroup"). A <u>quasigroup</u> is a groupoid in which the equations $ax = b$, $ya = b$ have unique solutions x, y for any given $a, b \in G$. A <u>loop</u> is a quasigroup with unit (a "nonassociative group").

1. Show a groupoid G is a quasigroup iff left and right multiplications L_x, R_x are bijective maps for all $x \in G$.

An <u>isotopy</u> $G \xrightarrow{(\rho,\sigma,\tau)} \tilde{G}$ between groupoids is a triple of bijective maps ρ, σ, τ, such that $\rho(x \cdot y) = \sigma(x) \tilde{\cdot} \tau(y)$ for all $x, y \in G$. If $\rho = \sigma = \tau$ this is just ordinary isomorphism of groupoids.

2. Show that isotopy is an equivalence relation among groupoids.
3. Show that if a groupoid G is isotopic to a quasigroup, it must itself be a quasigroup.
4. Show that if ρ, σ, τ are 3 bijections $G \to S$ onto a set S, then the groupoid structure on G induces in a natural way a groupoid structure on S, isotopic to that on G. If G is a quasigroup so is S.

This last gives a method for constructing new groupoids. For example, if u,v are elements of a quasigroup G such that R_u, L_v are bijective (if G is a quasigroup, any two elements will do) we define the <u>principal u,v isotope</u> $G^{[u,v]}$ by taking $\rho = 1$, $\sigma = R_u^{-1}$, $\tau = L_v^{-1}$: $x \cdot_{u,v} y = (R_u^{-1} x)(L_v^{-1} y)$. (Beware the inverses! We use square brackets to avoid confusion with alternative isotopes $x \cdot_{u,v} y = (R_u x)(L_v y)$. In the very special case where $R_u^{-1} = R_{u^{-1}}$, $L_v^{-1} = L_{v^{-1}}$ we would have $G^{[u,v]} = G^{(u^{-1},v^{-1})}$.)

5. If G is associative, show $G^{[u,v]} = G^{(u^{-1},v^{-1})}$.

6. Is principal isotopy transitive? Symmetric? Reflexive?

7. Show $G^{[u,v]}$ always has unit, whether G does or not. Show that if G is a quasigroup, all its principal isotopes $G^{[u,v]}$ are loops. Conclude that every quasigroup is isotopic to a loop. Thus we can obtain all quasigroups by starting with all loops and taking isotopes.

8. If $G \xrightarrow{(\rho,\sigma,\tau)} \tilde{G}$ is an isotopy of groupoids such that $\tilde{G}$ has unit $\tilde{e}$, show ρ is an isomorphism of the principal isotope $G^{[u,v]}$ with G ($\tau(u) = \tilde{e}$, $\sigma(v) = \tilde{e}$). Thus for unital groupoids one needs only consider principal isotopes.

9. If G is isotopic to $\tilde{G}$, where G is associative and $\tilde{G}$ unital, show G and $\tilde{G}$ are isomorphic (hence both are unital and both are associative).

10. Try to define $G^{(u,v)}$ by $x \cdot_{u,v} y = (xu)(vy)$ when R_u, L_v are bijective, so $(1,R_u,L_v)$ is an isotopy $G^{(u,v)} \to G$. Show $G^{(u,v)}$ has unit e iff $L_{eu}L_v = I = R_{ve}R_u$, i.e. $L_v^{-1} = L_{eu}$, $R_u^{-1} = R_{ve}$ (in a quasigroup). But if $L_x^{-1} = L_y$ (in a loop) show $y = x^{-1}$, so $L_{x^{-1}} = L_x^{-1}$, $R_{x^{-1}} = R_x^{-1}$. We have seen in Problem Set #2 this forces alternativity in an algebra, and in a loop it forces Moufangitivity. Thus only in the alternative case does R_u, L_v lead to a suitable theory.