§3 Basic identities

So far most of our results have depended very little on alternativity, and were valid for arbitrary linear algebras. Now we start investigating properties peculiar to the alternative case.

In addition to the defining identities, we will need to know the important Moufang identities for alternative algebras

(3.1) $(xyx)z = x(y(xz))$ (left Moufang identity)

(3.2) $z(xyx) = ((zx)y)x$ (Right Moufang identity)

(3.3) $x(yz)x = (xy)(zx)$ (Middle Moufang identity)

These play at least as important a role in the theory of alternative algebras as do the alternative laws themselves. Observe that they are generalizations of the alternative laws in the sense that if we set $y = 1$ in (3.1), (3.2), (3.3) we get (1.1), (1.2), (1.3).

In operator notation these laws are

(3.1op) $L_{xyx} = L_x L_y L_x$

(3.2op) $R_{xyx} = R_x R_y R_x$

(3.3op) $U_x L_y = L_{xy} R_x, \quad U_x R_z = R_{zx} L_x$.

In terms of associators, the Moufang identities correspond to the bumping formulas

(3.1a) $[x,y,zx] = x[y,z,x]$ (Left bumping formula)

(3.2a) $[xy,z,x] = [x,y,z]x$ (Right bumping formula)

(3.3a) $[y,x^2,z] = x[y,x,z] + [y,x,z]x$ (Middle bumping formula).

These give a method for bumping a factor out of an associator (or absorbing one into an associator), and they are easy to remember: in the first two, line the variables up with an x on each end, then bump the unattached x outside; in the third, bump an x out on either side.

Rather than bother lining the variables up, we can rewrite left and right bumping as

(3.1a') $[y,zx,x] = x[y,z,x]$

(3.2a') $[x,xy,z] = [x,y,z]x$

More generally: in any associator of x,y,z with a factor x on y or z, bump the factor x outside (leaving the other x and y,z alone) to the <u>left</u> if x is a <u>right</u> factor or to the <u>right</u> if x is a <u>left</u> factor. Examples: $[x,zx,y] = x[x,z,y]$, $[x,z,xy] = [x,z,y]x$, etc.

It's time to stop explaining these identities and start proving them. We first show all the Moufang and bumping formulas are equivalent in an alternative algebra: (3.3)$\Longleftrightarrow$(3.1a)$\Longleftrightarrow$(3.1), hence (3.3) $\Longleftrightarrow$ (3.2a)$\Longleftrightarrow$(3.2) by symmetry. Indeed, $(xy)(zx) - x\{(yz)x\} = [x,y,zx] + x\{y(zx)\} - x\{(yz)x\} = [x,y,zx] - x[y,z,x] = -[x,zx,y] - x[z,x,y]$ (by alternativity) $= -\{x(zx)\}y + x\{(zx)y\} - x\{(zx)y\} + x\{z(xy)\} = -(xzx)y + x\{z(xy)\}$ so that $(xy)(zx) - x(yz)x$ vanishes identically iff $[x,y,zx] - x[y,z,x]$ vanishes identically iff $(xzx)y - x\{z(xy)\}$ vanishes identically.

To see that (3.1), say, is valid, observe $[x,xy,y] = 0$ since $\{x(xy)\}y - x\{(xy)y\} = \{x^2y\}y - x\{xy^2\} = x^2y^2 - x^2y^2 = 0$ by alternativity;

linearizing with respect to y yields $0 = [x,xy,z] + [x,xz,y] = -[xy,x,z] - [x,y,xz]$ (alternativity) $= -\{(xy)x\}z + (xy)(xz) - (xy)(xz) + x\{y(xz)\} = x\{y(xz)\} - \{(xy)x\}z$.

Finally, for (3.3a) we linearize (3.1) in the form $z[y,x,z] = [z,y,xz]$ with respect to z to get $z[y,x,x] + x[y,x,z] = [z,y,x^2] + [x,y,xz]$, which by alternativity is $x[y,x,z] = [y,x^2,z] + [x,y,z]x = [y,x^2,z] - [y,x,z]x$. ∎

In addition to these versions of the Moufang identities we will need the following <u>fundamental formulas</u>

(3.4) $U_{xy} = L_x U_y R_x$ (Left fundamental formula)

(3.5) $U_{xy} = R_y U_x L_y$ (Right fundamental formula)

(3.6) $U_{xyx} = U_x U_y U_x$ (Middle fundamental formula)

(The element versions of these identities are not memorable!) To establish the left fundamental formula,

$$
\begin{aligned}
(U_{xy} - L_x U_y R_x)z &= (xy)\{z(xy)\} - x(y\{(zx)y\}) \\
&= [x,y,z(xy)] + x(y\{z(xy)\}) - x(y\{(zx)y\}) \\
&= [x,y,z(xy)] - x(y[z,x,y]) \\
&= [x,y,z(xy)] - x[y,z,xy] \quad \text{(left bumping)} \\
&= -[xy,y,zx] + (xy)[y,z,x] \quad \text{(linearized left bumping)} \\
&= -(xy^2)(zx) + (xy)\{y(zx)\} + (xy)\{(yz)x\} - (xy)\{y(zx)\}
\end{aligned}
$$

$$= - x(y^2z)x + x\{y(yz)\}x \quad \text{(middle Moufang)}$$

$$= 0 \quad \text{(left alternativity).}$$

The right fundamental formula follows by symmetry, and together these imply the middle fundamental formula:

$$U_{x(yx)} = L_x U_{yx} R_x = L_x \{R_x U_y L_x\} R_x$$

$$= \{L_x R_x\} U_y \{L_x R_x\} = U_x U_y U_x . \quad \square$$

As an application of Moufangitivity, let us define the powers of an element x recursively by

$$x^o = 1, \; x^1 = x, \; x^{n+2} = U_x x^n .$$

(Note $x^2 = U_x 1 = x^2$!!) Then $L_{x^o} = L_1 = I$, $L_{x^1} = L_x$, and $L_{x^{n+2}} = L_{xx^n x} = L_x L_{x^n} L_x$ by left Moufang, so by induction $L_{x^n} = L_x^n$ for all n. We have the same result for the R's by right Moufang and the U's by middle fundamental. Therefore we can strengthen alternativity $L_{x^2} = L_x^2$, $R_{x^2} = R_x^2$ to

(3.7) $$L_{x^n} = L_x^n \; , \; R_{x^n} = R_x^n \; , \; U_{x^n} = U_x^n \; .$$

This immediately implies power-associativity

(3.8) $$x^n x^m = x^{n+m}$$

since $x^n x^m = L_{x^n} L_{x^m} 1 = L_x^n L_x^m 1 = L_x^{n+m} 1 = L_{x^{n+m}} 1 = x^{n+m}$. Alternately we could prove (3.8) by induction on n+m using middle Moufang: $x^n x^m = (xx^{n-1})(x^{m-1}x)$ (induction on (3.8)) $= U_x(x^{n-1}x^{m-1}) = U_x(x^{n+m-2})$ (by induction) $= x^{n+m}$ (by definition).

Once we have (3.8) we see that any reasonable definition of powers leads to the same result; for example, we could have used the left powers defined by $x^1 = x$, $x^{n+1} = xx^n$. We also see that the subalgebra $\Phi[x]$ generated by a single element x is a commutative associative algebra isomorphic to the ordinary polynomial ring in one variable.

Once we have a reasonable notion of powers we can talk about idempotents and nilpotents. An element e is idempotent if $e^2 = e$; trivially 0 and 1 are idempotents, anything else is a proper idempotent. Idempotents are discussed more fully in Chapter VII. An element z is nilpotent if $z^n = 0$ for some n, the smallest such n being the index of nilpotency. An algebra is nil if all its elements are nilpotent. These are discussed in Chapter VI on radicals.

Formulas (3.1)-(3.6) are essentially all that we shall need to develop the theory of alternative algebras. Since they are so useful, and so easily formulated, it is desirable to commit them to memory. We recapitulate:

Jordan Structure

(Left Moufang)	$L_{xyx} = L_x L_y L_x$	$L_{x^2} = L_x^2$	(left alternative)
(Right Moufang)	$R_{xyx} = R_x R_y R_x$	$R_{x^2} = R_x^2$	(Right alternative)
(Middle fundamental)	$U_{xyx} = U_x U_y U_x$	$U_{x^2} = U_x^2$	(Middle fundamental)

U-operators

(Left fundamental) $U_{xy} = L_x U_y R_x = R_y U_x L_y$ (Right fundamental)

(Middle Moufang) $U_x(yz) = (xy)(zx)$ $U_x = L_x R_x = R_x L_x$ (Middle alternative)

<u>Associators</u>

$[x,y,z]$ is an alternating function of its arguments

(Left bumping) $[x,y,zx] = x[y,z,x]$, $[xy,z,x] = [x,y,z]x$ (Right bumping)

$$[y,x^2,z] = x[y,x,z] + [y,x,z]x \qquad \text{(Middle bumping)}$$

To be memorable, an identity must posses a pattern. Once you see a pattern you can memorize the pattern. One thing which makes for a simple pattern is symmetry. The identities on Jordan structure are highly symmetric, and consequently easy to remember: L of a Jordan product (be it x^2 or xyx) is that Jordan product of the L's, and the same for R and U. To see a pattern in the U operator identities, read them as $U_{L(x)y} = L_x U_y R_x$, i.e. L_x (sometimes $L(x)$ for typographical reasons) can be moved outside of a U operator if we put its opposite R_x on the right, similarly in $U_{R(y)x} = R_y U_x L_y$ we can more an R_y outside on the left, at the same time putting an L_y on the right.

Exercises

3.1 Linearize the left, right, and middle Moufang formulas.

3.2 In an alternative algebra show $x\circ[x,y,z] = [x,x\circ y,z]$ and $[x,[x,y,z]] = [x,[x,y],z]$.

3.3 If we let $U_{x,z} \equiv U_{x+z} - U_x - U_z$ denote the linearization of the U operator, linearize (3.4) in x. Set $z = 1$ in the linearized version ; show the resulting identity (acting on an element w) could be derived directly from the Moufang formulas.

3.4 Linearize the Middle Moufang formula from x to x, w, then set $w = 1$. Write this as an operator identity acting on x; on y; on z.

3.5 Show $U_x 1 = x^2$, $U_x x = x^3$ $(= x^2 x = xx^2)$. Find ways of rewriting $(xy)^2$; $(xyx)^2$; $(xy)^3$.

3.6 Prove $[x^2,x,y] = 0$ in an alternative algebra. Prove $[x^n,x^m,y] = 0$.

3.7 Prove $U_x U_y - U_y U_x = U_{yx,x} L_y - L_y U_{xy,x} = U_{x\circ y,x} L_y - \{U_{xy,x} L_y + L_y U_{xy,x}\}$ in an alternative algebra.

3.8 An element z is <u>trivial</u> if $U_z = 0$. If z is trivial show z is nilpotent; show any xz or zx is trivial too. What about $z+w$?

3.9 An element x is <u>regular</u> if $x \in \operatorname{Im} U_x$, i.e. $x = U_x y$ for some y. Show y can be chosen so that also $y = U_y x$. Show any idempotent is regular.

3.10 Prove power associativity (3.8) using left powers $x^{n+1} = xx^n$.

3.11 If e is idempotent show L_e, R_e, U_e are too; if z is nilpotent show L_z, R_z, U_z are.