Quadratic Jordan superpairs covered by grids

Dedicated to Ottmar Loos
on the occasion of his 60th birthday

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Abstract. In this paper we are proposing a theory of Jordan superpairs defined over (super)commutative superrings. Our framework has two novelties: we allow scalars of even and odd parity and we do not assume that $\frac{1}{2}$ lies in our base superring. To demonstrate that it is possible to work in this generality we classify Jordan superpairs covered by a grid.

There has recently been a lot of interest in linear Jordan superstructures. One of the major advances in this area is the classification of simple finite-dimensional Jordan superalgebras over algebraically closed fields of characteristic $\neq 2$, due to Racine-Zelmanov [**39**],[**38**] and Martínez-Zelmanov [**25**], extending Kac's classification [**15**] ([**10**], [**16**]) of the characteristic 0 case. Another important achievement is the classification of infinite-dimensional graded-simple Jordan superalgebras whose graded components are uniformly bounded, due to Kac-Martínez-Zelmanov [**14**]. Most the recent research has been devoted to Jordan superalgebras, but one now has a classification of simple finite-dimensional Jordan superpairs over algebraically closed fields of characteristic 0, due to Krutelevich [**18**] and based on Kac's determination of $\mathbb{Z}$-gradings of simple finite-dimensional Lie superalgebras [**15**].

It is remarkable that most (probably all) examples of linear Jordan superalgebras and superpairs in the papers mentioned above can in fact be defined over arbitrary superrings. For some of them this was verified in the recent preprint [**17**] by King, for others like the Kantor double with a bracket of vector field type or the Cheng-Kac Jordan superalgebra one can use the speciality results of McCrimmon [**27**] and Martínez-Shestakov-Zelmanov [**24**] to give a model over superrings. To the best of my knowledge, King's preprint [**17**], which I received after the research for this paper had been finished, is the only publication devoted to quadratic Jordan superstructures. In this paper King introduces a notion of quadratic Jordan superalgebras. Apart from the fact that King works over commutative rings while we work over commutative superrings, there is also a difference in "characteristic 2": King's Jordan triple product is only skew-symmetric in the outer two odd variables,

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hence $2\{x_{\bar 1} y x_{\bar 1}\} = 0$ for an odd $x_{\bar 1}$, while we require that it is even alternating and hence $\{x_{\bar 1} y x_{\bar 1}\} = 0$ always holds in our setting. Our reasons for imposing the stronger condition is that it holds for all reasonable Jordan superstructures we know of, for example for special Jordan superstructures (see 2.14) or for King's quadratic version of Kac's 10-dimensional Jordan superalgebra K_{10}. (There is a small exception for Jordan superalgebras associated to quadratic forms since King requires the form on the odd part to be only skew symmetric and not necessarily alternating, as we do.)

Why Jordan superstructures over superrings? It is of course true that any Jordan superpair over a superring is also a Jordan superpair over a ring, for example over the even part of the base superring. Nevertheless, there are good reasons for working over superrings. This setting naturally occurs in the class of Jordan superpairs classified in this paper, Jordan superpairs covered by a grid (§4). For example, a Jordan superpair V covered by an even quadratic form grid is in a natural way a quadratic form superpair over a superring, even if one originally considered V only over a ring (see 4.14).

A description of the paper's contents follows. Due to a lack of an appropriate reference, the following section §1 provides the necessary background from the theory of supermodules over superrings in as far as this is needed later on. This section also contains the fundamental and new definition of a quadratic map between supermodules over superrings. In the next section §2 we define (quadratic) Jordan superpairs, Jordan supertriples and unital Jordan superalgebras over superrings. We develop some basic theory and give examples. This section could be considered as a super version of [**20**, §1]. In the following section §3 we introduce grids in Jordan superpairs and refined root gradings. The final section §4 gives the classification of Jordan superpairs covered by a grid, the super version of results from [**35**], and – more generally – the description of refined root gradings. This latter description is new even in the case of Jordan pairs. Our interest in Jordan superpairs with a refined root grading comes from their connection to Lie (super)algebras which have a refined root grading, see [**41**] for the case of Lie algebras graded by a simply-laced root system.

There are three sequels to this paper, all jointly with E. García. Semiprimeness, primeness and simplicity of Jordan superpairs covered by grids are characterized in [**8**]. The corresponding Tits-Kantor-Koecher superalgebras are described in [**9**], while [**7**] studies the Gelfand-Kirillov dimension of Jordan superpairs and their associated Lie superalgebras.

I would like to thank Ottmar Loos and Michel Racine for helpful hints on a preliminary version of this paper. In particular, it was Ottmar Loos who convinced me to include the requirement $\{x_{\bar 1} y x_{\bar 1}\} = 0$ in the definition of a Jordan superpair, motivated by his crucial example of a quadratic map in the supersetting (1.9).

1. Supermodules and their multilinear and quadratic maps.

In this section we introduce our terminology regarding supermodules and multilinear and quadratic maps. With the exception of quadratic maps, these concepts have already been introduced in the literature ([**6**, Ch. 1], [**19**, Ch. 1] and [**23**, Ch. 3]), but not in the form and generality suitable for this paper. One of the main differences is that our objects will be defined over a superring not necessarily containing $\frac{1}{2}$.

1.1. Base superrings. We write $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and use its standard field structure. We put $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. Most objects studied here will be $\mathbb{Z}_2$-graded in a natural sense. For example, a $\mathbb{Z}_2$-graded abelian group M is just a direct sum $M = M_{\bar{0}} \oplus M_{\bar{1}}$ of two subgroups M_α, $\alpha \in \mathbb{Z}_2$. In this case, elements in $M_{\bar{0}} \cup M_{\bar{1}}$ are called *homogenous*. For a homogenous $m \in M_\alpha$, $\alpha \in \mathbb{Z}_2$, its *degree* is denoted by $|m| = \alpha \in \mathbb{Z}_2$. We adopt the convention that whenever the degree function occurs in a formula, the corresponding elements are supposed to be homogeneous.

An arbitrary (not necessarily associative) ring S is called $\mathbb{Z}_2$*-graded* or a *superring* if $S = S_{\bar{0}} \oplus S_{\bar{1}}$ as abelian group and $S_\alpha S_\beta \subset S_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$. A superring is called *commutative* if $st = (-1)^{|s||t|} ts$ holds for $s, t \in S$. Some authors would call such a superring supercommutative, but we have tried to minimize the usage of the adjective "super". In a commutative superring S we always have $2s_{\bar{1}}^2 = 0$ for any $s_{\bar{1}} \in S_{\bar{1}}$, creating a sometimes exceptional situation if 2 is not invertible in S. (One could think of adding the condition $s_{\bar{1}}^2 = 0$ for $s_{\bar{1}} \in S_{\bar{1}}$ to the definition of a commutative superring. This, however, would impose restrictions elsewhere: several of the natural examples of Jordan superpairs, e.g. quadratic form superpairs, are defined over a commutative superring not necessarily satisfying $s_{\bar{1}}^2 = 0$.)

A superring S is called *unital* if there exists $1 \in S_{\bar{0}}$ such that $1s = s$ for all $s \in S$, and it is called *associative* if it is so as ungraded ring: $(ab)c = a(bc)$ for all $a, b, c \in S$. We will call S a *base superring* if S is a commutative associative unital superring. Analogously, a *base ring* is a commutative associative unital ring.

Unless specified otherwise, S will always denote a base superring and all structures considered here will be defined over S in a sense to be explained in the following.

1.2. Supermodules. An S*-supermodule* is a left module M over (the associative ring) S whose underlying abelian group is $\mathbb{Z}_2$-graded such that $S_\alpha M_\beta \subset M_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$. It will be convenient to consider S-supermodules also as S-bimodules by defining the right action as

$$ms = (-1)^{|s||m|} sm \tag{1}$$

for $s \in S$ and $m \in M$. Alternatively, one can define S-supermodules as S-bimodules satisfying (1), or as right S-modules and then define the left action by (1).

Let M be an S-supermodule. A *submodule* of M is a submodule N of the S-module M which respects the $\mathbb{Z}_2$-grading, i.e., $N = (N \cap M_{\bar{0}}) \oplus (N \cap M_{\bar{1}})$. Then N is an S-supermodule with the induced actions. The *quotient* of M by a submodule N is again an S-supermodule with respect to the canonical S-module structure and $\mathbb{Z}_2$-grading: $(M/N)_\alpha = M_\alpha / N_\alpha$ for $\alpha \in \mathbb{Z}_2$. The *direct sum* of a family $(M^i)_{i \in I}$ of S-supermodules is an S-supermodule, denoted $\oplus_{i \in I} M^i$, with homogenous parts $(\oplus_{i \in I} M^i)_\alpha = \oplus_{i \in I} M^i_\alpha$ for $\alpha \in \mathbb{Z}_2$. In case all $M^i = M$ this supermodule is denoted $M^{(I)}$.

A new S-supermodule $\prod M$ is obtained from M by interchanging the parity of M: $\prod M = M$ as abelian groups, but $(\prod M)_\alpha = M_{\alpha+\bar{1}}$ for $\alpha \in \mathbb{Z}_2$ and $(\prod m)s = \prod(ms)$ where $s \in S$ and $\prod m$ is the element of $\prod M$ corresponding to $m \in M$. It follows that $s(\prod m) = (-1)^{|s|} \prod(sm)$ indicating that $\prod$ can be viewed as an entity of degree $\bar{1}$, called the *parity change functor*. A *free* S-supermodule is an S-supermodule isomorphic (in the sense of 1.3) to

$$S^{(I_{\bar{0}}|I_{\bar{1}})} := S^{(I_{\bar{0}})} \oplus (\textstyle\prod S)^{(I_{\bar{1}})}$$

for suitable sets I_α. Thus, M is free if and only if M is free as a module over the ring S and has a homogenous basis.

1.3. Multilinear maps. Let $M^1, \ldots, M^n$ and N be S-supermodules, and let $\alpha \in \mathbb{Z}_2$. An S*-multilinear map of degree* α from $M^1, \ldots, M^n$ to N is a map $f\colon M^1 \times \cdots \times M^n \to N$ satisfying

(i) $f(M^1_{\beta_1}, \ldots, M^n_{\beta_n}) \subset N_{\alpha+\beta_1+\cdots+\beta_n}$ for all $\beta_i \in \mathbb{Z}_2$,
(ii) f is additive in each variable, and
(iii) for $s \in S$, $m_j \in M^j$ and $1 < i \le n$ we have

$$f(m_1, \ldots, m_{i-1}s, m_i, \ldots, m_n) = f(m_1, \ldots, m_{i-1}, sm_i, \ldots, m_n) \quad \text{and}$$
$$f(m_1, \ldots, m_n s) = f(m_1, \ldots, m_n)s.$$

For readers preferring left modules we note that the conditions (iii) are equivalent to

$$\begin{aligned} &f(m_1, \ldots, m_{i-1}, sm_i, \ldots, m_n) \\ &= (-1)^{|s|(|f|+|m_1|+\cdots+|m_{i-1}|)} sf(m_1, \ldots, m_{i-1}, m_i, \ldots, m_n). \end{aligned}$$

We denote by $\mathcal{L}_S(M^1, \ldots, M^n; N)_\alpha$ the abelian group of S-multilinear maps of degree α and put

$$\mathcal{L}_S(M^1, \ldots, M^n; N) := \mathcal{L}_S(M^1, \ldots, M^n; N)_{\bar{0}} \oplus \mathcal{L}_S(M^1, \ldots, M^n; N)_{\bar{1}}.$$

We endow $\mathcal{L}_S(M^1, \ldots, M^n; N)$ with an S-supermodule structure by $(s.f)(m_1, \ldots, m_n) = sf(m_1, \ldots, m_n)$.

As usual, the elements of $\mathcal{L}_S(M^1, M^2; S)$ are called *bilinear forms*. We will use the abbreviation $\mathrm{Hom}_S(M, N) = \mathcal{L}_S(M; N)$, and call its elements *homomorphisms* or S*-linear maps*. Specializing the definition above, an additive map $f\colon M \to N$ is a homomorphism of supermodules if $f(ms) = f(m)s$ for $m \in M$ and $s \in S$ or, equivalently, $sf(m) = (-1)^{|s||f|} f(sm)$. The concept of an isomorphism is then just the usual one. It is easily verified that S-supermodules together with S-linear maps form a category. It is in fact a tensor category with respect to the tensor product defined in 1.4 ([**23**, Ch.3 §2]).

1.4. Tensor products. For two S-supermodules M and N we denote by $M \otimes_S N$ the tensor product of M and N in the category of S-bimodules. To recognize $M \otimes_S N$ as an

S-supermodule we recall the construction of $M \otimes_S N$ (see e.g. [**4**, §11.5]). The $S_{\bar{0}}$-module $M \otimes_{S_{\bar{0}}} N$ has a $\mathbb{Z}_2$-grading given by

$$(M \otimes_{S_{\bar{0}}} N)_{\bar{0}} = (M_{\bar{0}} \otimes_{S_{\bar{0}}} N_{\bar{0}}) \oplus (M_{\bar{1}} \otimes_{S_{\bar{0}}} N_{\bar{1}}),$$
$$(M \otimes_{S_{\bar{0}}} N)_{\bar{1}} = (M_{\bar{0}} \otimes_{S_{\bar{0}}} N_{\bar{1}}) \oplus (M_{\bar{1}} \otimes_{S_{\bar{0}}} N_{\bar{0}}).$$

By definition,

$$M \otimes_S N = (M \otimes_{S_{\bar{0}}} N)/Q = \big((M \otimes_{S_{\bar{0}}} N)_{\bar{0}}/Q_{\bar{0}}\big) \oplus \big((M \otimes_{S_{\bar{0}}} N)_{\bar{1}}/Q_{\bar{1}}\big) \tag{1}$$

where $Q = Q_{\bar{0}} \oplus Q_{\bar{1}}$ is the $S_{\bar{0}}$-submodule of $M \otimes_{S_{\bar{0}}} N$ spanned by homogeneous elements of type $ms_{\bar{1}} \otimes_{S_{\bar{0}}} n - m \otimes_{S_{\bar{0}}} s_{\bar{1}}n$ with $s_{\bar{1}} \in S_{\bar{1}}$. We denote by $m \otimes_S n$ the image of $m \otimes_{S_{\bar{0}}} n$ in $M \otimes_S N$ under the quotient map $M \otimes_{S_{\bar{0}}} N \to M \otimes_S N$ of (1). Then S acts on $M \otimes_S N$ by $s.(m \otimes_S n) = (sm) \otimes_S n$, $ms \otimes_S n = m \otimes_S sn$ and $(m \otimes_S n)s = m \otimes_S (ns)$. This action fulfills the condition 1.2.1 with respect to the $\mathbb{Z}_2$-grading (1), thus giving $M \otimes_S N$ the structure of an S-supermodule. By abuse of notation, we will occasionally write $M_\alpha \otimes_S N_\beta$ ($\alpha, \beta \in \mathbb{Z}_2$) for the span of all $m_\alpha \otimes_S n_\beta$ where $m_\alpha \in M_\lambda$ and $n_\beta \in N_\beta$. We then have $(M \otimes_S N)_{\bar{0}} = M_{\bar{0}} \otimes_S N_{\bar{0}} + M_{\bar{1}} \otimes_S N_{\bar{1}}$, which is in general not a direct sum of S-supermodules, and similarly for $(M \otimes_S N)_{\bar{1}}$.

For S-supermodules M, N and P there are canonical isomorphisms of S-supermodules

$$\begin{aligned}
(M \otimes_S N) \otimes_S P &\xrightarrow{\cong} M \otimes_S (N \otimes_S P), &(2)\\
\mathfrak{L}(M, N; P) &\xrightarrow{\cong} \mathrm{Hom}_S(M \otimes_S N, P), &(3)\\
\psi_{M,N} : M \otimes_S N &\xrightarrow{\cong} N \otimes_S M, &(4)\\
S \otimes_S M &\xrightarrow{\cong} M, &(5)
\end{aligned}$$

given by the maps $(m \otimes_S n) \otimes_S p \mapsto m \otimes_S (n \otimes_S p)$, $b \mapsto [(m \otimes_S n) \mapsto b(m, n)]$, $m \otimes_S n \mapsto (-1)^{|m||n|} n \otimes_S m$ and $s \otimes_S m \mapsto sm$.

1.5. S-superalgebras. An *S-superalgebra*, also called a *superalgebra over S*, is an S-supermodule A together with an S-bilinear map m: $A \times A \to A$ of degree $\bar{0}$. It is usual to abbreviate $m(a, b) =: ab$ and call ab the *product* of A. A *homomorphism* of S-superalgebras is an S-linear map f: $A \to B$ of degree $\bar{0}$ such that $f(aa') = f(a)f(a')$ for all $a, a' \in A$.

Let A be an S-superalgebra. It is in particular a superring as defined in 1.1, hence the concepts defined there (commutative, associative and unital) apply to A. Let Λ be an abelian group. A *Λ-grading* of A is a family $(A_\lambda : \lambda \in \Lambda)$ of S-submodules of A satisfying $A = \oplus_{\lambda \in \Lambda} A_\lambda$ and $A_\lambda A_\mu \subset A_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$. Note that the Λ-grading is compatible with the $\mathbb{Z}_2$-grading of A by our definition of S-submodules. The *opposite* of an S-superalgebra A is the S-superalgebra A^{op} with product $\cdot$ defined on the S-supermodule underlying A by the formula $a \cdot b = (-1)^{|a||b|}\, ba$ where the product on the right side is calculated in A.

The tensor product $A \otimes_S B$ of two S-superalgebras A and B is again an S-superalgebra with respect to the product

$$(a \otimes_S b)(a' \otimes_S b') = (-1)^{|a'||b|} aa' \otimes_S bb'. \tag{1}$$

To see that this is indeed a well-defined product, one can, for example, use 1.4.3. In the following, tensor products of superalgebras will always be equipped with the product (1). We note that the S-supermodule isomorphism

$$\psi_{A,B} : A \otimes_S B \longrightarrow B \otimes_S A : a \otimes_S b \mapsto (-1)^{|a||b|} b \otimes_S b$$

of 1.4.4 is an isomorphism of S-superalgebras. The following lemma is easily verified.

1.6. Lemma. *Let* P *be one of the properties commutative, associative or unital, and let A and B be S-superalgebras. If both A and B have property* P*, then so does $A \otimes_S B$.*

1.7. Superextensions. An *S-superextension* is a commutative, associative and unital S-superalgebra. Superextensions of S form a category whose morphisms are the superalgebra homomorphisms preserving the unit elements. It is tensor category by Lemma 1.6: $A \otimes_S B$ is an S-superextension if A and A are S-superextension.

An example of a $\mathbb{Z}$-superextension is the algebra of dual numbers $\mathbb{Z}[\varepsilon] = \mathbb{Z} \oplus \mathbb{Z}\varepsilon$ where ε is a homogenous element satisfying $\varepsilon^2 = 0$. It gives rise to the *S-superalgebra of dual numbers* $S[\varepsilon] = S \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]$. We have

$$S[\varepsilon]_{\bar{0}} = \begin{cases} S_{\bar{0}} \oplus S_{\bar{0}}\varepsilon & \text{if } |\varepsilon| = \bar{0}, \\ S_{\bar{0}} \oplus S_{\bar{1}}\varepsilon & \text{if } |\varepsilon| = \bar{1}, \end{cases} \quad \text{and} \quad S[\varepsilon]_{\bar{1}} = \begin{cases} S_{\bar{1}} \oplus S_{\bar{1}}\varepsilon & \text{if } |\varepsilon| = \bar{0}, \\ S_{\bar{1}} \oplus S_{\bar{0}}\varepsilon & \text{if } |\varepsilon| = \bar{1}. \end{cases} \tag{1}$$

Another example is the Grassmann algebra over S, to be discussed in 1.11.

We note that an S-superextension T can serve as a new base superring. If A is an S-superalgebra, the tensor product superalgebra $A_T := T \otimes_S A$ (1.5) becomes a T-superalgebra, called the *base superring extension.* In particular, A_T is a T-superextension if A is an S-superextension.

1.8. Superextensions of supermodules and multilinear maps. Let T be an S-superextension, and let M be an S-supermodule. Then the *T-superextension of M*

$$M_T := T \otimes_S M$$

has a canonical left T-module, namely $t(t' \otimes_S m) = (tt') \otimes_S m)$ for $t, t' \in T$ and $m \in M$, with respect to which it is a T-supermodule.

Taking extensions of supermodules is transitive: If U is a T-superextension, then because of 1.4.2 and 1.4.5 we have $(M_T)_U \cong M_U$. Moreover, using the isomorphisms for $\otimes_S$ exhibited in 1.4 one easily verifies that there is an isomorphism of T-supermodules

$$M_T \otimes_T N_T \xrightarrow{\cong} (M \otimes_S N)_T$$

given by $(t \otimes_S m) \otimes_T (t' \otimes_S n) \mapsto (-1)^{|m||t'|}\, tt' \otimes_S (m \otimes_S n)$.

Let $M^1, \dots, M^n$ and N be S-supermodules. For $t \in T$ and $f \in \mathcal{L}_S(M^1, \dots, M^n; N)$ there exists a unique T-multilinear map $\widetilde{t \otimes_S f}\colon M^1_T \times \cdots \times M^n_T \to N_T$ satisfying, with obvious notation,

$$
\begin{aligned}
&(\widetilde{t \otimes f})(t_1 \otimes_S m_1, \dots, t_n \otimes_S m_n) \\
&\quad = (-1)^{|f||t_1 \cdots t_n| + \sum_{i=2}^n |t_i||m_1 \otimes \cdots \otimes m_{i-1}|} \; t t_1 \cdots t_n \otimes_S f(m_1, \dots, m_n)
\end{aligned}
$$

where of course $|t_1 \cdots t_n| = \sum_{i=1}^n |t_i|$ and $|m_1 \otimes \cdots \otimes m_{i-1}| = \sum_{j=1}^{i-1} |m_j|$. Moreover,

$$
\widetilde{} : T \otimes_S \mathcal{L}_S(M^1, \dots, M^n; N) \longrightarrow \mathcal{L}_T(M^1_T, \dots, M^n_T; N_T) : t \otimes_S f \mapsto \widetilde{t \otimes_S f} \tag{1}
$$

is a T-linear map of the corresponding T-supermodules. We call $f_T := \widetilde{1 \otimes_S f}$ the *T-superextension of* $f \in \mathcal{L}_S(M^1, \dots, M_n; N)$.

In particular, for every $t \in T$ and $f \in \operatorname{Hom}_S(M, N)$ there exists a unique T-linear map

$$
\widetilde{t \otimes_S f} : M_T \longrightarrow N_T : (t' \otimes_S m) \mapsto (-1)^{|f||t'|} t t' \otimes_S f(m)
$$

This gives rise to a T-linear map of degree $\bar{0}$

$$
\widetilde{} : T \otimes_S \operatorname{Hom}_S(M, N) \longrightarrow \operatorname{Hom}_T(M_T, N_T) : t \otimes_S f \mapsto \widetilde{t \otimes_S f}. \tag{2}
$$

1.9. Quadratic maps. Let M and N be S-supermodules. A homogeneous S-bilinear map $b\colon M \times M \to N$ is called *symmetric-alternating* if

$$
\begin{aligned}
b(m, m') &= (-1)^{|m||m'|} b(m', m) \quad \text{and} \\
b(m_{\bar{1}}, m_{\bar{1}}) &= 0
\end{aligned}
$$

for $m, m' \in M$ and $m_{\bar{1}} \in M_{\bar{1}}$. We note that the second condition on b follows from the first as soon as it holds for a spanning set of $M_{\bar{1}}$. It is of course satisfied if $\frac{1}{2} \in S$.

An *S-quadratic map from M to N*, written in the form $q\colon M \to N$, is a pair $q = (q_{\bar{0}}, b)$, where $q_{\bar{0}}\colon M_{\bar{0}} \to N_{\bar{0}}$ is an $S_{\bar{0}}$-quadratic map and where $b\colon M \times M \to N$ is a symmetric-alternating S-bilinear map of degree $\bar{0}$ such that

$$
b(m_{\bar{0}}, m'_{\bar{0}}) = q_{\bar{0}}(m_{\bar{0}} + m'_{\bar{0}}) - q_{\bar{0}}(m_{\bar{0}}) - q_{\bar{0}}(m'_{\bar{0}}) \tag{1}
$$

for all $m_{\bar{0}}, m'_{\bar{0}} \in M_{\bar{0}}$, i.e., $b|M_{\bar{0}} \times M_{\bar{0}}$ is polar of $q_{\bar{0}}$ in the usual sense. We therefore call b the *polar of q*. An S-quadratic map $q\colon M \to S$ will be called an *S-quadratic form*.

We note that $2q_{\bar{0}}(m_{\bar{0}}) = b(m_{\bar{0}}, m_{\bar{0}})$ and hence $q_{\bar{0}}$ is determined by b if $\frac{1}{2} \in S$. Also $4q_{\bar{0}}(s_{\bar{1}} m_{\bar{1}}) = 2b(s_{\bar{1}} m_{\bar{1}}, s_{\bar{1}} m_{\bar{1}}) = -2s_{\bar{1}}^2 b(m_{\bar{1}}, m_{\bar{1}}) = 0$. For a finite family $(s_i, m_i)_{i \in F} \subset (S_{\bar{0}} \times M_{\bar{0}}) \cup (S_{\bar{1}} \times M_{\bar{1}})$ we have

$$
q_{\bar{0}}\Big(\sum_{i \in F} s_i m_i\Big) = \sum_{|m_i| = \bar{0}} s_i^2 q_{\bar{0}}(m_i) + \sum_{|m_i| = \bar{1}} q_{\bar{0}}(s_i m_i) + \sum_{|\{i,j\}| = 2} (-1)^{|s_j||m_i|} s_i s_j b(m_i, m_j) \tag{2}
$$

where $\sum_{|\{i,j\}|=2}$ is the sum over all two-element subsets of F. This makes sense since

$$
(-1)^{|s_j||m_i|} s_i s_j b(m_i, m_j) = (-1)^{|s_i||m_j|} s_j s_i b(m_j, m_i)
$$

is symmetric on i and j.

1.10. Examples of quadratic maps. (a) (O. Loos) For an S-bilinear map a: $M \times M \to N$ of degree $\bar{0}$ define

$$q^a_{\bar{0}}(m_{\bar{0}}) = a(m_{\bar{0}}, m_{\bar{0}}) \quad \text{and} \quad b^a(m, m') = a(m, m') + (-1)^{|m||m'|} a(m', m).$$

Then $q^a = (q^a_{\bar{0}}, b^a)$: $M \to N$ is an S-quadratic map, called the *quadratic map associated to a*. Over a free supermodule every quadratic form is obtained in this way (cf. [**3**, §3.4, Prop. 2] for the classical case).

(b) Let q: $M \to N$ be an S-quadratic map and let f: $N \to P$ be an S-linear map of degree $\bar{0}$. Then $f \circ q = (f \circ q_{\bar{0}}, f \circ b)$: $M \to P$ is an S-quadratic map. Similarly, if g: $L \to M$ is an S-linear map of degree $\bar{0}$ then $q \circ g = (q_{\bar{0}} \circ g, b \circ (g \times g))$: $L \to N$ is an S-quadratic map.

(c) For an S-quadratic map q: $M \to N$ define $\operatorname{Rad} q = \{m \in M : q_{\bar{0}}(m_{\bar{0}}) = 0 = b(m, M)\}$ where $m_{\bar{0}}$ denotes the $M_{\bar{0}}$-component of m. Then $\operatorname{Rad} q$ is an S-submodule of M. If F is a submodule of $\operatorname{Rad} q$ then q induces an S-quadratic map $\bar{q}$: $M/F \to N$ given by $\bar{q}_{\bar{0}}(m_{\bar{0}} + F) = q_{\bar{0}}(m_{\bar{0}})$ and $\bar{b}(m + F, m' + F) = b(m, m')$.

1.11. Grassmann algebras. We let $G_{\mathbb{Z}}$ be the exterior algebra of the free $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$, i.e., the unital $\mathbb{Z}$-algebra generated by the odd generators $\xi_i, i \in \mathbb{N}$ and subject to the relations $\xi_i^2 = 0 = \xi_i \xi_j + \xi_j \xi_i$ for $i, j \in \mathbb{N}$. For a finite non-empty subset I of $\mathbb{N}$, written in the form $I = \{i_1, i_2, \ldots, i_r\}$, $i_1 < i_2 < \ldots < i_r$, we put $\xi_I = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_r}$, and let $\xi_\emptyset = 1_{G_{\mathbb{Z}}}$. Then $(\xi_I : I \subset \mathbb{N} \text{ finite})$ is a $\mathbb{Z}$-basis of $G_{\mathbb{Z}}$, satisfying

$$\xi_I \xi_J = (-1)^{|I||J|} \xi_J \xi_I = \left\{ \begin{array}{ll} \pm \xi_{I \cup J} & I \cap J = \emptyset \\ 0 & I \cap J \neq \emptyset \end{array} \right\} \tag{1}$$

(the sign on the right hand side is described explicitly in [**5**, §7.8 (19)]). Let $G_{\mathbb{Z}\bar{0}}$ (respectively $G_{\mathbb{Z}\bar{1}}$) be the $\mathbb{Z}$-span of all ξ_I with $|I|$ even (respectively odd). Using (1) it follows easily that $G_{\mathbb{Z}} = G_{\mathbb{Z}\bar{0}} \oplus G_{\mathbb{Z}\bar{1}}$ is a superextension of $\mathbb{Z}$.

For a base superring S we put

$$G_S = G_{\mathbb{Z}} \otimes_{\mathbb{Z}} S = \bigoplus_{\alpha, \beta \in \mathbb{Z}_2} G_{\mathbb{Z}\alpha} \otimes_{\mathbb{Z}} S_\beta.$$

G_S is a free S-supermodule with basis $(\xi_I : I \subset \mathbb{N} \text{ finite})$. By 1.7, G_S is also an S-superextension with respect to the $\mathbb{Z}_2$-grading

$$G_{S\bar{0}} = (G_{\mathbb{Z}\bar{0}} \otimes_{\mathbb{Z}} S_{\bar{0}}) \oplus (G_{\mathbb{Z}\bar{1}} \otimes_{\mathbb{Z}} S_{\bar{1}}) \quad \text{and} \quad G_{S\bar{1}} = (G_{\mathbb{Z}\bar{0}} \otimes_{\mathbb{Z}} S_{\bar{1}}) \oplus (G_{\mathbb{Z}\bar{1}} \otimes_{\mathbb{Z}} S_{\bar{0}})$$

(direct sum of $S_{\bar{0}}$-modules). In particular

$$G(S) := G_{S\bar{0}}$$

is a commutative associative unital $S_{\bar{0}}$-algebra with a $\mathbb{Z}_2$-grading (note: $G(S)$ is in general not a commutative superalgebra).

1.12. Grassmann envelopes of supermodules. Let M be an S-supermodule. Because of 1.4.2 and 1.4.5 we have

$$G_S \otimes_S M = G_{\mathbb{Z}} \otimes_{\mathbb{Z}} S \otimes_S M \cong G_{\mathbb{Z}} \otimes_{\mathbb{Z}} M = \bigoplus_{\alpha,\beta\in\mathbb{Z}_2} G_{\mathbb{Z}\alpha} \otimes_{\mathbb{Z}} M_\beta.$$

In the future we will consider the isomorphism above as an equality. The G_S-action on $G_S \otimes_S M$ is then given by

$$(g \otimes_{\mathbb{Z}} s)(g' \otimes_{\mathbb{Z}} m) = (-1)^{|s||g'|}\, gg' \otimes_{\mathbb{Z}} sm. \tag{1}$$

The *Grassmann envelope of an S-supermodule M* is defined as the $G(S)$-module

$$G_S(M) := (G_S \otimes_S M)_{\bar 0} = (G_{\mathbb{Z}\bar 0} \otimes_{\mathbb{Z}} M_{\bar 0}) \oplus (G_{\mathbb{Z}\bar 1} \otimes_{\mathbb{Z}} M_{\bar 1})$$

with $G(S)$-module action given by (1).

Example. Let k be a base ring, M a k-module and S a k-superextension. Then $G_S \otimes_S M_S = (G_{\mathbb{Z}} \otimes_{\mathbb{Z}} S) \otimes_S (S \otimes_k M) = G_{\mathbb{Z}} \otimes_{\mathbb{Z}} (S \otimes_S (S \otimes_k M)) = G_{\mathbb{Z}} \otimes_{\mathbb{Z}} (S \otimes_k M) = (G_{\mathbb{Z}} \otimes_{\mathbb{Z}} S) \otimes_k M = G_S \otimes_k M$. Hence the Grassmann envelope of the S-superextension M_S can be identified with the $G(S)$-extension of M:

$$G_S(M_S) = (G_{\mathbb{Z}} \otimes_{\mathbb{Z}} S)_{\bar 0} \otimes_k M = G(S) \otimes_k M. \tag{2}$$

1.13. Grassmann envelopes of multilinear maps. Let $M^1, \ldots, M^n$ and N be S-supermodules. Restricting the map 1.8.1 to the Grassmann envelopes, yields a $G(S)$-linear map

$$\widetilde{} : G_S(\mathcal{L}_S(M^1, \ldots, M^n; N)) \longrightarrow \mathcal{L}_{G(S)}(G_S(M^1), \ldots, G_S(M^n); G_S(N)). \tag{1}$$

In particular, for $f \in \mathcal{L}(M^1, \ldots, M^n; N)_{\bar 0}$ the restriction of the G_S-superextension f_{G_S} (see 1.8) to the Grassmann envelopes $G(M^i)$ is a $G(S)$-multilinear map

$$G_S(f)\colon G_S(M^1) \times \cdots \times G_S(M^n) \to G_S(N),$$

called the *Grassmann envelope of f*. For example, the Grassmann envelope of an $f \in \mathrm{Hom}_S(M,N)_{\bar 0}$ is the $G(S)$-linear map

$$G(f)\colon G(M) \to G(N) : g \otimes_{\mathbb{Z}} m \mapsto g \otimes_{\mathbb{Z}} f(m), \tag{2}$$

and by restricting of the map 1.8.2 we obtain a $G(S)$-linear map

$$\widetilde{} : G_S(\mathrm{Hom}_S(M,N)) \longrightarrow \mathrm{Hom}_{G(S)}(G_S(M), G_S(N)) : t \otimes_S f \mapsto \widetilde{t \otimes_S f}. \tag{3}$$

If $f \in \mathrm{Hom}_S(M,N)_{\bar 0}$ is invertible it is immediate from (2) that $G(f)$ is invertible too. More precisely, we have

$$f \text{ is invertible} \quad \Longleftrightarrow \quad G(f) \text{ is invertible}. \tag{4}$$

Indeed, if $G(f)$ is invertible its inverse leaves all spaces $\xi_I \otimes_{\mathbb{Z}} N_{|I|}$ invariant. Since G is free we have an imbedding

$$M_{|I|} \hookrightarrow G_S(M)\colon m \mapsto \xi_I \otimes_{\mathbb{Z}} m \tag{5}$$

for any finite $I \subset \mathbb{N}$. Now invertibility of f follows from $M \cong M_{\bar 0} \oplus (\xi_1 \otimes M_{\bar 1})$.

1.14. Grassmann envelopes of quadratic maps. The *Grassmann envelope of an S-quadratic map* $q = (q_{\bar 0}, b)\colon M \to N$ is the $G(S)$-quadratic map $G_S(q)\colon G_S(M) \to G_S(N)$ defined as follows:

$$G_S(q)(\sum_I \xi_I \otimes_{\mathbb{Z}} m_I) \;=\; 1_G \otimes_{\mathbb{Z}} q_{\bar 0}(m_\emptyset) + \sum_{|\{I,J\}|=2} (-1)^{|\xi_J||m_I|}\, \xi_I \xi_J \otimes_{\mathbb{Z}} b(m_I, m_J) \qquad (1)$$

where the second sum is taken over all sets consisting of two distinct finite subsets of $\mathbb{N}$, including the possibility $I = \emptyset$. It has the following properties:

(i) The polar of $G_S(q)$ is the Grassmann envelope of the bilinear form b.

(ii) $G_S(q)|G_{\mathbb{Z}\bar 0} \otimes_{\mathbb{Z}} M_{\bar 0}$ is the $G_{S\bar 0}$-extension of the $S_{\bar 0}$-quadratic form $q_{\bar 0}$.

Example. Let k be a base ring, M and N k-modules and $q\colon M \to N$ a k-quadratic map. Assume further that S is a k-superextension. As explained in 1.12.2, the Grassmann envelopes of M_S and N_S can be identified with the $G(S)$-extensions of M and N. It is well-known (see e.g. [**3**, §3.4, Prop.3]) that there exists a unique extension of q to a $G(S)$-quadratic map $q_{G(S)}\colon G(S) \otimes_k M \to G(S) \otimes_k N$. We claim that there exists a unique S-quadratic map $q_S\colon M_S \to N_S$ whose Grassmann envelope makes the following diagram commutative:

$$\begin{array}{ccc} G(M_S) & \overline{\overline{\qquad\qquad}} & G(S) \otimes M \\ \Big\downarrow{\scriptstyle G(q_S)} & & \Big\downarrow{\scriptstyle q_{G(S)}} \\ G(N_S) & \overline{\overline{\qquad\qquad}} & G(S) \otimes M \end{array} \qquad (2)$$

Indeed, the map $q_S = (b_S, q_{\bar 0 S})$ is given as follows: $q_{\bar 0 S}\colon S_{\bar 0} \otimes M \to S_{\bar 0} \otimes N$ is the $S_{\bar 0}$-extension of q, while b_S is the S-extension of b.

Remark. The definition of the Grassmann envelope of an S-quadratic map and the definition of q_S in (2) are special cases of the general fact that every S-quadratic map $q\colon M \to N$ can be extended to a T-quadratic map $q_T\colon M_T \to N_T$ for every S-extension T. Since this result is not needed in the paper, we omit its proof which can be given along the lines of the corresponding extension result for quadratic forms over rings ([**3**, §3.4, Prop.3]).

1.15. Varieties of superalgebras. Let A be an S-superalgebra. It follows from 1.13.1 that the Grassmann envelope $G_S(A)$ is a $G(S)$-algebra. Moreover, 1.13.5 allows one to compare identities in A and $G(A)$. For example, it is easily (and well-known) that

$$A \text{ is associative (commutative)} \Longleftrightarrow G_S(A) \text{ is associative (commutative).} \qquad (1)$$

In general, let $\mathcal{V}$ be a homogeneous variety of algebras, i.e., a variety of algebras whose T-ideal is generated by homogeneous elements [**42**, 1.3]. An S-superalgebra A is called a *$\mathcal{V}$-superalgebra* if $G_S(A)$ belongs to $\mathcal{V}$. Because of 1.13.5, $\mathcal{V}$-superalgebras can be defined by a set of homogeneous identities obtained from the defining identities of $\mathcal{V}$. Rather than doing the precise transfer from $G_S(A)$ to A one can simply apply the *sign rule* to obtain the super version of an identity: Whenever the order of two symbols x, y is changed from $x \dots y$ to $y \dots x$, one must introduce a sign $(-1)^{|x||y|}$ in front of $y \dots x$.

Let T be an S-superextension and let A be a $\mathcal{V}$-superalgebra over S. If A satisfies the super version of a homogeneous identity f defining $\mathcal{V}$, the T-superextension A_T will also satisfy f, because of the uniqueness of superextensions of multilinear maps (1.8).

Example: alternative superalgebras. Recall that an algebra A is alternative if $(a,a,b) = 0 = (b,a,a)$ for all $a,b \in A$ where $(a,b,c) = (ab)c - a(bc)$ is the *associator*, which can of course be defined in any S-superalgebra. Hence, an S-superalgebra A is an *alternative superalgebra* if it satisfies the following identities:

(i) $(a_{\bar 0}, a_{\bar 0}, b) = 0 = (b, a_{\bar 0}, a_{\bar 0})$ for all $a_{\bar 0} \in A_{\bar 0}, b \in A$ and

(ii) $(a,b,c) + (-1)^{|a||b|}\,(b,a,c) = 0 = (a,b,c) + (-1)^{|b||c|}\,(a,c,b)$ for all $a,b,c \in A$.

2. Jordan superpairs: basic definitions and examples.

2.1. Quadratic maps and supertriple products. The notation introduced here will be used throughout the paper.

Let $V = (V^+, V^-)$ be a pair of S-supermodules and let $Q^\sigma : V^\sigma \to \operatorname{Hom}_S(V^{-\sigma}, V^\sigma)$ be a pair of S-quadratic maps. We write $Q^\sigma = (Q^\sigma_{\bar 0}, Q^\sigma(.,.))$ and recall that Q^σ is S-quadratic if and only if the following holds:

(a) $Q^\sigma(.,.)$: $V^\sigma \times V^\sigma \to \operatorname{Hom}_S(V^{-\sigma}, V^\sigma)$: $(u,w) \mapsto Q^\sigma(u,w)$ is a map that is

 (a.1) additive in each variable,

 (a.2) of degree $\bar 0$, i.e., $Q^\sigma(V^\sigma_\alpha, V^\sigma_\beta) \subset \operatorname{Hom}_S(V^{-\sigma}, V^\sigma)_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$,

 (a.3) symmetric-alternating: $Q^\sigma(u,w) = (-1)^{|u||w|} Q^\sigma(w,u)$ and $Q(u_{\bar 1}, u_{\bar 1}) = 0$ for $u, w \in V^\sigma$, $u_{\bar 1} \in V_{\bar 1}$, and

 (a.4) S-bilinear: $Q^\sigma(su, w) = sQ^\sigma(u,w)$ for $s \in S$.

(b) The map $Q^\sigma_{\bar 0}$: $V^\sigma_{\bar 0} \to (\operatorname{Hom}_S(V^{-\sigma}, V^\sigma))_{\bar 0}$ has the following properties:

 (b.1) $Q^\sigma_{\bar 0}(s_{\bar 0} u_{\bar 0}) = s_{\bar 0}^2 Q^\sigma_{\bar 0}(u_{\bar 0})$ for $s \in S_{\bar 0}$, $u_{\bar 0} \in V^\sigma_{\bar 0}$,

 (b.2) $Q^\sigma_{\bar 0}(u+w) - Q^\sigma_{\bar 0}(u) - Q^\sigma_{\bar 0}(w) = Q^\sigma(u,w)$ for $u, w \in V^\sigma$.

Given such maps Q^σ we define a *supertriple product*

$$\{\ldots\}\colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma\colon (u,v,w) \mapsto \{u\,v\,w\} \tag{1}$$

and an S-bilinear map of degree 0

$$D^\sigma(.,.)\colon V^\sigma \times V^{-\sigma} \to \operatorname{End}_S V^\sigma$$

by the formula

$$\{u\,v\,w\} = D^\sigma(u,v)w = (-1)^{|v||w|} Q^\sigma(u,w)v. \tag{2}$$

The triple product $\{...\}$ is an S-trilinear map of degree 0 which is symmetric in the outer variables,

$$\{u\,v\,w\} = (-1)^{|u||v|+|u||w|+|v||w|}\,\{w\,v\,u\} \quad \text{and} \quad \{u_{\bar 1}\,v\,u_{\bar 1}\} = 0. \tag{3}$$

We note that, conversely, given S-trilinear maps $\{...\} : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ of degree $\bar 0$, which are symmetric in the outer two variables, one can define S-bilinear symmetric maps $Q^\sigma(.,.)$ of degree $\bar 0$ by (2).

In the situation above we consider the Grassmann envelopes of Q^σ, see 1.12. Abbreviating $G(.) = G_S(.)$, we have a $G(S)$-quadratic map

$$G(Q^\sigma)\colon G(V^\sigma) \to G(\mathrm{Hom}_S(V^{-\sigma}, V^\sigma))$$

which we compose with the $G(S)$-linear map (see 1.13.3)

$$\widetilde{\ }\colon G(\mathrm{Hom}_S(V^{-\sigma}, V^\sigma)) \to \mathrm{Hom}_{G(S)}(G(V^{-\sigma}), G(V^\sigma))$$

to obtain, by Example 1.10(b), a $G(S)$-quadratic map

$$\widetilde{Q}^\sigma := \widetilde{G(Q^\sigma)}\colon G(V^\sigma) \to \mathrm{Hom}_{G(S)}(G(V^{-\sigma}), G(V^\sigma)). \tag{4}$$

Let $\widetilde{Q}^\sigma(.,.)$ be its polar. As usual, we associate to the pair $(\widetilde{Q}^+, \widetilde{Q}^-)$ a $G(S)$-trilinear triple product $G(V^\sigma) \times G(V^{-\sigma}) \times G(V^\sigma) \to G(V^\sigma)$ and $G(S)$-bilinear maps $\widetilde{D}^\sigma(.,.)\colon G(V^\sigma) \times G(V^{-\sigma}) \to \mathrm{End}_S\, G(V^\sigma)$. (We leave out the $\widetilde{\ }$ in the notation for the triple product since this will most likely not lead to confusion with the triple product of V.) We then have the following formulas for homogeneous $u, w \in V^\sigma, v \in V^{-\sigma}$ and $g \in G_S$ such that $g_u \otimes u \in G(V^\sigma)$ etc.

$$\widetilde{Q}^\sigma(\xi_I \otimes u) = \begin{Bmatrix} 0 & \text{for } I \neq \emptyset \\ \mathrm{Id} \otimes Q_{\bar 0}^\sigma(u) & \text{for } I = \emptyset \end{Bmatrix}, \tag{5}$$

$$\widetilde{Q}^\sigma(g_u \otimes u, g_w \otimes w) = (-1)^{|u||w|}\, g_u g_w \otimes Q^\sigma(u,w), \tag{6}$$

$$\widetilde{D}^\sigma(g_u \otimes u, g_v \otimes v) = (-1)^{|u||v|}\, g_u g_v \otimes D^\sigma(u,v), \tag{7}$$

$$\{(g_u \otimes u)\ (g_v \otimes v)\ (g_w \otimes w)\} = (-1)^{|u||v|+|u||w|+|v||w|}\, g_u g_v g_w \otimes \{u\, v\, w\}. \tag{8}$$

Of course, $\otimes = \otimes_{\mathbb{Z}}$ in the formulas above. In particular, it follows from (8) that the triple product on the Grassmann envelope

$$G_S(V) := G(V) := (G(V^+), G(V^-))$$

is just the Grassmann envelope of the triple product of V (1.12). *In the following we will leave out the superscript σ if it can be inferred from the context or if it is unimportant.*

2.2. Jordan superpairs. A *Jordan S-superpair*, also called a *Jordan superpair over S*, is a pair $V = (V^+, V^-)$ of S-supermodules together with a pair (Q^+, Q^-) of S-quadratic maps $Q^\sigma\colon V^\sigma \to \mathrm{Hom}_S(V^{-\sigma}, V^\sigma)$ such that its Grassmann envelope $G_S(V)$ together with the quadratic maps $(\widetilde{Q}^+, \widetilde{Q}^-)$ of 2.1.4 is a Jordan pair over $G(S)$.

The condition that $G(V)$ be a Jordan pair can be expressed in terms of identities as follows. Using the notation of [**20**], $G(V)$ is a Jordan pair if and only if the identities (JP1)–(JP3) and all their linearizations hold when substituting elements from the spanning set $\xi_I \otimes v^\sigma$ ($\xi_I \in G_{\mathbb{Z}}$, $v \in V^\sigma_{|I|}$) of $G(V)$ (a total of 15 identities). Since $v \mapsto \xi_I \otimes_{\mathbb{Z}} v$ is an imbedding, we can pull back the identities to V. It follows that V is a Jordan superpair if and only if the super versions of (JP1)–(JP3) and all their linearizations V. One obtains the super version (JSPx) of the Jordan identity (JPx) by using the sign rule (1.15) and by replacing any quadratic operator $Q(x)$ by $Q_{\bar 0}(x_{\bar 0})$ with an even $x_{\bar 0}$. For example:

$$\begin{aligned} &D(x,y)D(u,v) - (-1)^{(|x|+|y|)(|u|+|v|)} D(u,v)D(x,y) \\ &\quad = D(\{x\,y\,u\},v) - (-1)^{|x||y|+|x||u|+|y||u|} D(u,\{y\,x\,v\}) \end{aligned} \qquad \text{(JSP15)}$$

As in the classical theory the definition of a Jordan superpair simplifies if $\frac{1}{2}, \frac{1}{3} \in S$. Indeed, assuming this, let $V = (V^+, V^-)$ be a pair of S-supermodules with a pair of triple products 2.1.1 which is supersymmetric in the outer variables. Define $D(.,.)$ by 2.1.2. Since (JSP15) for V is equivalent to (JP15) for $G(V)$ it follows from [**20**, 2.2] that

If $\frac{1}{2}, \frac{1}{3} \in S$ then V is a Jordan superpair if and only if (JSP15) *holds for V.* (1)

This characterization is taken as the definition in Krutelevich's paper [**18**] which contains a classification of simple finite-dimensional Jordan superpairs over algebraically closed fields of characteristic 0.

2.3. Basic concepts. A *homomorphism* $f\colon V \to W$ of Jordan S-superpairs is a pair $f = (f^+, f^-)$ of S-linear maps $f^\sigma\colon V^\sigma \to W^\sigma$ of degree $\bar 0$ satisfying for $x_{\bar 0} \in V^\sigma_{\bar 0}$ and arbitrary $u, w \in V^\sigma$ and $v \in V^{-\sigma}$

$$f^\sigma(Q_{\bar 0}(x_{\bar 0})v) = Q_{\bar 0}(f^\sigma x_{\bar 0}) f^{-\sigma} v \quad \text{and} \quad f^\sigma(\{u\,v\,w\}) = \{f^\sigma(u)\, f^{-\sigma}(v)\, f^\sigma(w)\}. \qquad (1)$$

There is a useful *homomorphism criterion.* Suppose $f\colon V \to W$ is a pair of S-linear maps of degree 0 and let $G(f)\colon G(V) \to G(W)$ be its Grassmann envelope (1.13.2). Then

$$f \text{ is a homomorphism} \quad \Leftrightarrow \quad G(f) \text{ is a homomorphism.} \qquad (2)$$

The definition of an *isomorphism* respectively *automorphism* between Jordan superpairs is obvious, and clearly (2) also holds for them.

A pair $U = (U^+, U^-)$ of $\mathbb{Z}_2$-graded S-submodules of a Jordan superpair V over S is a *subpair* of V if

$$Q_{\bar 0}(U^\sigma_{\bar 0})U^{-\sigma} \subset U^\sigma \text{ and } \{U^\sigma\, U^{-\sigma}\, U^\sigma\} \subset U^\sigma. \qquad (3)$$

In this case $G_S(U)$ imbeds as a subpair of $G_S(V)$ and hence U is a Jordan S-superpair with the induced grading and the induced quadratic maps. In particular, $V_{\bar 0}$ is a subpair of V considered as Jordan superpair over $S_{\bar 0}$. Similarly, a pair $U = (U^+, U^-)$ of $\mathbb{Z}_2$-graded submodules is an *ideal* of V if

$$Q_{\bar 0}(U^\sigma_{\bar 0})V^{-\sigma} + Q_{\bar 0}(V^\sigma_{\bar 0})U^{-\sigma} + \{V^\sigma\, V^{-\sigma}\, U^\sigma\} + \{V^\sigma\, U^{-\sigma}\, V^\sigma\} \subset U^\sigma\,.$$

In this case, $V/U = (V^+/U^+, V^-/U^-)$ is a Jordan superpair with the induced operations. It is clear that $U \subset V$ is a subpair (respectively ideal) if and only if $G_S(U) \subset G_S(V)$ is a subpair (respectively ideal). Once calls V *simple* if V has only the trivial ideals and if $Q^\sigma \neq 0$.

Let Γ be an abelian group. A *Γ-grading of a Jordan S-superpair* $V=(V^+,V^-)$ is a family $(V^\sigma[\alpha];\sigma=\pm,\alpha\in\Gamma)$ of S-submodules such that

$$V^\sigma=\oplus_{\gamma\in\Gamma}V^\sigma[\gamma]$$

and the following multiplications rules hold for all $\alpha,\beta,\gamma\in\Gamma$

$$Q_{\bar 0}(V^\sigma_{\bar 0}[\alpha])V^{-\sigma}[\beta]\ \subset\ V^\sigma[2\alpha+\beta],\quad\text{and}\tag{4}$$
$$\{V^\sigma[\alpha]\,,\,V^{-\sigma}[\beta]\,,\,V^\sigma[\gamma]\}\ \subset\ V^\sigma[\alpha+\beta+\gamma],\tag{5}$$

see [**21**] for the classical situation. In this case, V will be called *Γ-graded* and the $V[\alpha]=(V^+[\alpha],V^-[\alpha])$ will be referred to as *homogeneous spaces*. If V and W are Γ-graded Jordan superpairs we will say that they are *graded-isomorphic* and denote this by $V\cong_\Gamma W$, if there exists an isomorphism $f\colon V\to W$ with $f^\sigma(V^\sigma[\gamma])=W^\sigma[\gamma]$ for $\sigma=\pm$ and all $\gamma\in\Gamma$.

We call $u_{\bar 0}\in V^\sigma_{\bar 0}$ *invertible* if $Q^\sigma_{\bar 0}(u)\in\mathrm{Hom}_S(V^{-\sigma},V^\sigma)$ is invertible. In this case, its *inverse* is defined by $u^{-1}=Q^\sigma_{\bar 0}(u)^{-1}(u)\in V^{-\sigma}_{\bar 0}$. Since $\widetilde{Q}^\sigma(1\otimes u)$ is the Grassmann envelope of $Q_{\bar 0}(u)$, it follows from 1.13.4 that u is invertible if and only if $1\otimes u$ is invertible in the Jordan pair $G(V)$. In this case, $1\otimes u^{-1}=(1\otimes u)^{-1}$, $Q^{-\sigma}_{\bar 0}(u^{-1})=Q^\sigma_{\bar 0}(u)^{-1}$ and u^{-1} is again invertible and has inverse u.

For $(x,y)\in V^\sigma_{\bar 0}\times V^{-\sigma}_{\bar 0}$ the *Bergman operator* is defined as

$$B(x,y)=\mathrm{Id}-D^\sigma(x,y)+Q^\sigma_{\bar 0}(x)Q^{-\sigma}_{\bar 0}(y)\in\mathrm{End}_S(V^\sigma).$$

Observe that the Grassmann envelope of $B(x,y)$ is the Bergman operator of the pair $(1\otimes x,1\otimes y)$. Hence, by 1.13.4 and the elemental characterization of quasi-invertibility in Jordan pairs we see that the following conditions are equivalent:

(i) $B(x,y)$ is invertible;
(ii) $(1\otimes x,1\otimes y)$ is quasi-invertible in the Jordan pair $G(V)$;
(iii) (x,y) is quasi-invertible in the Jordan pair $V_{\bar 0}$;

In this case, we call $(x,y)\in V$ *quasi-invertible*, and note that

$$\beta(x,y)=(B(x,y),B(y,x)^{-1})\text{ is an automorphism of }V,\tag{6}$$

called the *inner automorphism* defined by (x,y). Indeed, this follows from the homomorphism criterion (2) and the corresponding fact for Jordan pairs ([**20**, 3.9]).

2.4. Proposition (Split null extensions). *Let U be a Jordan pair over k, $M=(M^+,M^-)$ a pair of k-modules, $d^\sigma\colon U^\sigma\times U^{-\sigma}\to\mathrm{End}_k(M^\sigma)$ bilinear maps and $q^\sigma\colon U^\sigma\to\mathrm{Hom}_k(M^{-\sigma},M^\sigma)$ quadratic maps. On $V=U\oplus M=(U^+\oplus M^+,U^-\oplus M^-)$ we define quadratic maps $Q^\sigma\colon V^\sigma\to\mathrm{Hom}_k(V^{-\sigma},V^\sigma)$ by*

$$Q^\sigma(x\oplus m)(y\oplus n)=Q^\sigma(x)y\oplus q^\sigma(x)n+d^\sigma(x,y)m.\tag{1}$$

Then the following are equivalent:

(i) *V is a Jordan pair with respect to Q defined in* (1).
(ii) *V is a Jordan superpair over k with homogeneous parts $V_{\bar 0}=U$, $V_{\bar 1}=M$ and quadratic maps $(Q^\sigma|U^\sigma,Q^\sigma(.,.))$ where $Q^\sigma(.,.)$ is the polar of Q^σ.*
(iii) *(M,d,q) is a U-module in the sense of* [**20**, 2.3].

One calls V the *split null extension of* U *by* M ([**20**, 2.7]).

Proof. We start out with a general observation. For a fixed Grassmann generator, say ξ_1, the pair $W = (W^+, W^-) \subset G(V)$ given by $W^\sigma = (1 \otimes U^\sigma) \oplus (\xi_i \otimes M^\sigma)$ is a subpair of $G(V)$, i.e., 2.3.3 holds which makes sense even if $G(V)$ is not necessarily a Jordan pair. Moreover, the canonical map

$$V \to W\colon u \oplus m \mapsto (1 \otimes u) \oplus (\xi_i \otimes m) \tag{2}$$

is an isomorphism of pairs in the sense that 2.3.1 holds which, again, makes sense for arbitrary pairs.

(i) $\Rightarrow$ (ii): To prove that $G(V)$ is a Jordan pair we have to verify that the Jordan pair identities and all their linearizations hold for elements from the spanning set $\xi_I \otimes u_I^\sigma$ of $G(V)$. The product formula (1) implies that in $G(V)$ all products with more than one factor from $G_{\bar{1}} \otimes M$ vanish. Thus, it is sufficient to check that the identity holds in W. But this is indeed the case, since $W \approx V$ by (2) and since V is a Jordan pair by assumption.

(ii) $\Rightarrow$ (iii): By the observation above, W is a subpair of the Jordan pair $G(V)$ and hence itself a Jordan pair. Using the enumeration of [**20**, 2.3], the defining identities (1), (3), (4) and (5) of a representation, follow by evaluating the Jordan pair identities (JP1) – (JP3) on W while (2) is a consequence of (JP12).

(iii) $\Rightarrow$ (i): this is [**20**, 2.7].

2.5. Corollary (First approximation of Jordan superpairs). *Let* V *be a Jordan superpair over a base superring* S*. Then* $V_{\bar{1}}$ *is a* $V_{\bar{0}}$*-module, and hence the split null extension* V' *of* $V_{\bar{0}}$ *by* $V_{\bar{1}}$ *is a Jordan superpair over* S *as well as a Jordan pair over* $S_{\bar{0}}$*, called the first approximation of* V*.*

Proof. Let V' be the pair obtained from V by putting all products with more than one factor from $V_{\bar{1}}$ equal to zero. By 2.4.2 the pairs V' and W are isomorphic. Since by assumption $G(V)$ is a Jordan pair, so is $W \approx V'$. By 2.4 we then know that $V'_{\bar{1}}$ is a $V'_{\bar{0}}$-module with respect to the canonical maps which, by definition of V', means that $V_{\bar{1}}$ is a $V_{\bar{0}}$-module.

2.6. Proposition (Superextensions of Jordan pairs). *Let* V *be a Jordan pair over a base ring* k *and let* S *be a* k*-superextension. We put* $V_S = (S \otimes_k V^+, S \otimes_k V^-)$ *and denote by* $V_{G(S)} = G(S) \otimes_k V$ *the base ring extension of* V *by* $G(S)$*. By* 1.12.2 *we can identify* $G(V_S^\sigma) = G(S) \otimes_k V^\sigma = V_{G(S)}^\sigma$ *as* $G(S)$*-modules.*

There exist a unique Jordan S*-superpair structure on* $V_S = (S \otimes V^+, S \otimes V^-)$*, called the* S-extension of V*, such that* $G(V_S) = V_{G(S)}$*.*

Proof. We let $Q_S = (Q_S^+, Q_S^-)$ be the S-extensions of the structure maps Q^σ (see Example 1.14) followed by the S-linear map $\eta\colon S \otimes_k \mathrm{Hom}_k(V^{-\sigma}, V^\sigma) \to \mathrm{Hom}_S(V_S^{-\sigma}, V_S^\sigma)$ of 1.8.2. It is then straightforward to verify that $G(V_S) = V_{G(S)}$.

2.7. Jordan supertriples. Let T be an S-supermodule with an S-quadratic map $P\colon T \to \mathrm{End}_S(T)$. As in 2.1.4 this gives rise to a $G(S)$-quadratic map $\widetilde{P}\colon G(T) \to \mathrm{End}_{G(S)}(G(T))$. We call T a *Jordan supertriple over S* if $G(T)$ together with $\widetilde{P}$ is a Jordan triple (system), as for example defined in [**20**, 1.13]. Homomorphisms of Jordan supertriples, ideals and simplicity are defined in the obvious way.

The relation between Jordan supertriples and Jordan superpairs is the same as in the classical theory. To explain this, we need some more definitions. The *opposite* of a Jordan S-superpair $V = (V^+, V^-)$ is the Jordan superpair $V^{\mathrm{op}} = (V^-, V^+)$ with quadratic maps (Q^-, Q^+). That V^{op} is indeed a Jordan superpair follows from $G(V^{\mathrm{op}}) = G(V)^{\mathrm{op}}$. An *involution* of V is a homomorphism $\eta\colon V \to V^{\mathrm{op}}$ such that $(\eta^- \circ \eta^+, \eta^+ \circ \eta^-) = \mathrm{Id}_V$. It is clear that η is an involution of V if and only if its Grassmann envelope $G(\eta)$ is an involution of $G(V)$. One can now easily verify:

(a) If (T, P) is a Jordan supertriple then $V(T) = (T, T)$ with the quadratic maps (P, P) is a Jordan superpair with involution $\eta = (\mathrm{Id}, \mathrm{Id})$.
(b) Conversely, if V is a Jordan superpair with involution η then $T = V^+$ together with P defined by $P(x) = Q^+(x)\eta^+$ is a Jordan supertriple whose associated Jordan superpair (T, T) is isomorphic to V via $(\mathrm{Id}, \eta^+)\colon (T, T) \to V$.

As in the classical theory one can, conversely, imbed the category of Jordan superpairs in the category of Jordan supertriples by associating to a Jordan superpair $V = (V^+, V^-)$ the Jordan supertriple $T(V) = V^+ \oplus V^-$ with quadratic maps determined by

$$\begin{aligned} P_{\bar 0}(x^+_{\bar 0} \oplus x^-_{\bar 0})(y^+ \oplus y^-) &= Q^+_{\bar 0}(x^+_{\bar 0})y^- \oplus Q^-_{\bar 0}(x^-_{\bar 0})(y^+) \quad \text{and} \\ \{x^+ \oplus x^-\,,\, y^+ \oplus y^-\,,\, z^+ \oplus z^-\} &= \{x^+\; y^-\; z^+\} \oplus \{x^-\; y^+\; z^-\}. \end{aligned}$$

That $T(V)$ is indeed a Jordan supertriple follows from $G(T(V)) = T(G(V))$ and the corresponding fact for Jordan pairs ([**20**, 1.14]). One then has the super version of the well-known simplicity transfer (see for example [**30**, 1.5]):

2.8. Lemma (a) *A Jordan superpair V is simple if and only if the Jordan supertriple $T(V)$ is simple.*

(b) *A Jordan supertriple T is simple if and only the Jordan superpair $V(T)$ is either simple or a direct sum of two simple ideals, $V(T) = W \oplus W^{\mathrm{op}}$, such that $T = T(W)$.*

2.9. Example: Quadratic form supertriples. To motivate the definition below we first recall quadratic form triples. Let k be a base ring, X a k-module and $q\colon X \to k$ a quadratic form with polar b. Then X becomes a Jordan triple system over k, called a *quadratic form triple*, with quadratic map $P(x)y = b(x, y)x - q(x)y$.

Let now S be a base superring and let $q = (q_{\bar 0}, b)\colon M \to S$ be an S-quadratic form. Define for $m_{\bar 0} \in M_{\bar 0}$ and arbitrary homogeneous $m, n, p \in M$

$$\begin{aligned} P_{\bar 0}(m_{\bar 0})n &= b(m_{\bar 0}, n)m_{\bar 0} - q_{\bar 0}(m_{\bar 0})n, \\ \{m\, n\, p\} &= b(m, n)p + mb(n, p) - (-1)^{|n||p|}b(m, p)n. \end{aligned}$$

Then M together with the quadratic map and triple products defined above is a Jordan supertriple over S, called the *Jordan supertriple associated to* q or sometimes simply a *quadratic form supertriple*. Indeed, the Grassmann envelope of the supertriple M is the quadratic form triple on $G(M)$ with respect to the $G(S)$-quadratic form $G(q)$ of 1.14. The Jordan superpair (M, M), see 2.7, will be called the *quadratic form superpair associated to* q.

The radical $\operatorname{Rad} q$ of q, see 1.10(c), is an ideal of the quadratic form supertriple M defined by q whose multiplication is trivial. Hence, a necessary condition for simplicity of M or (M, M) is that q is *nondegenerate* in the sense that $\operatorname{Rad} q = 0$. The techniques to establish the following simplicity criterion are well-known, see e.g. [**13**, Th. 11] and [**17**, Th. 6.1] for the case of Jordan algebras and superalgebras. Its proof will therefore be left to the reader, but we note that because of 2.8 it is sufficient to consider the quadratic form superpair (M, M).

2.10. Lemma. *Let $S = S_{\bar 0}$ be a field, let M be a non-zero S-supermodule and let q: $M \to S$ be a nondegenerate S-quadratic form. Exclude the following situation: S is a field of characteristic 2, $M = M_{\bar 1}$ and $\dim_S M = 2$.*

Then the quadratic form triple M is simple, while the quadratic form pair $V = (M, M)$ is either simple or $M = M_{\bar 0}$ has dimension 2 and q is hyperbolic. In the latter case, if $h_\pm$ is a hyperbolic basis of M, the Jordan pair $(M, M) = W \oplus W^{\mathrm{op}}$ is a direct sum of two ideals W and W^{op} for $W = (Sh_+, Sh_-)$.

2.11. Unital Jordan superalgebras. A *unital Jordan superalgebra over* S is a triple $(J, U, 1_J)$, where J is an S-supermodule, U: $J \to \operatorname{End}_S(J)$ is an S-quadratic map and 1_J is a distinguished element in $J_{\bar 0}$ such that the Grassmann envelope $G(J)$ together with the $G(S)$-quadratic map $\widetilde{G(U)}$: $G(J) \to \operatorname{End}_{G(S)}(G(J))$ is a (quadratic) Jordan algebra with unit element $1_G \otimes 1_J$, as for example defined in [**12**, 1.3.4]. It follows that

$$U_{\bar 0}(1_J) = \mathrm{Id}. \tag{1}$$

Since a unital Jordan algebra is the same as a Jordan triple with an element satisfying (1), unital Jordan superalgebras can also be characterized as Jordan supertriples containing an element 1_J satisfying (1).

Basic concepts like homomorphism, ideal and simplicity are defined in an analogous manner as in 2.3 for Jordan superpairs. Details can be left to the reader but, for later use, we mention explicitly the definition of a grading. Let Λ be an abelian group. A *Λ-grading* of a unital Jordan S-superalgebra J is a family $(J_\lambda : \lambda \in \Lambda)$ of S-submodules such that $J = \oplus_{\lambda \in \Lambda} J_\lambda$ and the following multiplication rules hold for $\lambda, \mu, \nu \in \Lambda$:

$$U_{\bar 0}(J_\lambda) J_\mu \subset J_{2\lambda+\mu} \quad \text{and} \quad \{J_\lambda\ J_\mu\ J_\nu\} \subset J_{\lambda+\mu+\nu} \tag{2}$$

where $\{...\}$ denotes the Jordan triple product of the Jordan supertriple underlying J.

Remarks. (a) If $\frac{1}{2} \in S$ one can define a *linear Jordan superalgebra* as an S-superalgebra with the property that its Grassmann envelope is a linear Jordan algebra (1.15). As in the classical case, they coincide with quadratic Jordan superalgebras defined above. The relation between the quadratic structure and the linear Jordan superalgebra product is given by

$$U_{\bar{0}}(a_{\bar{0}})b = 2a_{\bar{0}}(a_{\bar{0}}b) - a_{\bar{0}}^2 b \tag{3}$$

$$\{a\,b\,c\} = 2\big(a(bc) + (ab)c - (-1)^{|b||c|}(ac)b\big) \tag{4}$$

(b) The same approach that we have used to define Jordan superpairs leads to a definition of not necessarily unital (quadratic) Jordan superalgebras: one requires that the Grassmann envelope is a non-unital quadratic Jordan algebra. For the case of base rings, details can be found in the recent paper [**17**] which also contains a discussion of some of the standard examples of Jordan superalgebras.

The relation between Jordan superalgebras and Jordan superpairs is the same as in the non-super case [**20**, 1.6, 1.11]:

2.12. Lemma (Isotopes). (a) *Let J be a unital Jordan superalgebra over S. Then $V = (J, J)$ with $Q^\sigma = U$ is a Jordan superpair with invertible element $1_J \in V^-_{\bar{0}}$ and inverse $1_J \in V^+_{\bar{0}}$. If J is simple then so is V.*

(b) *Conversely, let V be a Jordan superpair over S and suppose that $v \in V^-_{\bar{0}}$ is invertible with inverse $u \in V^+_{\bar{0}}$. Then $J = V^+$ together with $1_J = u$ and quadratic maps given by $U_{\bar{0}}(x) = Q^+_{\bar{0}}(x)Q^-_{\bar{0}}(v)$ and $U(x,y) = Q^+(x,y)Q^-_{\bar{0}}(v)$ is a unital Jordan superalgebra, called the v-isotope of V. Moreover, $(\mathrm{Id}_J, Q^-_{\bar{0}}(v))\colon (J,J) \to V$ is an isomorphism of Jordan superpairs. If V is simple then so is J.*

2.13. Example: quadratic form superalgebras. Let $V = (M, M)$ be the quadratic form superpair associated to an S-quadratic form $q = (b, q_{\bar{0}}) : M \to S$. If $1 \in M_{\bar{0}}$ is a *base point*, i.e., $q_{\bar{0}}(1) = 1$, then $1 \in V^- = M$ is invertible with inverse $1 \in V^+ = M$: the map $Q_{\bar{0}}(1)m = b(1, m)1 - m =: \overline{m}$ satisfies $\overline{\overline{m}} = m$. Hence, by 2.12.b, the S-module M together with the quadratic map

$$\begin{aligned} U_{\bar{0}}(m_{\bar{0}})n &= b(m_{\bar{0}}, \overline{n})m_{\bar{0}} - q_{\bar{0}}(m_{\bar{0}})\overline{n}, \\ \{m\,n\,p\} &= b(m, \overline{n})p + mb(\overline{n}, p) - (-1)^{|n||p|}b(m,p)\overline{n}. \end{aligned}$$

is a unital Jordan superalgebra with identity element 1. For $S = S_{\bar{0}}$ these superalgebras are studied in [**17**, 6].

2.14. Example: special Jordan supertriples. Every associative algebra A becomes a Jordan algebra, denoted $A^{(+)}$, with respect to the quadratic operation $U(x)y = xyx$, where the product on the right hand side is calculated in the associative algebra A. The corresponding triple product is $\{a\,b\,c\} = abc + cba$. We will describe the super version of this example but since we did not define non-unital Jordan superalgebras we will work with Jordan supertriples instead.

Let A be an associative superalgebra over some base superring S with multiplication ab for $a, b \in A$. For $a \in A$ we define the *left multiplication* $L(a)$ respectively *right multiplication* $R(a)$ by

$$L(a)b = ab \quad , \quad R(a)b = (-1)^{|a||b|}\, ba.$$

Then $L(a), R(a) \in \mathrm{End}_S(A)$ and $L(a)R(b) = (-1)^{|a||b|}R(b)L(a)$ for $a, b \in A$. We have an S-quadratic map P: $A \to \mathrm{End}_S(A)$ given by

$$\begin{aligned} &P_{\bar{0}}(a_{\bar{0}}) = L(a_{\bar{0}})R(a_{\bar{0}}) \quad \text{and} \\ &P(a,b) = L(a)R(b) + (-1)^{|a||b|}L(b)R(a) = L(a)R(b) + R(a)L(b). \end{aligned}$$

Indeed, P is the quadratic map associated to the S-bilinear map $A \times A \to \mathrm{End}_S(A)$ defined by $(a,b) \mapsto L(a)R(b)$, see Example 1.10(a). The corresponding triple product 2.1.2 is

$$\{a\, b\, c\} = abc + (-1)^{|a||b|+|a||c|+|b||c|}\, cba.$$

These formulas imply that the Grassmann envelope of (A, P) is the Jordan triple system $G(A)^+$, hence (A, P) is a Jordan supertriple, denoted again $A^{(+)}$. Note that $A^{(+)}$ is a Jordan superalgebra if A is unital. In any case, by 2.7.a, (A, A) is always a Jordan superpair.

An *involution* of an S-superalgebra A is an S-linear map π: $A \to A$ of degree 0 satisfying for $a, b \in A$

$$(ab)^\pi = (-1)^{|a||b|}\, b^\pi a^\pi \quad \text{and} \quad (a^\pi)^\pi = a.$$

Obviously, π is an involution if and only if its Grassmann envelope $G(\pi)$ is an involution of the algebra $G(A)$. Any involution π of an associative A is also an involution of the supertriple $A^{(+)}$ in the following sense

$$(P_{\bar{0}}(a_{\bar{0}})b)^\pi = P_{\bar{0}}(a_{\bar{0}}^\pi)b^\pi \quad \{a\, b\, c\}^\pi = \{a^\pi\, b^\pi\, c^\pi\}, \tag{1}$$

and hence induces an involution of the associated Jordan superpair (A, A) as defined in 2.7 . We denote by $H(A, \pi) = \{a \in A : a^\pi = a\}$ the symmetric elements and by $S(A, \pi) = \{a \in A : a^\pi = -a\}$ the skew symmetric elements of A. Then (1) implies that

$$(H(A,\pi), H(A,\pi)) \textit{ and } (S(A,\pi), S(A,\pi)) \textit{ are subpairs of } (A, A). \tag{2}$$

Special quadratic Jordan superalgebras are also considered in [**17**]. For a description of involutions of simple or primitive associative superalgebras see [**8**, Th.2.10] and [**37**].

3. Grids in Jordan superpairs.

Unless stated otherwise, $V = V_{\bar 0} \oplus V_{\bar 1}$ *will denote a Jordan superpair over some base superring* S*. We will write* Q *for* Q^σ *and* D *for* D^σ *if* σ *can be inferred from the context. We will frequently consider elements* $e = (e^+, e^-)$, $f = (f^+, f^-)$ *or* $g = (g^+, g^-)$*, in which case it is often useful to employ the following abbreviations*

$$\begin{aligned} Q_{\bar 0}(e) &:= (Q_{\bar 0}(e^+), Q_{\bar 0}(e^-)) \quad \text{(for even } e\text{)}, \\ D(e,f) &:= (D(e^+, f^-), D(e^-, f^+)) \quad \text{and} \\ \{e\, f\, g\} &:= (\{e^+\, f^-\, g^+\}, \{e^-\, f^+\, g^-\}). \end{aligned}$$

3.1. Idempotents. This subsection is the super version of [**20**, 5.4]. All unexplained results follow from there. Using the abbreviations above, an *idempotent of* V is an element $e = (e^+, e^-) \in V_{\bar 0}$ satisfying $Q_{\bar 0}(e)e = e$. To an idempotent e we associate *Peirce projections* $E_i = (E_i^+, E_i^-), i = 0, 1, 2$, given by

$$E_2^\sigma = Q_{\bar 0}(e^\sigma)Q_{\bar 0}(e^{-\sigma}), \;\; E_1^\sigma = D(e^\sigma, e^{-\sigma}) - 2E_2^\sigma, \;\; E_0^\sigma = B(e^\sigma, e^{-\sigma}).$$

Let V' be the first approximation of V (2.5). Since the E_i are the same for V and the Jordan pair V', the classical theory implies that they form a complete system of orthogonal projections onto the *Peirce spaces* of e,

$$V_i(e) = (V_i^+(e), V_i^-(e)), \quad V_i^\sigma = E_i^\sigma(V^\sigma),$$

and hence give rise to the *Peirce decomposition* $V = V_2(e) \oplus V_1(e) \oplus V_0(e)$. Of course this direct sum has to be understood componentwise. We will abbreviate $V_i(e)$ by V_i if the idempotent e is clear from the context. The Peirce spaces are S-submodules, and they are the same for V and V'. Therefore we have the following characterizations:

$$\begin{aligned} V_2^\sigma &= \mathrm{Im}(Q_{\bar 0}(e^\sigma)), \; V_1^\sigma \oplus V_0^\sigma = \mathrm{Ker}(Q_{\bar 0}(e^{-\sigma})), \\ V_1^\sigma &= \mathrm{Ker}(\mathrm{Id} - D(e^\sigma, e^{-\sigma})), \\ V_0^\sigma &= \mathrm{Ker}(Q_{\bar 0}(e^{-\sigma})) \,\cap\, \mathrm{Ker}(D(e^\sigma, e^{-\sigma})), \\ V_i^\sigma &\subset \{v \in V^\sigma : \{e^\sigma\, e^{-\sigma}\, v\} = iv\}, \quad (i = 0, 1, 2) \end{aligned}$$

where the inclusion above is an equality if either $i = 1$ or $i = 0, 2$ and V has no 2-torsion. The element $1 \otimes e = (1 \otimes e^+, 1 \otimes e^-)$ is an idempotent of the Grassmann envelope $G(V)$. Since the Grassmann envelopes of the Peirce projections E_i are the Peirce projections of the idempotent $1 \otimes e \in G(V)$ it follows that

$$G(V_i(e)) = G(V)_i(e), \quad (i = 0, 1, 2). \tag{1}$$

Using (1), the multiplication rules between the Peirce spaces of $1 \otimes e$ can be pulled back to V. Setting $V_i = 0$ for $i \neq 0, 1, 2$ we therefore have

$$Q_{\bar 0}(V_i)V_j \subset V_{2i-j} \quad \text{and} \quad \{V_i\, V_j\, V_k\} \subset V_{i-j+k}\,, \tag{2}$$

$$D(V_2, V_0) = 0 = D(V_0, V_2). \tag{3}$$

In particular, (2) says that every $V_i(e)$ is a subpair of V.

For two idempotents e and f in Jordan superpair V we say

(i) e and f are *associated* $(e \approx f)$ if $e \in V_2(f)$ and $f \in V_2(e)$ or, equivalently, the Peirce spaces of e and f coincide,

(ii) e and f are *collinear* $(e \top f)$ if $e \in V_1(f)$ and $f \in V_1(e)$,

(iii) e and f are *orthogonal* $(e \perp f)$ if $e \in V_0(f)$ or, equivalently, $f \in V_0(e)$,

(iv) e *governs* f $(e \vdash f)$ if $e \in V_1(f)$ and $f \in V_2(e)$.

3.2. McCrimmon-Meyberg superalgebras. Let e, f be two collinear idempotents in in a Jordan pair U. By a result of McCrimmon-Meyberg ([**28**, 1.1]) the pair $(e^+ + f^+, e^- + f^-) \in U$ is quasi-invertible and gives rise to the *exchange automorphism* $t_{e,f} = \beta(e^+ + f^+, e^- + f^-)$ which has period 2, and satisfies $t_{e,f}(e) = f$ and $t_{e,f}(f) = e$. We also recall from [**28**, 2.2] that the algebra A, defined on $U_2^+(e) \cap U_1^+(f)$ by

$$A : \qquad ab = \{\{a\, e^-\, f^+\} f^-\, b\} \qquad (a, b \in U_2^+(e) \cap U_1^+(f)), \tag{1}$$

is an alternative algebra with identity element e^+. We will call A the *McCrimmon-Meyberg algebra of the pair* (e, f).

These results immediately generalize to the setting of Jordan superpairs. Indeed, let V be a Jordan superpair and assume that $e, f \in V_{\bar 0}$ are two collinear idempotents. Applying the above to the Jordan pair V', the first approximation of V, we have the exchange automorphism $t_{e,f}$ of order 2. Also, $A = V_2^+(e) \cap V_1^+(f)$ together with the product (1) is an S-superalgebra. By 3.1.1 and the definition of the algebra respectively triple product in the Grassmann envelopes (1.5.1, 2.1.8) the Grassmann envelope of A is the McCrimmon-Meyberg algebra of the collinear pair $(1 \otimes e, 1 \otimes f)$ in $G(V)$. Therefore, by 1.15, A is an alternative superalgebra. It is unital with identity element e^+ and will be called the *McCrimmon-Meyberg superalgebra of the collinear pair* (e, f).

3.3. Grids. Grids in Jordan triple systems have been studied in [**31**] and [**33**]. By considering the polarized Jordan triple system associated to any Jordan pair, this theory can be applied to Jordan pairs, see [**35**, §1] for a review of grids in Jordan pairs. Since by definition an idempotent in a Jordan superpair V lies in the Jordan pair $V_{\bar 0}$, the theory of grids is also available for Jordan superpairs, by considering the subpair $V_{\bar 0} \subset V$. For the sake of completeness we give a short review below. We will use some concepts from the theory of 3-graded root systems for which the reader is referred to [**22**, §17 and §18]. A summary of some results is also given in [**32**], [**33**, §1] and [**35**, 1.1], but note the following changes: in [**22**] 0 is considered a root and the Cartan integers are denoted $\langle \alpha, \beta^\vee \rangle$.

A *cog* in V is a family $\mathcal{E} \subset V$ of non-zero idempotents such that two distinct idempotents $e, f \in \mathcal{E}$ satisfy exactly one of the *Peirce relations* $e \top f, e \perp f, e \vdash f$ or $e \dashv f$. A cog $\mathcal{E}$ is *closed* if there exists a 3-graded root system (R, R_1) and a bijection $R_1 \to \mathcal{E}$: $\alpha \mapsto e_\alpha$ which preserves the Peirce relations $\top, \perp$ and $\vdash$. Such a 3-graded root system is uniquely determined up to isomorphism and called the *associated 3-graded root system of* $\mathcal{E}$ [**33**, 3.2]. We fix one such bijection and enumerate $\mathcal{E} = \{e_\alpha; \alpha \in R_1\}$. Since $e_\alpha \top e_\beta \Leftrightarrow \alpha \top \beta$ and similarly for $\perp$ and $\vdash$ we have

$$e_\alpha \in V_{\langle \alpha, \beta^\vee \rangle}(e_\beta)\,, \qquad \text{in particular } \{e_\alpha\, e_\alpha\, e_\beta\} = \langle \beta, \alpha^\vee \rangle e_\beta\,. \tag{1}$$

A cog $\mathcal{E}$ in V is called *connected* if every two idempotents $e, f \in \mathcal{E}$ can be connected by a finite chain $(e = f_1, f_2, \dots, f_n = f) \subset \mathcal{E}$ with $f_i \not\perp f_{i+1}$ for every $1 \le i < n$. A closed cog is connected if and only if its associated 3-graded root system is irreducible [**33**, 3.4]. One calls two cogs $\mathcal{E}$ and $\mathcal{E}'$ *associated* ($\mathcal{E} \approx \mathcal{E}'$) if there exists a bijection $\varphi\colon \mathcal{E} \to \mathcal{E}'$ such that $\varphi(e) \approx e$ for every $e \in \mathcal{E}$. Two associated closed cogs have isomorphic associated 3-graded root systems ([**33**, Thm. 3.4.a]).

A closed cog $\mathcal{G} \subset V$ is a *grid* if it has the following two properties:

(G1) whenever $(g_1, g_2, g_3) \subset \mathcal{G}$ is a family of pairwise collinear idempotents such that $\{g_1\, g_2, g_3\} \neq 0$ then there exists $h \in \mathcal{G}$ such that $g_1 \vdash h \dashv g_3$ and $h \perp g_2$, i.e., the Peirce relations in $(h; g_1, g_2, g_3)$ are the same as in a diamond of roots, and

(G2) if $g_1 \dashv g_2 \vdash g_3 \top g_1$ then $\{g_1\, g_2\, g_3\} = 0$.

For covering grids another characterization will be given in 3.4.2. Special examples of grids will be studied in detail in section 4.

A *collinear family* is a family of pairwise collinear non-zero idempotents. A cog $\mathcal{E}$ is called *pure* if $\{e\, f\, g\} = 0$ for any collinear family $(e, f, g) \subset \mathcal{E}$. A collinear family is a grid if and only if it is pure. It follows from the classification of grids in [**31**, Ch. II] that any connected non-pure grid is associated to a so-called hermitian grid as defined in 4.8.

3.4. Covering grids. For a closed cog $\mathcal{G} = \{e_\alpha : \alpha \in R_1\} \subset V$ and $\alpha \in R_1$ we define the *(joint) α-Peirce space of* $\mathcal{G}$ by

$$V_\alpha := \cap_{\beta \in R_1} V_{\langle \alpha, \beta^\vee \rangle}(e_\beta)$$

where, of course, the intersection has to be taken componentwise. Observe that $e_\alpha \in V_\alpha \subset V_2(e_\alpha)$. The sum of the joint Peirce spaces is always direct, and one says $\mathcal{G}$ *covers* V if

$$V = \oplus_{\alpha \in R_1} V_\alpha. \tag{1}$$

By [**31**, Thm. I.4.14],

$$\textit{a covering closed cog is necessarily a grid,} \tag{2}$$

and hence in the future we will only speak of covering grids instead of covering closed cogs. Recall that two associated closed cogs have the same Peirce spaces ([**33**, (3.8.1)]). In particular, one is a covering grid if and only if both are covering grids.

In view of 3.1.1, the Grassmann envelope of the joint α-Peirce space of a grid $\mathcal{G}$ is the α-Peirce space of the closed cog $1 \otimes \mathcal{G} = \{1 \otimes g : g \in \mathcal{G}\} \subset G(V)$, from which it easily follows that

$$\mathcal{G} \textit{ covers } V \iff 1 \otimes \mathcal{G} \textit{ covers } G(V). \tag{3}$$

The Peirce multiplication rules 3.1.2 and 3.1.3 for a single idempotent imply

$$Q_{\bar 0}(V_\alpha)V_\beta \subset V_{2\alpha-\beta}, \quad \{V_\alpha\, V_\beta\, V_\gamma\} \subset V_{\alpha-\beta+\gamma} \quad \text{and} \tag{4}$$

$$\{V_\alpha\, V_\beta\, V\} = 0 \quad \text{if} \quad \alpha \perp \beta. \tag{5}$$

where $Q_{\bar 0}(V_\alpha)V_\beta = 0$ if $2\alpha - \beta \notin R_1$ and analogously for the triple product $\{V_\alpha\, V_\beta\, V_\gamma\}$.

Suppose that $\mathcal{G}$ is a covering grid. The multiplication rule (4) can also be interpreted by saying that (1) is a grading of V by the root lattice $\mathbb{Z}[R]$ of R. Indeed, (4) becomes 2.3.4 if one defines

$$V^\sigma[\alpha] = \begin{cases} V_\alpha^+ & \sigma = +,\, \alpha \in R_1, \\ V_{-\alpha}^- & \sigma = -,\, \alpha \in R_{-1} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

This grading will be denoted by $\mathfrak{R}$ and called the *root grading induced by* $\mathcal{G}$.

Let $(\alpha, \beta) \subset R_1$ be a pair of collinear roots, hence e_α, e_β are collinear idempotents. (Such a pair does not exist if and only if $R = \dot{\mathrm{A}}_1$ or $R = \mathrm{B}_2$). The McCrimmon-Meyberg superalgebra of (e_α, e_β), as defined in 3.2, is defined on the Peirce space V_α^+ since

$$V_\alpha = V_2(e_\alpha) \cap V_1(e_\beta). \tag{7}$$

Indeed, $V_\alpha \subset V_2(e_\alpha) \cap V_1(e_\beta)$ since $\langle \alpha, \alpha^\vee \rangle = 2$ and $\langle \beta, \alpha^\vee \rangle = 1$. For the other inclusion, we note that always $V_2(e_\alpha) \cap V_1(e_\beta) = \oplus\{V_\gamma : \gamma \in R_1, \langle \gamma, \alpha^\vee \rangle = 2, \langle \gamma, \beta^\vee \rangle = 1\}$. For any $\gamma \in R_1$ satisfying $\gamma \neq \alpha$ and $\langle \gamma, \alpha^\vee \rangle = 2$ we have $\gamma \dashv \alpha \top \beta$ and therefore $\gamma \dashv \beta$ or $\gamma \perp \beta$ by length considerations ([**22**, 18.6.b(ii)]). In particular, $\langle \gamma, \beta^\vee \rangle \neq 1$ which implies (7).

The following lemma is immediate from (5). It reduces the classification of Jordan superpairs covered by a grid to the case of connected grids.

3.5. Lemma (Direct sums). *Let V be a Jordan superpair with a covering grid $\mathcal{G}$ whose associated 3-graded root system (R, R_1) is an orthogonal sum of 3-graded root systems $(R^{(i)}, R_1^{(i)})$, e.g. the decomposition of (R, R_1) into its irreducible components. Put $V^{(i)} = \oplus_{\alpha \in R_1^{(i)}} V_\alpha$. Then $V = \oplus_i V^{(i)}$ is a direct sum of ideals.*

3.6. Standard grids. In an arbitrary grid $\mathcal{G}$ the relations between idempotents are controlled by the associated 3-graded root system, but products of type $Q_{\bar{0}}(e)f$ or $\{efg\}$ for $e, f, g \in \mathcal{G}$ may fall outside of $\mathcal{G}$ even if $Q_{\bar{0}}(e)f$ or $\{efg\}$ are idempotents. Roughly speaking, standard grids are characterized by the condition that Jordan products of idempotents in $\mathcal{G}$ which are idempotents lie in $\pm\mathcal{G}$. To define standard grids, we need the following concepts.

A family $(e_0; e_1, e_2)$ of non-zero idempotents in V is a *triangle of idempotents* if
- (i) $e_0 \vdash e_1 \perp e_2 \dashv e_0$, and
- (ii) $Q_{\bar{0}}(e_0)e_1 = e_2$, $Q_{\bar{0}}(e_0)e_2 = e_1$ and $\{e_1\, e_0\, e_2\} = e_0$ (by [**31**, I.2.5], the first of these three equations implies the remaining two).

A family (e_1, e_2, e_3, e_4) of non-zero idempotents in a Jordan superpair V is a *quadrangle of idempotents* if for all indices mod 4 we have
- (i) $e_i \top e_{i+1} \perp e_{i+3}$ and
- (ii) $\{e_i\; e_{i+1}\; e_{i+2}\} = e_{i+3}$.

A family $(e_0; e_1, e_2, e_3)$ of non-zero idempotents in V is a *diamond of idempotents* if
- (i) (e_1, e_2, e_3) is a collinear family and $e_1 \vdash e_0 \dashv e_3$, $e_0 \perp e_2$;
- (ii) $\{e_0\, e_1\, e_2\} = e_3$, $\{e_1\, e_2\, e_3\} = 2e_0$, $\{e_2\, e_3\, e_0\} = e_1$, $\{e_3\, e_0\, e_1\} = e_2$ (the first of these four equations actually implies the remaining three, see [**31**, I.2.8]).

In the three definitions above the conditions (i) coincide with the definition of a triangle, quadrangle or diamond of roots ([**22**, 18.3]). To distinguish them from triangle of idempotents etc., we will refer to the configurations of roots as *root triangle*, *root quadrangle* or *root diamond* respectively.

A grid $\mathcal{G} = \{e_\alpha : \alpha \in R_1\}$ in a Jordan superpair is a *standard grid* ([**33**, 3.5]) if

(SG1) the idempotents corresponding to a root triangle $(\alpha; \beta, \gamma) \subset R_1$ form a triangle of idempotents;

(SG2) the idempotents corresponding to a root diamond $(\alpha; \beta, \gamma, \delta) \subset R_1$ form a diamond of idempotents;

(SG3) for every root quadrangle $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \subset R_1$ there exists a sign $\varepsilon \in \{\pm 1\}$ such that $(e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3}, \varepsilon e_{\alpha_4})$ is a quadrangle of idempotents.

Clearly, triangles and quadrangles of idempotents are examples of standard grids. A diamond creates a standard grid, namely a hermitian grid $\mathcal{H}(3)$ as defined in 4.8, see [**31**, Thm. I.2.11].

Every grid $\mathcal{G}$ is associated to a standard grid ([**33**, 3.7 and 3.8]). Such a standard grid is not unique, but one example can be constructed as follows. We choose a grid base B of (R, R_1) (see [**32**] or [**33**, 1.5]) and define $\widetilde{\mathcal{G}} = \{\tilde{g}_\alpha : \alpha \in R_1\}$ by induction on the height. For $\beta \in B$ we put $\tilde{g}_\beta = e_\beta$. For $\alpha \in R_1$ with $\mathrm{ht}(\alpha) \geq 3$ we choose a decomposition $\alpha = \gamma - \beta_1 + \beta_2$ with $\beta_i \in B$ and $\gamma \in R_1, \mathrm{ht}(\gamma) = \mathrm{ht}(\alpha) - 2$, and define $\tilde{g}_\alpha$ by

a) $\tilde{g}_\alpha = Q_{\bar{0}}(e_\gamma)e_{\beta_1}$ in case $\gamma = \beta_2$ and $(\gamma; \beta_1, \alpha)$ is a root triangle;

b) $\tilde{g}_\alpha = \{e_\gamma\, e_{\beta_1}\, e_{\beta_2}\}$ in case $(\beta_1, \beta_2, \alpha, \gamma)$ is a root quadrangle or $(\beta_1; \beta_2, \alpha, \gamma)$ is a root diamond.

Then $\widetilde{\mathcal{G}}$ is a standard grid with $\widetilde{\mathcal{G}} \approx \mathcal{G}$. It is called the *standard grid generated by* $\{e_\beta : \beta \in B\}$. It is unique in the following sense: if $\mathcal{G}'$ is another standard grid with $\{e_\beta : \beta \in B\} \subset \mathcal{G}'$ and with the same 3-graded root system as $\mathcal{G}$ (and $\widetilde{\mathcal{G}}$) then the idempotents in $\widetilde{\mathcal{G}}$ and in $\mathcal{G}'$ differ by a sign only ([**33**, 3.7]).

3.7. Refined root gradings of Jordan superpairs. Suppose that V is covered by a standard grid $\mathcal{G} = \{e_\alpha : \alpha \in R_1\}$ with associated 3-graded root system (R, R_1). We then have an induced root grading $\mathfrak{R}$ of V with grading group $\mathbb{Z}[R]$ as defined in 3.4.6.

A *refined root grading of* $(V, \mathcal{G})$ is a grading $(V^\sigma[\gamma] : \sigma = \pm, \gamma \in \mathit{\Gamma})$ of V with grading group $\mathit{\Gamma}$, written additively, such that the following two properties hold:

(i) There exists a group homomorphism $\varphi\colon \mathit{\Gamma} \to \mathbb{Z}[R]$ such that for every $\alpha \in R_1$ we have
$$V^\sigma_\alpha = \oplus_{\gamma \in \varphi^{-1}(\alpha)} V^\sigma[\sigma\gamma]\,.$$

(ii) Every e_α is homogeneous: $e_\alpha \in (V^+[\dot\alpha^+], V^-[\dot\alpha^-])$ for suitable $\dot\alpha^\pm \in \mathit{\Gamma}$.

Throughout we will use the following notation for a refined root grading with grading group $\mathit{\Gamma}$. Since $0 \neq e^\sigma_\alpha = Q_{\bar{0}}(e^\sigma_\alpha)e^{-\sigma}_\alpha \in V^\sigma[2\dot\alpha^\sigma + \dot\alpha^{-\sigma}]$ it follows that $\dot\alpha^\sigma = -\dot\alpha^{-\sigma}$. Hence, with $\dot\alpha := \dot\alpha^+$, we have $e_\alpha \in (V^+[\dot\alpha], V^-[-\dot\alpha])$. We put

$$\dot{\mathit{\Gamma}} := \text{ the subgroup of } \mathit{\Gamma} \text{ generated by } \{\dot\alpha : \alpha \in R_1\}$$
$$\mathit{\Gamma}^0 := \operatorname{Ker}\varphi.$$

We can therefore write

$$V^\sigma_\alpha = \oplus_{\lambda\in\Gamma^0} V^\sigma[\sigma\dot\alpha + \lambda]$$

for $\alpha \in R_1$. Our notation is influenced by the notations used in the theory of extended affine Lie algebras, see for example [**2**] or [**1**]. Indeed, for suitable choices of Γ and V the TKK-algebra of V is the core of an extended affine Lie algebra.

Remarks. 1) The definition above makes perfectly sense for an arbitrary covering grid $\mathcal{G}$ which is not necessarily a standard grid. This will however not lead to a more general structure. Indeed, let B be a grid base of (R, R_1) and let $\widetilde{\mathcal{G}}$ be the standard grid generated by $\{e_\beta : \beta \in B\}$, see 3.6. Then $\widetilde{\mathcal{G}}$ has the same root grading as $\mathcal{G}$ since $\widetilde{\mathcal{G}} \approx \mathcal{G}$. By the description of $\widetilde{\mathcal{G}}$ given in 3.6, every idempotent of $\widetilde{\mathcal{G}}$ is Γ-homogeneous. It is therefore no loss of generality to assume in the definition of a refined root grading that $\mathcal{G}$ is a standard grid. In fact, we can even assume that $\mathcal{G}$ is the standard grid generated by $\{e_\beta : \beta \in B\}$ for some grid base B of (R, R_1).

2) Refined root gradings naturally occur in the following set-up. Suppose, for simplicity, that k is a field of characteristic 0. Let $\mathfrak{h}_\mathcal{G}$ denote the span of all inner derivations $(D(e^+_\alpha, e^-_\alpha), -D(e^-_\alpha, e^+_\alpha)), \alpha \in R_1$. Then $\mathbb{Z}[R]$ imbeds as a subgroup of the dual space of $\mathfrak{h}_\mathcal{G}$ via $\alpha(\Delta(e^+_\beta, e^-_\beta)) = \langle \alpha, \beta^\vee\rangle$, [**34**, 3.2.c]. Assume further that $\mathfrak{h} \subset (\operatorname{Der} V)_{\bar 0}$ is an subalgebra of the derivation algebra $\operatorname{Der} V$ which acts diagonalizably on V and contains $\mathfrak{h}_\mathcal{G}$. The weight spaces of $\mathfrak{h}$ in V then define a refined root grading with grading group $\mathfrak{h}^*$. In this case the map φ can be taken to be the restriction of $\lambda \in \mathfrak{h}^*$ to $\mathbb{Z}[R] \subset \mathfrak{h}^*_\mathcal{G}$.

3) Generalizing 2), one can define refined root gradings of Lie algebras graded by a root system R ([**41**, §2]). For the case of a 3-graded R, refined root gradings are described in [**9**, 2.11].

4) For an easy example of refined root grading see 3.9. We will describe refined root gradings in terms of graded supercoordinate systems in section 4.

3.8. Lemma. (a) *Let $(V^\pm[\gamma] : \gamma \in \Gamma)$ be a refined root grading with grading group Γ. Then:*

(a.i) *$\varphi|\dot\Gamma$ is a group isomorphism onto $\mathbb{Z}[R]$ and $\Gamma = \dot\Gamma \oplus \Gamma^0$.*

(a.ii) *Put*

$$V^\sigma(\lambda) := \oplus_{\alpha\in R_1} V^\sigma[\sigma\dot\alpha + \lambda],\ \sigma = \pm\,.$$

Then $(V^\pm(\lambda); \lambda \in \Gamma^0)$ is a Γ^0-grading of the Jordan superpair V as defined in 2.3. In particular, $V(0)$ is a subpair of V containing $\mathcal{G}$:

(b) *Conversely, assume that Λ is an abelian group and that $(V^\pm\langle\lambda^\vee\rangle : \lambda \in \Lambda)$ is a Λ-grading of V which is compatible with the root grading $\mathfrak{R}$ in the following sense:*

(b.i) *$V^\sigma\langle\lambda^\vee\rangle = \oplus_{\alpha\in R_1} (V^\sigma\langle\lambda^\vee\rangle \cap V^\sigma_\alpha)$ for every $\lambda \in \Lambda$;*

(b.ii) *$\mathcal{G} \subset V\langle 0^\vee\rangle$.*

Put $\Gamma = \mathbb{Z}[R] \oplus \Lambda$ (direct sum of abelian groups), and for $\varrho \in \mathbb{Z}[R]$ and $\lambda \in \Lambda$ define

$$V^\sigma[\varrho \oplus \lambda] := \begin{cases} V^\sigma_\alpha \cap V^\sigma\langle\lambda^\vee\rangle & \text{if } \varrho = \sigma\alpha, \alpha \in R_1 \\ 0 & \text{otherwise} \end{cases}$$

Then $(V^\pm[\varrho \oplus \lambda] : \varrho \oplus \lambda \in \Gamma)$ is a refined root grading of $(V, \mathcal{G})$.

A refined root grading of $(V, \mathcal{G})$ with grading group $\mathbb{Z}[R] \oplus \Lambda$ will be called a *refined root grading of type* $(\mathfrak{R}, \Lambda)$. We will say V *has a refined root grading of type* $(\mathfrak{R}, \Lambda)$ if there exists a covering grid $\mathcal{G}$ with 3-graded root system $(R, R_1) = \mathfrak{R}$ such that $(V, \mathcal{G})$ has a refined root grading of type $(\mathfrak{R}, \Lambda)$. Because of the result above, every refined root grading of V is of type $(\mathfrak{R}, \Lambda)$ for some suitable $\mathfrak{R}$ and Λ.

Proof. (a.i) Let B be a grid base of (R, R_1). Because of the uniqueness of standard grids we may assume that $\mathcal{G}$ is the standard grid generated by $\{e_\beta : \beta \in R_1\}$, see 3.6. Then an induction on the height, using the formulas of [**22**, 18.4], shows that $\dot{\Gamma}$ is spanned by $\{\dot{\beta} : \beta \in B\}$. Since $\mathbb{Z}[R] = \oplus_{\beta \in B} \mathbb{Z}\beta$, it follows that $\varphi|\dot{\Gamma}$ is an isomorphism onto $\mathbb{Z}[R]$, and this then implies the second claim.

(a.ii) It is clear that $V^\sigma = \oplus_{\lambda \in \Gamma^0} V^\sigma(\lambda)$. The multiplication rules 2.3.4 and 2.3.5 hold because for $\alpha, \beta, \gamma \in R_1$ and $\lambda, \mu, \nu \in \Gamma^0$ there exists $\delta \in R_1$ such that

$$Q(V^\sigma[\sigma\dot{\alpha} + \lambda])V^{-\sigma}[-\sigma\dot{\beta} + \mu] \subset V^\sigma[\sigma\dot{\delta} + 2\lambda + \mu] \quad \text{and} \tag{1}$$

$$\{V^\sigma[\sigma\dot{\alpha} + \lambda]\, V^{-\sigma}[-\sigma\dot{\beta} + \mu]\, V^\sigma[\sigma\gamma + \nu]\} \subset V^\sigma[\sigma\dot{\delta} + \lambda + \mu + \nu]\,. \tag{2}$$

Indeed, since we have a Γ-grading the left side of (1) lies in $V^\sigma[\sigma(2\dot{\alpha} - \dot{\beta}) + 2\lambda + \mu]$. We can assume that it is non-zero. Then, because of 3.4.4, we have $\varphi(2\dot{\alpha} - \dot{\beta}) = 2\alpha - \beta =: \delta \in R_1$ whence $2\dot{\alpha} - \dot{\beta} = \dot{\delta}$ by injectivity of $\varphi|\dot{\Gamma}$. (2) is proven similarly.

(b) is a straightforward verification.

3.9. Split Jordan superpairs. Because of [**31**, Thm I.4.3] and the defining properties of standard grids, the $\mathbb{Z}$-span of any standard grid $\mathcal{G}$ in V,

$$\mathbb{Z}[\mathcal{G}] = \oplus_{g \in \mathcal{G}} (\mathbb{Z}g^+, \mathbb{Z}g^-)$$

is a subpair of the Jordan superpair V considered as a superpair over the integers. It follows easily from the properties mentioned above that the following are equivalent for a Jordan superpair V over some base superring S:

(a) there exists a grid $\mathcal{G} \subset V_{\bar{0}}$ such that $\{g^\sigma : g \in \mathcal{G}\}$ is a basis of the S-supermodule V^σ;

(b) there exists a standard grid $\mathcal{G} \subset V_{\bar{0}}$ such that $\{g^\sigma : g \in \mathcal{G}\}$ is a basis of the S-supermodule V^σ;

(c) there exists a standard grid $\mathcal{G} \subset V_{\bar{0}}$ such that V is isomorphic to the S-superextension $\mathbb{Z}[\mathcal{G}]_S$ of $\mathbb{Z}[\mathcal{G}]$ by S as defined in 2.6.

Generalizing a concept from [**36**, 3], V is called *split* or *split of type* $\mathcal{G}$ if the conditions (a) – (c) are fulfilled. In this case, $\mathcal{G}$ is a covering grid of V.

Let $\mathcal{G} = \{e_\alpha : \alpha \in R_1\}$ be a standard grid and suppose that S has a Λ-grading as defined in 1.5. The split Jordan superpair $V = \mathbb{Z}[\mathcal{G}]_S$ then has a Λ-grading with homogeneous parts $S_\lambda \otimes \mathbb{Z}[\mathcal{G}]$ which is compatible with the root grading of V. Hence, by 3.8.b, V has a refined root grading with grading group $\mathbb{Z}[R] \oplus \Lambda$.

4. Refined root gradings of Jordan superpairs.

4.1. Preparation. *Unless stated otherwise, in this section V will denote a Jordan superpair over some base superring S.* Suppose V is covered by a grid $\mathcal{G}$ with associated 3-graded root system (R, R_1). Every grid is the union of connected, pairwise orthogonal grids or, equivalently, every 3-graded root system is the orthogonal sum of irreducible root systems. Hence, by 3.5, V is a direct sum of ideals each covered by a connected subgrid. For the purpose of classification we may therefore assume that $\mathcal{G}$ is connected, or equivalently, that R is irreducible.

Connected grids in Jordan triple systems are classified up to association in [**31**, II]. As explained in 3.3, this can be applied to $\mathcal{G} \subset V_{\bar{0}}$. Since idempotents are associated in V if and only if they are associated in $V_{\bar{0}}$ and since a grid associated to a covering grid is still covering, it follows from the classification of grids that we may assume that $\mathcal{G}$ is exactly one of the seven types of grids listed below. For the convenience of the reader the definition of these grids is given in the subsections indicated. All of these seven grids are connected standard grids. Their associated 3-graded root systems are the ones with the corresponding names, see for example [**22**, 17.8, 17.9].

To classify Jordan superpairs covered by a grid now means to define for each of these seven types a so-called *standard example* of a Jordan superpair covered by $\mathcal{G}$ and to prove a *coordinatization theorem*, i.e., to show that an abstract Jordan superpair covered by $\mathcal{G}$ is isomorphic to a standard example. For the convenience of the reader the list of the various coordinatization results is indicated in the column "coordinatization".

Name of grid	Definition	Coordinatization
rectangular grid $\mathcal{R}(M, N)$ $(1 \le \lvert M\rvert \le \lvert N\rvert)$	4.2	$4.3, 4.5, 4.7$
hermitian grid $\mathcal{H}(I)$ $(2 \le \lvert I\rvert)$	4.8	$4.9, 4.12$
even quadratic form grid $\mathcal{Q}_e(I)$ $(3 \le \lvert I\rvert)$	4.13	4.14
odd quadratic form grid $\mathcal{Q}_o(I)$ $(2 \le \lvert I\rvert)$	4.15	4.16
alternating (= symplectic) grid $\mathcal{A}(I)$ $(5 \le \lvert I\rvert)$	4.17	4.18
bi-Cayley grid $\mathcal{B}$	4.19	4.20
Albert grid $\mathcal{A}$	4.21	4.22

Once one knows the structure of a Jordan superpair V covered by a grid $\mathcal{G}$, i.e., a Jordan superpair with a root grading $\mathfrak{R}$, one can then easily describe the refined root gradings of $(V, \mathcal{G})$. We will employ the terminology of 3.8 and study refined root gradings of type $(\mathfrak{R}, \Lambda)$ where Λ is an abelian group.

Although refined root gradings are more general than root gradings, i.e., the gradings obtained from covering grids, we feel it is more natural to formulate our coordinatization results first for covering grids and then indicate the necessary "refinements" for refined root gradings – after all, this is what the terminology suggests. Let us point out that the coordinatization theorems for refined root gradings are new even in the case of Jordan pairs. Because of this they cannot be obtained by applying the Even Rules Principle ([**6**, 1.7]. Of course, with some good will this principle can be applied in the ungraded case, i.e., the coordinatization of Jordan superpairs covered by grids. However, since the proofs of the various coordinatization theorems are quite similar, we will only present three of them as representative examples (4.5, 4.9 and 4.16).

4.2. Rectangular grids. For arbitrary (possibly infinite) non-empty sets M, N with $|M| \le |N|$ a family $\mathcal{R}(M,N) = \{e_{mn} : m \in M, n \in N\}$ of non-zero idempotents in V is called a *rectangular grid of size* $M \times N$ if it has the following properties:

(i) if $|M| = 1$ then $\mathcal{R}(M,N)$ is a collinear family, 3.3,

(ii) for distinct $m, m' \in M$ and $n, n' \in N$ the subfamily $(e_{mn}, e_{mn'}, e_{m'n'}, e_{m'n})$ of $\mathcal{R}(M,N)$ is a quadrangle of idempotents, 3.6, and

(iii) $\mathcal{R}(M,N)$ is pure, 3.3.

For finite M, N with $|M| = m$ and $|N| = n$ we will write $\mathcal{R}(M,N) = \mathcal{R}(m,n)$. The 3-graded root system (R, R_1) associated to a rectangular grid $\mathcal{R}(M,N)$ is the rectangular grading $\dot{\mathrm{A}}_I^{M,N}$ for $I = M \dot\cup N$ as defined in [**22**, 17.8]. We have $R = \{\varepsilon_i - \varepsilon_j : i, j \in I\}$ and $R_1 = \{\varepsilon_m - \varepsilon_n : m \in M, n \in N\}$.

The classification of Jordan superpairs covered by a rectangular grid naturally leads to three subcases: $(|M|, |N|) = (1,1)$, $(|M|, |N|) = (1,2)$ and $|M| + |N| \ge 4$. The last one will be dealt with in 4.6 and 4.7, for the second see 4.4 and 4.5.

In the first case we have $R = \mathrm{A}_1$. The standard example for such a Jordan superpair is (J, J) where J is a unital Jordan superalgebra J, 2.12. Indeed, (J, J) is covered by the grid $\mathcal{G} = \{e\}$ for $e = (1_J, 1_J)$. A Λ-grading of J, as defined in 2.11, gives rise to a refined root grading of (J, J) of type (A_1, Λ). Conversely, if V is a Jordan superpair covered by a single idempotent e then $V = V_2(e)$. Since $Q_{\bar 0}(e^\sigma) Q_{\bar 0}(e^{-\sigma})$ projects onto $V_2^\sigma(e)$ it follows that $V = V_2(e)$ if and only if e^σ is invertible, and in this case we have $(e^\sigma)^{-1} = e^{-\sigma}$. Hence we can apply 2.12 and obtain $V \cong (J, J)$ via the isomorphism $(\mathrm{Id}_{V^+}, Q_{\bar 0}(e^-)) \colon V \to (J, J)$. In case V has a refined root grading, this isomorphism becomes a graded isomorphism, where J has the induced grading given by $J_\lambda = V_\lambda^+$. These results are summarized below.

4.3. A_1-Coordinatization. *A Jordan superpair V over S is covered by a single idempotent if and only if V is isomorphic to the superpair (J, J) of a unital Jordan superalgebra J over S. More generally, V has a refined root grading of type (A_1, Λ) if and only if V is graded-isomorphic to (J, J) where J is a unital Jordan superalgebra with a Λ-grading.*

4.4. $\mathcal{R}(1,2)$ and alternative 1×2-matrices. A rectangular grid $\mathcal{R}(1,2)$ is the same as a collinear pair (e, f). A collinear pair (e, f) covers a Jordan superpair V if and only if $V = V_2(e) \oplus V_2(f)$ and $V_2(e) = V_1(f), V_2(f) = V_1(e)$.

Before we describe Jordan superpairs covered by a collinear pair, let us recall the classical situation. One knows ([**28**, 2.2], [**35**, (3.2.3)]) that a Jordan pair U is covered by a collinear pair (e,f) if and only if U is isomorphic to the Jordan pair $\mathbb{M}_{12}(B) := (\mathrm{Mat}(1,2;B), \mathrm{Mat}(2,1;B))$ where B is a unital alternative algebra, which one can take to be the McCrimmon-Meyberg algebra of the collinear pair (e,f). For $(x,y) \in \mathbb{M}_{12}(B)$ written in the form $x = (x_1\, x_2), y^{\mathrm{T}} = (y_1\, y_2)$ the Jordan pair products are

$$\begin{aligned} Q^+(x)y &= x(yx) = \big(\, x_1(y_1x_1) + x_2(y_2x_1)\,,\, x_1(y_1x_2) + x_2(y_2x_2)\,\big) \\ Q^-(y)x &= (yx)x = \big(\,(y_1x_1)y_1 + (y_1x_2)y_2\,,\,(y_2x_1)y_1 + (y_2x_2)y_2\,\big)^{\mathrm{T}} \end{aligned}$$

Of course, because of the Moufang identity $a(ba) = (ab)a$, some of the brackets above are superfluous. They are included for easier comparison with the supercase discussed below.

In the supercase, we consider a unital alternative superalgebra A over S. For natural numbers m,n we denote by $\mathrm{Mat}(m,n;A)$ the $m \times n$-matrices with entries from A. This becomes an S-supermodule whose even part is $\mathrm{Mat}(m,n;A_{\bar 0})$ and whose odd part consists of those $m\times n$-matrices for which all entries lie in $A_{\bar 1}$. (Warning: Matrices over A are also defined in [**19**, §3] and [**23**, Chap.3 §1.7]. The matrices considered here all have even rows and columns in the terminology of [**19**] and [**23**].) In particular,

$$\mathbb{M}_{12}(A) := (\mathrm{Mat}(1,2;A), \mathrm{Mat}(2,1;A))$$

is a pair of S-supermodules. There are canonical S-quadratic maps $Q = (Q^+, Q^-)$ on $\mathbb{M}_{12}(A)$ such that the Grassmann envelope of $(\mathbb{M}_{12}(A), Q)$ is the Jordan pair $\mathbb{M}_{12}(G(A))$. Namely, for $x_{\bar 0} = (x_{\bar 01}\, x_{\bar 02}) \in \mathrm{Mat}(1,2;A_{\bar 0}), y \in \mathrm{Mat}(2,1;A)$ with $y^{\mathrm{T}} = (y_1\, y_2)$ and arbitrary homogeneous $x, z \in \mathrm{Mat}(1,2;A), y \in \mathrm{Mat}(2,1;A)$ we define

$$\begin{aligned} Q^+_{\bar 0}(x_{\bar 0})y &= \Big(\, x_{\bar 01}(y_1x_{\bar 01}) + x_{\bar 02}(y_2x_{\bar 01})\,,\, x_{\bar 01}(y_1x_{\bar 02}) + x_{\bar 02}(y_2x_{\bar 02})\,\Big) \\ \{x\,y\,z\} &= \Big(\, x_1(y_1z_1) + x_2(y_2z_1) + (-1)^{|x||y|+|x||z|+|y||z|}\,\big(\,z_1(y_1x_1) + z_2(y_2x_1)\,\big)\,, \\ &\qquad x_1(y_1z_2) + x_2(y_2z_2) + (-1)^{|x||y|+|x||z|+|y||z|}\,\big(z_1(y_1x_2) + z_2(y_2x_2)\,\big)\Big) \end{aligned}$$

One obtains $Q^-_{\bar 0}$ and the other supertriple product $\{...\}$: $V^- \times V^+ \times V^- \to V^-$ by shifting the brackets in the expressions above one position to the left and taking the transpose. With respect to this product

$$e = \left((1\,0), \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad \text{and} \quad f = \left((0\,1), \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

are collinear idempotents which cover $\mathbb{M}_{12}(A)$. Moreover, any Λ-grading of A, as defined in 1.5, gives rise to a refined root grading of $\mathbb{M}_{12}(A)$ of type (A_2, Λ) by defining the homogeneous λ-space as $\mathbb{M}_{12}(A)\langle\lambda\rangle = (\mathrm{Mat}(1,2;A_\lambda), \mathrm{Mat}(2,1;A_\lambda))$.

4.5. A_2**-Coordinatization Theorem.** *A Jordan superpair V over S is covered by a collinear pair (e,f) if and only if V is isomorphic to a Jordan superpair $\mathbb{M}_{12}(A)$ of a unital alternative superalgebra A over S. One can take A to be the McCrimmon-Meyberg superalgebra of (e,f).*

In this case, $(V,\{e,f\})$ has a refined root grading of type (A_2,Λ) if and and if A is Λ-graded. Then V and $\mathbb{M}_{12}(A)$ are graded-isomorphic.

Proof. Suppose V is covered by a collinear pair (e,f), and let A be its McCrimmon-Meyberg superalgebra. Thus $V = V_1 \oplus V_2$ where $V_i = V_i(f)$ and $A = V_1^+$ as S-modules. We define $\varphi\colon V \to \mathbb{M}_{12}(A)$ by

$$\varphi^+(x_1 \oplus x_2) = (x_1\,,\, \{e^+ f^- x_2\})\,, \quad \varphi^-(y_1 \oplus y_2) = \begin{pmatrix} Q_{\bar 0}(e^+)y_1 \\ Q_{\bar 0}(e^+)\{e^- f^+ y_2\} \end{pmatrix}$$

and claim that φ is an isomorphism of Jordan superpairs over S. By the homomorphism criterion 2.3.2 it suffices to show that the Grassmann envelope $G(\varphi)$ is an isomorphism. By 3.4.3 we know that $G(V)$ is covered by the collinear pair $(1\otimes e, 1\otimes f)$, and by 3.2 the Grassmann envelope of A is the McCrimmon-Meyberg algebra of $(1\otimes e, 1\otimes f)$. Since $\big(G(\mathrm{Mat}(1,2;A)), G(\mathrm{Mat}(2,1;A))\big) = \big(\mathrm{Mat}(1,2;G(A)), \mathrm{Mat}(2,1;G(A)\big)$ one then finds that $G(\varphi)\colon G(V) \to \mathbb{M}_{12}(G(A))$ is exactly the map used in the A_2-coordinatization of Jordan pairs ([**28**, 2.2] and [**35**, (3.2.3)]) and is therefore an isomorphism.

Now suppose that $(V,\{e,f\})$ has a refined root grading of type (A_2,Λ) with homogeneous spaces $V^\sigma\langle\lambda\rangle$ in the notation of 3.8.b. Define $A_\lambda = V_\lambda^+ \cap A$. Since $e,f \in V\langle 0\rangle$ it easily follows from the product formula 3.2.1 that $A = \oplus_{\lambda\in\Lambda} A_\lambda$ is a Λ-grading of A. Moreover, the isomorphism φ defined above is a graded isomorphism since $\varphi(V\langle\lambda\rangle) \subset \mathbb{M}_{12}(A)_\lambda$. This proves one direction of the theorem, the other has been established in 4.4.

4.6. Rectangular matrix superpairs. Let A be a unital associative superalgebra over S, and let M,N be arbitrary sets. A *finite matrix over A of size $M\times N$* is a matrix $x = (x_{mn})_{m\in M, n\in N}$ where all $x_{mn}\in A$ and $x_{mn}\neq 0$ for only a finite number of indices m,n. Generalizing the notation of 4.4 we denote by $\mathrm{Mat}(M,N;A)$ the left A-module of all finite matrices over A of size $M\times N$. By restriction of scalars, this becomes an S-supermodule with even part $\mathrm{Mat}(M,N;A_{\bar 0})$ and odd part $\mathrm{Mat}(M,N;A_{\bar 1})$ (in obvious notation).

Let P be the disjoint union $P = M \dot\cup N$. With respect to the usual matrix multiplication, $\mathrm{Mat}(P,P;A)$ is an associative superalgebra over S. By 2.14 we therefore have a Jordan superpair $(\mathrm{Mat}(P,P;A), \mathrm{Mat}(P,P;A))$ over S. The *rectangular matrix superpair* of size $M\times N$ and with coordinate algebra A is the pair

$$\mathbb{M}_{MN}(A) = (\mathrm{Mat}(M,N;A), \mathrm{Mat}(N,M;A))$$

which we consider as a subpair of $(\mathrm{Mat}(P,P;A), \mathrm{Mat}(P,P;A))$ via the imbedding of $\mathbb{M}_{MN}(A)$ in $\mathbb{M}_{PP}(A)$ given by

$$(x,y) \quad\mapsto\quad \left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right).$$

Thus, the structure maps of $\mathbb{M}_{MN}(A)$ are

$$Q_{\bar{0}}^{\sigma}(x_{\bar{0}})y = x_{\bar{0}}yx_{\bar{0}}\,; \quad \{x\,y\,z\} = xyz + (-1)^{|x||y|+|x||z|+|y||z|}\,zyx \tag{1}$$

where on the right hand side of the equations we have the usual matrix multiplication.

Let E_{ij} be the matrix whose (ij)-entry is 1 and whose other entries are zero. Then $e_{ij} = (E_{ij}, E_{ji}) \in \mathbb{M}_{MN}(A)$ is an idempotent and $\mathcal{R}(M,N) = \{e_{mn} : m \in M, n \in N\}$ is a rectangular grid of size $M \times N$ which covers $\mathbb{M}_{MN}(A)$. If we choose the obvious map $R_1 \to \mathcal{R}(M,N)$ which sends $\varepsilon_i - \varepsilon_j$ to e_{ij}, then the joint Peirce spaces of $\mathcal{R}(M,N)$ are

$$\mathbb{M}_{MN}(A)_{\varepsilon_i - \varepsilon_j} = (AE_{ij}, AE_{ji}).$$

In particular, for $(|M|,|N|) = (1,1)$ or $(1,2)$ we obtain special cases of 4.3 and 4.4: $J = A^+$ in the first case and A associative in the second. It follows from 4.7 below that associative coordinates are necessary and sufficient for $|I| + |J| \geq 4$. In the ungraded case, this coordinatization result is the super version of [**28**, 3.4] and [**35**, (3.2.3)].

We have seen that $\mathbb{M}_{MN}(A)$ has a root grading of type $\dot{\mathrm{A}}_I^{MN}$. If $A = \oplus_{\lambda \in \Lambda} A_\lambda$ is a Λ-grading, we obtain a refined root grading of type $(\dot{\mathrm{A}}_I^{MN}, \Lambda)$ by putting $\mathbb{M}_{MN}(A)\langle\lambda\rangle = (\mathrm{Mat}(M,N;A_\lambda), \mathrm{Mat}(N,M;A_\lambda))$.

The proof of the following coordinatization theorem is analogous to the proof of 4.5, using the rectangular coordinatization theorems of Jordan pairs [**35**, (3.2.3)].

4.7. Rectangular Coordinatization Theorem. *Let V be a Jordan superpair over S. Then V is covered by a rectangular grid $\mathcal{R}(M,N)$ with $|M| + |N| \geq 4$ if and only if, as a Jordan superpair over S, V is isomorphic to a rectangular matrix superpair $\mathbb{M}_{MN}(A)$ where A is a unital associative S-superalgebra. As A we can take the McCrimmon-Meyberg superalgebra of a collinear pair $(e_{mn}, e_{mn'})$ for some choice of $m \in M$ and $n, n' \in N, n \neq n'$.*

In this case, $(V, \mathcal{R}(M,N))$ has a refined root grading of type $(\dot{\mathrm{A}}_I^{MN}, \Lambda)$ if and only if A is Λ-graded, and we then even have a graded isomorphism $V \cong_\Lambda \mathbb{M}_{MN}(A)$.

4.8. Hermitian grids. Let I be an arbitrary set with $|I| \geq 2$. A *hermitian grid of size I* is a family $\mathcal{H}(I) = \{h_{ij} = h_{ji} : i,j \in I\} \subset V$ of non-zero idempotents built out of triangles and diamonds, as defined in 3.6: for distinct $i,j,k \in I$ we have
(i) $(h_{ij}; h_{ii}, h_{jj})$ is a triangle of idempotents, and
(ii) $(h_{ii}; h_{ij}, h_{jk}, h_{ki})$ is a diamond of idempotents.
A hermitian grid is a connected (in general non-pure) standard grid. Its associated 3-graded root system is isomorphic to the hermitian grading $\mathrm{C}_I^{\mathrm{her}}$ determined on the root system $R = \mathrm{C}_I = \{\pm\varepsilon \pm \varepsilon_j : i,j \in I\}$ by $R_1 = \{\varepsilon_i + \varepsilon_j : i,j \in I\}$. The canonical bijection between R_1 and $\mathcal{H}(I)$ is given by $\varepsilon_i + \varepsilon_j \mapsto h_{ij}$.

As we will see, the description of Jordan superpairs V covered by a hermitian grid naturally falls into two cases: $|I| = 2$ and $|I| \geq 3$. The latter case will be dealt with in 4.12. In the first case, $R = \mathrm{C}_2 = \mathrm{B}_2$ and $\mathcal{H}(I)$ is a triangle of idempotents, say $\mathcal{H}(I) = (h_{12}; h_{11}, h_{22})$.

The standard example of a Jordan superpair covered by such a triangle is (J, J) where J is a unital Jordan superalgebra containing a pair (c_1, c_2) of orthogonal even idempotents which are supplementary, i.e., $c_1 + c_2 = 1_J$, and strongly connected, i.e., there exists an even element u in the Peirce space J_{12} such that $u^2 = c_1 + c_2$. Indeed, in this case $h_{12} = (u, u), h_{11} = (c_1, c_1), h_{22} = (c_2, c_2)$ form a triangle which covers (J, J). For this example a refined root grading is obtained by taking a Λ-grading of the Jordan superalgebra J, 2.11, which is compatible with the C_2-grading: we have $J = \oplus_{\lambda\in\Lambda} J_\lambda$ such that each $J_\lambda = (J_{11} \cap J_\lambda) \oplus (J_{12} \cap J_\lambda) \oplus (J_{22} \cap J_\lambda)$. The following coordinatization theorem says that this example is in fact the general case.

4.9. C_2-Coordinatization Theorem. *Let V be a Jordan superpair over S. Then V is covered by a hermitian grid $\mathcal{H}(I)$, $|I| = 2$ if and only if $V \cong (J, J)$ where J is a Jordan superalgebra over S which contains two strongly connected supplementary orthogonal idempotents.*

In this case, $(V, \mathcal{H}(I))$ has a refined root grading of type $(\mathrm{C}_2^{\mathrm{her}}, \Lambda)$ if and J has a Λ-grading compatible with the $\mathrm{C}_2^{\mathrm{her}}$-grading, and then $V \cong_\Lambda (J, J)$.

Proof. Suppose V is a Jordan superpair covered by a triangle $(h_{12}; h_{11}, h_{22})$. Then $V = V_{11} \oplus V_{12} \oplus V_{22}$ where V_{ij} are the Peirce spaces of the orthogonal system (h_{11}, h_{22}). It follows that $c = h_{11} + h_{22}$ is an invertible idempotent in V. Hence, by 2.12, $V \cong (J, J)$ where J is the c^--isotope of V. It is then easily checked that $c_1 = h_{11}^+$ and $c_2 = h_{22}^-$ are supplementary orthogonal idempotents which are strongly connected by $u = h_{12}^+$. In view of what has been said in 4.8, this proves the coordinatization theorem for root gradings. The proof for refined root gradings is then immediate (compare the proof of 4.3).

Remark. Examples of Jordan superalgebras with a covering triangle will be given in 4.11 and 4.15. Even in the classical case, the structure of Jordan pairs covered by a triangle is unknown in general. However, one has a classification in the case of a simple Jordan pair ([**29**]) and also in the case of the coordinate algebra of an extended affine Lie algebra of type C_2 ([**1**, §4]).

4.10. Ample subspaces. Let A be a unital alternative superalgebra over S. The *nucleus of* A is the submodule $N(A) = \{n \in A : (n, A, A) = 0\}$ where $(.\,, .\,, .)$ denotes the associator, see 1.15. Let π be an involution of A, as defined in 2.14. A S-submodule $A_0 \subset A$ is called an *ample subspace of* (A, π) if

(i) $1 \in A_0 \subset (H(A, \pi) \cap N(A))$,

(ii) $a_{\bar 0} A_0 a_{\bar 0}^\pi \subset A_0$ for all $a_{\bar 0} \in A_{\bar 0}$ and

(iii) $a(b_0 c^\pi) + (-1)^{|a||b_0|+|a||c|+|b_0||c|}\, c(b_0 a^\pi) \in A_0$ for all homogeneous $a, c \in A$ and $b_0 \in A_0$.

Note that (i) and (iii) imply $a + a^\pi \in N(A)$, from which it easily follows that $a(ba^\pi) = (ab)a^\pi$ for all $a, b \in A$. We can therefore leave out the brackets in an expression aba^π, as we have done in (ii).

All concepts in the definition of an ample subspace are compatible with taking Grassmann envelopes: π is an involution of A if and only if $G(\pi)$ is an involution of $G(A)$, $G(H(A, \pi)) = H(G(A), G(\pi))$ and $G(N(A)) = N(G(A))$. It is then easily seen that A_0 is an ample subspace for (A, π) if and only if $G(A_0)$ is an ample subspace for $(G(A), G(\pi))$ in the classical sense, i.e. (i) and (ii) hold with obvious meaning. Because of this connection and [**11**, page 1.47] we have the following criterion for the existence of an ample subspace: An ample subspace exists if and only if π is a *nuclear involution* in the sense that

(a) $a_{\bar 0} a_{\bar 0}^{\pi} \in N(A)$ for all $a_{\bar 0} \in A_{\bar 0}$, and

(b) $ab^{\pi} + (-1)^{|a||b|}\, ba^{\pi} \in N(A)$ for all homogeneous $a, b \in A$.

In this case,

$$\begin{aligned} A_{0,\min} &= S\text{-span}(\{a_{\bar 0} a_{\bar 0}^{\pi} : a_{\bar 0} \in A_{\bar 0}\} \cup \{ab^{\pi} + (-1)^{|a||b|}\, ba^{\pi} : a, b \in A\}) \\ \text{and} \quad A_{0,\max} &= H(A, \pi) \cap N(A) \end{aligned}$$

are ample subspaces, and hence $A_{0,\min} \subset A_0 \subset A_{0,\max}$ holds for every ample subspace A_0. In particular, if $\frac{1}{2} \in k$ then $A_{0,\min} = A_{0,\max}$ is the only ample subspace.

Examples. Let (A, π, A_0) be an alternative algebra over some base ring k with involution π and ample subspace A_0. If S is a k-superextension then the canonical S-superextensions $(S \otimes_k A, \mathrm{Id} \otimes_k \pi, S \otimes_k A_0)$ are an example of an alternative S-superalgebra with involution and ample subspace. More genuine super examples have been found by Shestakov in [**40**]. With the notation of that paper, the superalgebras $\mathbf{B(1,2)}$ and $\mathbf{B(4,2)}$ are simple alternative superalgebras defined over fields of characteristic 3 (!). Both have a nuclear (even central) involution. The corresponding Jordan superalgebras of 3×3-hermitian matrices are simple Jordan superalgebras ([**40**, Th.3] – these are examples ix) and x) in the Racine-Zelmanov list [**38**]). The corresponding hermitian matrix superpair of 4.11 are simple Jordan superpairs (2.12(a) or [**8**, 3.10]).

4.11. Hermitian matrix superpairs. To motivate the construction below we will start with an example of a Jordan superpair covered by a hermitian grid $\mathcal{H}(I), |I| \geq 2$, which, however, will turn out to be the general case for $|I| \geq 4$.

Let A be a unital associative S-superalgebra with involution π. We have then seen in 4.6 that $\mathrm{Mat}(I, I; A)$ is an associative superalgebra over S. The map $x = (x_{ij}) \mapsto x^* := x^{\pi\,\mathrm{T}} = (x_{ji}^{\pi})$ is an involution of the superalgebra $\mathrm{Mat}(I, I; A)$. Hence, if we define $\mathrm{H}_I(A, \pi) = \{x \in \mathrm{Mat}(I, I; A) : x = x^*\}$ then, by 2.14.2,

$$\mathbb{H}_I(A, \pi) := (H_I(A, \pi), \mathrm{H}_I(A, \pi))$$

is a Jordan S-superpair with quadratic maps given by matrix multiplication. Note that the diagonal elements of $x \in H_I(A, \pi)$ lie in $H(A, \pi)$. More generally, let A_0 be an ample subspace for (A, π) and define $\mathrm{H}_I(A, A_0, \pi) = \{x = (x_{ij}) \in \mathrm{Mat}(I, I; A) : x = x^*,\ \text{all } x_{ii} \in A_0\}$. Then

$$\mathbb{H}_I(A, A_0, \pi) := (\mathrm{H}_I(A, A_0, \pi), \mathrm{H}_I(A, A_0, \pi))$$

is a subpair of $\mathbb{M}_{II}(A)$ and hence itself a Jordan S-superpair. We recall from 4.10 that $\mathbb{H}_I(A, A_0, \pi) = \mathbb{H}_I(A, \pi)$ if $\frac{1}{2} \in k$.

The S-module $\mathrm{H}_I(A, A_0, \pi)$ is spanned by elements of type

$$a[ij] = aE_{ij} + a^{\pi} E_{ji},\ (a \in A, i \neq j) \quad \text{and} \quad a_0[ii] = a_0 E_{ii}\ (a_0 \in A_0).$$

The Jordan superpair product of $\mathbb{H}_I(A, A_0, \pi)$ is therefore known once it is known for this spanning set. Because all products of elements in our spanning set lie in an $\mathrm{H}_{I'}(A, A_0, \pi)$ for finite I' it is sufficient to consider I finite, in which case $\mathbb{H}_I(A, A_0, \pi)$ is the Jordan superpair associated to a unital Jordan superalgebra J, whose quadratic map we will denote by U. In the formulas below, $a_{0\bar 0} \in A_0 \cap A_{\bar 0}, a_0, b_0, c_0 \in A_0, a_{\bar 0} \in A_{\bar 0},\ a, b, c \in A$ (homogeneous if necessary) and $i, j, k, l \in I$ are pairwise distinct.

$$\begin{aligned}
U_{\bar 0}(a_{0\bar 0}[ii])b_0[ii] &= a_{0\bar 0}b_0a_{0\bar 0}[ii],\\
\{a_0[ii]\ b_0[ii]\ c_0[ii]\} &= (a_0b_0c_0 + (-1)^{|a_0||b_0|+|a_0||c_0|+|b_0||c_0|}c_0b_0a_0)[ii],\\
U_{\bar 0}(a_{\bar 0}[ij])b[ji] &= a_{\bar 0}ba_{\bar 0}[ij],\\
\{a[ij]\ b[ji]\ c[ij]\} &= \big(a(bc) + (-1)^{|a||b|+|a||c|+|b||c|}\, c(ba)\big)[ij],\\
U_{\bar 0}(a_{\bar 0}[ij])b_0[jj] &= a_{\bar 0}b_0a_{\bar 0}^{\pi}[ii],\\
\{a[ij]\ b_0[jj]\ c[ji]\} &= \big(a(bc) + (-1)^{|a||b_0|+|a||c|+|b_0||c|}\, c(b_0a)\big)[ij],\\
\{a_0[ii]\ b_0[ii]\ c[ij]\} &= a_0b_0c[ij],\\
\{a_0[ii]\ b[ij]\ c_0[jj]\} &= a_0bc_0[ij],\\
\{a[ij]\ b[ji]\ c_0[ii]\} &= \big(abc_0 + (-1)^{|a||b|+|a||c_0|+|b||c_0|}\ c_0b^{\pi}a^{\pi}\big)[ii],\\
\{a[ij]\ b[ji]\ c[ik]\} &= a(bc)[ik],\\
\{a[ij]\ b_0[jj]\ c[jk]\} &= ab_0c[ik],\\
\{a[ij]\ b[jk]\ c[ki]\} &= \big(a(bc) + (-1)^{|a||b|+|a||c|+|b||c|}(c^{\pi}b^{\pi})a^{\pi}\big)[ii],\\
\{a[ij]\ b[jk]\ c[kl]\} &= abc[il].
\end{aligned}$$

(Some of the parentheses in the products are of course not necessary since A is associative, but they will get their meaning below.) The formulas in particular imply that for $i, j \in I$ the elements

$$h_{ii} = (1[ii], 1[ii]) \quad \text{and} \quad h_{ij} = (1[ij], 1[ij]) = h_{ji},\ i \neq j,$$

are idempotents such that $\mathcal{H}(I) = \{h_{ij} : i, j \in I\}$ is a hermitian grid which covers $\mathbb{H}_I(A, A_0, \pi)$. The joint Peirce spaces are $(A[ij], A[ij])$ for $i \neq j$ and $(A_0[ii], A_0[ii])$.

We now consider the case $2 \leq |I| \leq 3$, and replace the associative superalgebra A by a unital alternative S-superalgebra, also denoted A. As before, we assume that π is an involution and that A_0 is an ample subspace for (A, π). We put $J = H_I(A, A_0, \pi)$ and use the formulas above (with the exception of the last one since $|I| \leq 3$) to define a quadratic map $U\colon J \to \mathrm{End}_S\, J$. The Grassmann envelope of this U satisfies all the formulas of [**35**, 4.1] (or [**11**, page 2.15]) and hence J is a unital Jordan superalgebra over S. (That the Grassmann envelope of J is a unital Jordan algebra has been proven by McCrimmon, see [**11**, Ch. II.2, page 2.17]; for the special case when π is a central involution, i.e., all norms aa^{π} are central, one can find a published proof in [**26**, Thm. 3].) As in the associative case, the Jordan superpair (J, J) is covered by a hermitian grid.

The Jordan superpairs $\mathbb{H}_I(A, A_0, \pi) = (H_I(A, A_0, \pi), \mathrm{H}_I(A, A_0, \pi))$ with A alternative for $2 \leq |I| \leq 3$ and A associative for $|I| \geq 4$ will be called *hermitian matrix superpairs of* *rank I and with coordinate algebra* (A, A_0, π).

Suppose $|I| \geq 3$ and let $1, 2, 3 \in I$ be three distinct elements. The algebra A can then be described as the McCrimmon-Meyberg superalgebra of the collinear pair h_{12}, h_{13}. We point out that the McCrimmon-Meyberg superalgebra of the collinear pair h_{12}, h_{23} is A^{op}.

To obtain a refined root grading of $\mathbb{H}_I(A, A_0, \pi)$ we take a *Λ-grading of* (A, A_0, π) in the following sense: we have a Λ-grading of A, say $A = \oplus_{\lambda \in \Lambda} A_\lambda$, which respects A_0 and π, i.e.,

$$A_0 = \oplus_{\lambda \in \Lambda} A_0 \cap A_\lambda \quad \text{and} \quad A_\lambda^\pi = A_\lambda \text{ for all } \lambda \in \Lambda.$$

Let $H_I(A, A_0, \pi)\langle\lambda\rangle$ be the matrices in $H_I(A, A_0, \pi)$ with all entries in A_λ. It is then easily checked that $\mathbb{H}_I(A, A_0, \pi)\langle\lambda\rangle = (H_I(A, A_0, \pi)\langle\lambda\rangle, H_I(A, A_0, \pi)\langle\lambda\rangle)$ defines a Λ-grading which is compatible with the root grading induced by the covering grid $\mathcal{H}(I)$.

The proof of the following coordinatization theorem can be given along the lines of the proof in 4.5, using the classical Hermitian Coordinatization Theorem [**35**, (4.1.2)].

4.12. Hermitian Coordinatization Theorem. *Let $|I| \geq 3$. A Jordan superpair V over S is isomorphic to a hermitian matrix superpair $\mathbb{H}_I(A, A_0, \pi)$ if and only if V is covered by a hermitian grid $\mathcal{H}(I) = \{h_{ij} : i, j \in I\}$ such that for all $i, j \in I, i \neq j$ the maps*

$$D(h_{ij}^\sigma, h_{jj}^{-\sigma}) \colon V_{jj}^\sigma \to V_{ij}^\sigma \quad \textit{are injective,} \tag{1}$$

where V_{ij} denotes the joint Peirce spaces of $\mathcal{H}(I)$. In this case, we may take

(i) *as A the McCrimmon-Meyberg superalgebra of a fixed collinear pair (h_{ij}, h_{ik}),*
(ii) *as ample subspace $A_0 = D(h_{ij}^+, h_{jj}^-)V_{jj}^+$ and*
(iii) *as involution π the map $a^\pi = Q_{\bar{0}}^+(h_{ij})\{h_{ii}^- \; a \; h_{jj}^-\}$.*

In this case, $(V, \mathcal{H}(I))$ has a refined root grading of type $(\mathrm{C}_I^{\mathrm{her}}, \Lambda)$ if and only if (A, A_0, π) is Λ-graded, and then $V \cong_\Lambda \mathbb{H}_I(A, A_0, \pi)$.

Concerning the condition (1) we note that (1) holds for all pairs (ij) if it holds for one pair (ij) and that (1) always holds if V has no 2-torsion or if the (suitable defined) extreme radical of V vanishes (see [**28**] or [**35**, 4.1.2]).

4.13. Even quadratic form grids. Let I be a set with $|I| \geq 2$. An *even quadratic form grid* is a family $\mathcal{Q}_e(I) = \{e_{\pm i} : i \in I\}$ of non-zero idempotents satisfying the following relations:

(i) $(e_{+i}, e_{+j}, e_{-i}, -e_{-j}), i \neq j$, is a quadrangle of idempotents, 3.6, and
(ii) $\mathcal{Q}_e(I)$ is pure, 3.3.

The reader should be warned that the terms "even" and "odd" quadratic form grids used here and in the following subsections do not refer to a $\mathbb{Z}_2$-grading but rather to the type of grid.

An even quadratic form grid is a connected standard grid. Its associated 3-graded root system (R, R_1) is the even quadratic form grading $\mathrm{D}_{I \cup \{0\}}^{\mathrm{qf}}$ as defined in [**22**, 17.8], where 0 is a symbol with $0 \notin I$. Thus, $R = \mathrm{D}_{I \cup \{0\}} = \{\pm\varepsilon_j \pm \varepsilon_k : j, k \in I \cup \{0\}, j \neq k\} \cup \{0\}$ and $R_1 = \{\varepsilon_0 \pm \varepsilon_i : i \in I\}$. A canonical bijection between R_1 and $\mathcal{Q}_e(I)$ preserving the Peirce relations is given by $\varepsilon_0 \pm \varepsilon_i \mapsto e_{\pm i}$. For $|I| = 2$, an even quadratic form grid is the same as a quadrangle of idempotents, after changing the sign of the fourth idempotent.

We will describe a realization of $\mathcal{Q}_e(I)$. For a base superring S we denote by $H(I,S)$ the free S-module with an even basis $\{h_{\pm i} : i \in I\}$, considered as an S-supermodule. Thus

$$H(I,S) = S^{(+I \cup -I)} = H_+(I,S) \oplus H_-(I,S) \;\text{ for }\; H_\pm(I,S) = \oplus_{i\in I} Sh_{\pm i}.$$

The *hyperbolic superspace over S of rank* $2|I|$ is the S-supermodule $H(I,S)$ together with the *hyperbolic form* q_I: $H(I,S) \to S$ which, by definition, is the quadratic form associated to the S-bilinear form h: $H(I,S) \times H(I,S) \to S$ given by

$$h\big(\textstyle\sum_i (a_{+i}h_{+i} + a_{-i}h_{-i}), \sum_i (b_{+i}h_{+i} + b_{-i}h_{-i})\big) = \sum_i a_{+i}b_{-i}\,,$$

see 1.10(a). The Grassmann envelope of the hyperbolic form q_I in the sense of1.12 is the usual hyperbolic space of rank $2|I|$ over the commutative ring $G(S)$. The quadratic form superpair

$$\mathbb{EQ}_I(S) := \big(H(I,S), H(I,S)\big)$$

associated to the hyperbolic form q_I, 2.9, will be called the *even quadratic form superpair over S of rank* $2|I|$. In $\mathbb{EQ}_I(S)$ the pairs

$$e_i = (h_{+i}, h_{-i}) \quad\text{and}\quad e_{-i} = (h_{-i}, h_{+i})$$

are idempotents, and the family $\mathcal{Q}_e(I) = \{e_{\pm i} : i \in I\}$ is an even quadratic form grid which covers $\mathbb{EQ}_I(S)$. Indeed, writing $e_{\pm i} = e_{\varepsilon_0 \pm \varepsilon_i}$ the joint Peirce spaces of $\mathcal{Q}_e(I)$ in $V = \mathbb{EQ}_I(S)$ are $V_{\varepsilon_0+\sigma\varepsilon_i} = V_2(e_{\sigma i}) = (Sh_{\sigma i}, Sh_{-\sigma i})$, $\sigma = \pm$. For $i,j \in I, i \neq j$, the idempotents e_{+i}, e_{+j} are collinear. The McCrimmon-Meyberg superalgebra of this collinear pair is defined on $V^+_{\varepsilon_0+\varepsilon_i} = Sh_i$ and can be canonically identified with S. Observe that $\mathbb{EQ}_I(S)$ is a split Jordan superpair of type $\mathcal{Q}_e(I)$ in the terminology of 3.9.

The Jordan superpairs occurring in the following coordinatization theorem are only formally more general: any S-superextension A can be considered as a base superring, and hence the above also defines a Jordan A-superpair $\mathbb{EQ}_I(A)$. The Jordan pair version of 4.14 is proven in [**35**, 5.2.3], based on the Jordan triple version of the quadratic form coordinatization [**31**, III Thm. 2.6 and Cor. 2.7].

4.14. Even Quadratic Form Coordinatization. *Suppose $|I| \geq 3$. A Jordan superpair V over S is covered by an even quadratic form grid $\mathcal{Q}_e(I)$ if and only if V is S-isomorphic to a quadratic form superpair $\mathbb{EQ}_I(A)$ for some S-superextension A. We may take A to be the McCrimmon-Meyberg superalgebra of some collinear pair in $\mathcal{Q}_e(I)$.*

In this case, $(V, \mathcal{Q}_e(I))$ has a refined root grading of type $(\mathrm{D}^{\mathrm{qf}}_{I\cup\{0\}}, \Lambda)$ if and only if V is graded-isomorphic to $\mathbb{EQ}_I(A)$ for some Λ-graded S-superextension A.

4.15. Odd quadratic form grids. Let I be a non-empty set. An *odd quadratic form grid* is a family $\mathcal{Q}_o(I) = \{e_0\}\dot\cup\{e_{\pm i} : i \in I\}$ of idempotents satisfying the following relations:

(i) $(e_0; e_{+i}, e_{-i}), i \in I$ arbitrary, is a triangle of idempotents, 3.6, and
(ii) if $|I| \geq 2$ then the subfamily $\{e_{\pm i} : i \in I\}$ is an even quadratic form grid.

An odd quadratic form grid is a connected standard grid. Its associated 3-graded root system (R, R_1) is the odd quadratic form grading $\mathrm{B}^{\mathrm{qf}}_{I\cup\{0\}}$ where 0 is a symbol with $0 \notin I$, as defined in [**22**, 17.8]. We have $R = \{0\}\cup\{\pm\varepsilon_j : j \in \{0\}\dot\cup I\}\cup\{\pm\varepsilon_j\pm\varepsilon_k : j, k \in \{0\}\dot\cup I, j \neq k\}$ and $R_1 = \{\varepsilon_0\} \cup \{\varepsilon_0 \pm \varepsilon_i : i \in I\}$. A canonical bijection between R_1 and $\mathcal{Q}_o(I)$ preserving the Peirce relations is given by $\varepsilon_0 \mapsto e_0$, $\varepsilon_0 \pm \varepsilon_i \mapsto e_{\pm i}$.

We will give a realization of odd quadratic form grids. Given two S-quadratic forms $q^i = (q^i_{\bar 0}, b^i)\colon M^i \to N$ we denote by $q^1 \oplus q^2$ their *orthogonal sum* , i.e., the S-quadratic map $(q_{\bar 0}, b)$ from the S-supermodule $M = M^1 \oplus M^2$ to N given by $q_{\bar 0}(m^1 \oplus m^2) = q^1_{\bar 0}(m^1) + q^2_{\bar 0}(m^2)$ and $b(m^1 \oplus m^2, n^1 \oplus n^2) = b^1(m^1, n^1) + b^2(m^2, n^2)$ for $m^i, n^i \in M^i$. For an S-superextension A we denote by

$$\mathbb{OQ}_I(A, q_X)$$

the quadratic form superpair associated to $q_I \oplus q_X$, where $q_A\colon H(I, A) \to A$ is the hyperbolic map defined in 4.13 and $q_X\colon X \to A$ is an S-quadratic map on some A-supermodule X with base point $h_0 \in X_{\bar 0}$, i.e., $q_{X\bar 0}(h_0) = 1$. We call $\mathbb{OQ}_I(A, q_X)$ an *odd quadratic form superpair.* This Jordan S-superpair contains the idempotents

$$e_0 = (h_0, h_0) \quad \text{and} \quad e_{+i} = (h_{+i}, h_{-i}),\ e_{-i} = (-h_{-i}, -h_{+i}).$$

Note the minus signs in the definition of e_{-i} which are needed to ensure that $\mathcal{Q}_o(I) = \{e_0\} \cup \{e_{\pm i} : i \in I\}$ is an odd quadratic form grid. It covers $V = \mathbb{OQ}_I(A, q_X)$. Writing $e_0 = e_{\varepsilon_0}$ and $e_{\pm i} = e_{\varepsilon_0\pm\varepsilon_i}$ the joint Peirce spaces of $\mathcal{Q}_o(I)$ in V are

$$V_{\varepsilon_0+\sigma\varepsilon_i} = V_2(e_{\sigma i}) = (Ah_{\sigma i}, Ah_{-\sigma i})\,,\ \sigma = \pm \quad \text{and} \quad V_{\varepsilon_0} = (X, X).$$

For $i, j \in I, i \neq j$, the idempotents e_{+i}, e_{+j} are collinear. The McCrimmon-Meyberg superalgebra of this collinear pair is defined on $V^+_{\varepsilon_0+\varepsilon_i} = Ah_i$ and can be canonically identified with A. A refined root grading of this superpair is obtained from a *Λ-grading of* (A, q_X) in the following sense:

(i) a Λ-grading of the S-superalgebra A, written in the form $A = \oplus_{\lambda\in\Lambda} A_\lambda$;
(ii) a Λ-grading of the A-supermodule X, i.e., a direct sum $X = \oplus_{\lambda\in\Lambda}X_\lambda$ such that $A_\lambda X_\mu \subset X_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$, and in addition
(iii) $h_0 \in X_0$ (so $h_0 \in X_{\bar 0} \cap X_0$),
(iv) $b_X(X_\lambda, X_\mu) \subset A_{\lambda+\mu}$ and $q_{X\bar 0}(X_\lambda) \subset A_{2\lambda}$.

If we have such a Λ-grading, we can build a Λ-grading of $\mathbb{OQ}_I(A, q_X)$ by defining the λ-homogenous space as the submodule where all components lie in A_λ respectively X_λ. This Λ-grading is compatible with the root grading and hence gives a refined root grading of type $(\mathrm{B}^{\mathrm{qf}}_{I\cup\{0\}}, \Lambda)$ where, as above, $0 \notin I$.

4.16. Odd Quadratic Form Coordinatization. *Let $|I| \geq 2$. A Jordan superpair V over S is covered by an odd quadratic form grid $\mathcal{Q}_o(I)$ if and only if there exists an S-superextension A, an A-supermodule X and an S-quadratic map $q_X\colon X \to A$ with base point such that V is S-isomorphic to the odd quadratic form superpair $\mathbb{OQ}_I(A, q_X)$.*

More precisely, if V is covered by $\mathcal{Q}_o(I)$ we denote by $1, 2$ two distinct elements of I and by ε_0 the unique long root in the 1-part of the 3-graded root system $\mathrm{B}^{\mathrm{qf}}_{I\cup\{0\}}$ associated to $\mathcal{Q}_o(I)$. Moreover, we let $e_{-1} \in \mathcal{Q}_o(I)$ be the unique idempotent satisfying $e_1 \perp e_{-1}$. The data A, X and q_X mentioned above can then be defined as follows:

(a) *A is the McCrimmon-Meyberg superalgebra of (e_1, e_2) (note that $A = V_2^+(e_1)$ as S-supermodule).*
(b) *X is the A-supermodule defined on the Peirce space $X = V^+_{\varepsilon_0}$ with the canonical induced $\mathbb{Z}_2$-grading and the A-action given by*

$$a.x = \{a\, e_1^-\, x\} \quad (a \in A, x \in X). \tag{1}$$

(c) *$q_X = (q_{X\bar{0}}, b_X)\colon X \to A$ is the S-quadratic map X given by*

$$q_{X\bar{0}}(x_{\bar{0}}) = Q_{\bar{0}}(x_{\bar{0}})e^-_{-1} \quad \text{and} \quad b_X(x, x') = \{x\, e^-_{-1}\, x'\}\,. \tag{2}$$

In this case, $(V, \mathcal{Q}_o(I))$ has a refined root grading of type $(\mathrm{B}^{\mathrm{qf}}_{I\cup\{0\}}, \varLambda)$ if and only if (A, q_X) is $\varLambda$-graded. Then V and $\mathbb{OQ}_I(A, q_X)$ are graded-isomorphic.

Proof. We know that $\widetilde{V} = G_S(V) = G(V)$ is covered by the odd quadratic form grid $\widetilde{\mathcal{Q}}_o(I) = \{\widetilde{e_0}\} \cup \{\widetilde{e}_{\pm i} : i \in I\}$, where $\widetilde{e_0} = 1 \otimes e_0$ and $\widetilde{e}_{\pm i} = (1 \otimes e^+_{\pm i}, 1 \otimes e^-_{\pm i})$. By the odd quadratic form coordinatization for Jordan pairs, [**35**, (5.3.1)] and [**31**, III Cor. 2.9], $\widetilde{V}$ is therefore isomorphic to an odd quadratic form pair $\mathbb{OQ}_I(\widetilde{A}, \widetilde{q})$ where $\widetilde{A}$ is a commutative associative unital $G(S)$-algebra and $\widetilde{q}$ is an $\widetilde{A}$-quadratic form on an $\widetilde{A}$-module $\widetilde{X}$. We will show that the data $\widetilde{A}, \widetilde{X}$ and $\widetilde{q}$ are in fact the Grassmann envelopes of the corresponding data A, X and q_X defined above, thereby also proving (a), (b) and (c).

First of all, by [**31**, Thm. 2.8], we may take $\widetilde{A}$ to be the McCrimmon-Meyberg superalgebra of the collinear pair $(\widetilde{e}_1, \widetilde{e}_2)$. Hence $\widetilde{A} = G(A)$ which proves (a). In the classical odd quadratic form coordinatization, the underlying abelian group of the $\widetilde{A}$-module $\widetilde{X}$ is the Peirce space $G(V)^+_{\varepsilon_0} = G(V^+_{\varepsilon_0})$ on which $\widetilde{A}$ acts by (1) interpreted for $\widetilde{A}, \widetilde{X}$. On the other side, we know that $V^+_{\varepsilon_0}$ is an S-supermodule. All properties of an A-supermodule are therefore clear, except that $(ab).x = a.(b.x)$ for $a, b \in A$ and $x \in X$. This means

$$\{\{\{a\, e_1^-\, e_2^+\}\, e_2^-\, b\}\, e_1^-\, x\} = \{a\, e_1^-\, \{b\, e_1^-\, x\}\}. \tag{3}$$

But since $\widetilde{X}$ is an $\widetilde{A}$-module, formula (3) holds for a, b, x replaced by $\xi_1^{|a|} \otimes a$, $\xi_2^{|b|} \otimes b$ and $\xi_3^{|x|} \otimes x$, which then implies (3). In other words, $\widetilde{X}$ is the Grassmann envelope of X.

Regarding (c), we observe that q_X is an S-quadratic map in view of properties of the quadratic map Q. Moreover, using 1.14.1, we find that the Grassmann envelope $G(q_X)$ of q_X is given by $G(q_X)(\widetilde{x}) = \widetilde{Q}(\widetilde{x})e^-_{-1}$ where $\widetilde{Q}$ is the quadratic map of the Jordan pair $\widetilde{V}$ and $\widetilde{x} \in \widetilde{V}^+$. On the other side, by [**31**, III Thm. 2.8], this is exactly the form $\widetilde{q}$ used in the coordinatization of $\widetilde{V}$, which proves $G(q_X) = \widetilde{q}$.

To show $V \cong \mathbb{OQ}_I(A, qX)$ we define an S-linear map $f\colon V \to \mathbb{OQ}_I(A, q_X)$ given as follows:

(i) on the subpair $\oplus_{i\in I}(V_{\varepsilon_0+\varepsilon_i} \oplus V_{\varepsilon_0-\varepsilon_i})$ it is the map used in the even quadratic form coordinatization, and hence it maps this subpair onto the obvious subpair $\mathbb{EQ}_I(A)$ of $\mathbb{OQ}_I(A, q_X)$;

(b) on V_{ε_0} it is defined by $f^+(v^+_{\varepsilon_0}) = v^+_{\varepsilon_0} \in X \subset \mathbb{OQ}_I(A, q_X)^+$ and $f^-(v^-_{\varepsilon_0}) = \{e^+_1\, v^-_{\varepsilon_0}\, e^+_{-1}\} \in X \subset \mathbb{OQ}_I(A, q_X)^-$.

The Grassmann envelope of this map is the isomorphism used in the classical odd quadratic form coordinatization (see the proof of [**31**, III Thm. 2.8]) and hence f is an isomorphism by the homomorphism criterion 2.3.2.

Finally suppose V has a Λ-grading compatible with the root grading induced by the covering odd quadratic form grid. The description of the data A, X and q_X given above then shows that (A, q_X) is Λ-graded in the sense of 4.15. Hence $\mathbb{OQ}_I(A, q_X)$ has a refined root grading. It is straightforward to check that the isomorphism $f\colon V \to \mathbb{OQ}_I(A, q_X)$ defined above is a graded isomorphism.

4.17. Alternating grids. Let I be a set with $|I| \geq 4$ and a total order $<$. An *alternating grid of size* I in a Jordan superpair V is a family $\mathcal{A}(I) = \{e_{ij} : i, j \in I, i < j\}$ of non-zero idempotents in V such that, putting $e_{ji} = -e_{ij}$, the following properties hold:

(i) $(e_{ij}, e_{kj}, e_{kl}, e_{il})$ for distinct $i, j, k, l \in I$ is a quadrangle of idempotents, and

(i) $\mathcal{A}(I)$ is pure.

Alternating grids were called *symplectic* in [**28**], [**31**], [**33**] and [**35**]. Following a suggestion of O. Loos, I have changed the name to "alternating", since the standard realization of these grids is in alternating matrices (see below), and since these grids have little to do with symplectic Lie algebras or symplectic groups.

An alternating grid $\mathcal{A}(I)$ is a connected standard grid. Identifying e_{ij} with $\varepsilon_i + \varepsilon_j$ one easily sees that the associated 3-graded root system of $\mathcal{A}(I)$ is the alternating grading $\mathrm{D}^{\mathrm{alt}}_I$ of the root system D_I, as defined in [**22**, 17.8]. An alternating grid $\mathcal{A}(I), |I| = 4$ is associated to an even quadratic form grid $\mathcal{Q}_e(J), |J| = 3$.

Let A be a superextension of S. Since the identity map is an involution of A the classical transpose map is an involution of the associative A-superalgebra $\mathrm{Mat}(I, I; A)$, see 4.11. A matrix $x = (x_{ij}) \in \mathrm{Mat}(I, I; A)$ is called *alternating* if $x^{\mathrm{T}} = -x$ and if all diagonal elements $x_{ii} = 0$. The set of all alternating matrices is an A-supermodule, denoted $\mathrm{Alt}(I, A)$. The pair

$$\mathbb{A}_I(A) := (\mathrm{Alt}(I, A), \mathrm{Alt}(I, A))$$

is a subpair of the Jordan superpair $\mathbb{M}_{II}(A)$ and hence itself a Jordan superpair over A (or over S) called the *alternating matrix superpair of rank I and with coordinate al-*

gebra A. Note that the product is given by 4.6.1. (One obtains an isomorphic Jordan superpair by taking the quadratic product $Q_{\bar 0}(x_{\bar 0})y = -x_{\bar 0} y x_{\bar 0}$ and $\{x\,y\,z\} = -xyz - (-1)^{|x||y|+|x||z|+|y||z|}\, zyx$, see [**35**, 6.1].) In the alternating matrix pair the family of all

$$e_{ij} = (E_{ij} - E_{ji}, E_{ji} - E_{ij}),\; i < j$$

forms a covering alternating grid. In fact, the alternating matrix pair $\mathbb{A}_I(A)$ is the split Jordan superpair of type $\mathcal{A}(I)$ over A. Conversely, using [**35**, (6.1)], we have:

4.18. Alternating Coordinatization. *Let $|I| \geq 4$ and let V be a Jordan superpair over S. Then V is covered by an alternating grid $\mathcal{A}(I)$ if and only if there exists a superextension A of S such that V is isomorphic to $\mathbb{A}_I(A)$. In this case, we may take A to be the McCrimmon-Meyberg superalgebra of some collinear pair in $\mathcal{A}(I)$.*

More generally, a Jordan superpair V has a refined root grading of type $(\mathrm{D}_I^{\mathrm{alt}}, \Lambda)$ if and only if V is graded-isomorphic to $\mathbb{A}_I(A)$ for some Λ-graded superextension A of S.

4.19. Bi-Cayley grids. A *Bi-Cayley grid* in a Jordan superpair V is a family $\mathcal{B} = (e_{\varepsilon i} : \varepsilon = \pm, 1 \leq i \leq 8)$ of 16 non-zero idempotents in V satisfying the following conditions:

(i) for $1 \leq i, j \leq 4, i \neq j$ and ε, μ arbitrary the following are quadrangles of idempotents:

  (1) $(e_{\varepsilon i}, e_{\mu j}, e_{-\varepsilon i}, -e_{-\mu j})$ and $(e_{\varepsilon(i+4)}, e_{\mu(j+4)}, e_{-\varepsilon(i+4)}, -e_{-\mu(j+4)})$,

  (2) $(e_{\varepsilon i}, e_{\varepsilon j}, e_{-\varepsilon(i+4)}, -e_{-\varepsilon(j+4)})$ and $(e_{\varepsilon i}, e_{\varepsilon j}, e_{\varepsilon(j+4)}, e_{\varepsilon(i+4)})$,

  (3) $(e_{-i}, e_{+j}, e_{-(k+4)}, \operatorname{sgn}\binom{1234}{ijkl} e_{+l})$, where $\operatorname{sgn}\binom{1234}{ijkl}$ is the signature of the permutation $\binom{1234}{ijkl}$;

(ii) $\mathcal{B}$ is pure.

An equivalent definition is given in [**31**, II§3.1]. A Bi-Cayley grid is a connected standard grid. Its associated 3-graded root system is the Bi-Cayley grading $\mathrm{E}_6^{\mathrm{bi}}$ of the root system E_6, see [**22**, 17.9].

We will indicate how to realize Bi-Cayley grids in Jordan superpairs. Let $\mathbb{O}_k$ be the split Cayley algebra over k, see e.g. [**42**, 2.2] or [**31**, III.3.1], obtained from the k-extension $k \oplus k$ by twice performing the Cayley-Dickson process using $1 \in k$ as structure constants. Let S be a base superring. The S-superring extension $\mathbb{O}_S := S \otimes \mathbb{O}_k$ is a unital alternative S-superalgebra, which we call the *split Cayley superalgebra over S*. By 4.4, it gives rise to a Jordan superpair

$$\mathbb{B}(S) := \mathbb{M}_{12}(\mathbb{O}_S)$$

called the *Bi-Cayley superpair over S*. It contains the Bi-Cayley pair $\mathbb{B}(k) = \mathbb{M}_{12}(\mathbb{O}_k)$ as a subpair. By [**35**, 7.2], $\mathbb{B}(k)$ is covered by a Bi-Cayley grid $\mathcal{B}$, in fact, $\mathbb{B}(k)$ is the split Jordan pair of type $\mathcal{B}$ over k. It follows that that $\mathbb{B}(S)$ is the split Jordan superpair of type $\mathcal{B}$ over S. In particular, $\mathbb{B}(S)$ is the S-extension of $\mathbb{B}(k)$ and is covered by a Bi-Cayley grid.

More generally, we can replace S in the above construction by any Λ-graded superextension A of S. We obtain a Jordan A-superpair $\mathbb{B}(A)$ which, by restriction of scalars, becomes a Jordan superpair over S. It has a refined root grading of type $(\mathrm{E}_6^{\mathrm{bi}}, \Lambda)$. Conversely, using the classical Bi-Cayley Coordinatization Theorems [**31**, III.3.3] and [**35**, (7.2.1)], one proves:

4.20. Bi-Cayley Coordinatization. *A Jordan superpair V over S is covered by a Bi-Cayley grid $\mathcal{B}$ if and only if there exists a superextension A of S such that V is isomorphic to the Bi-Cayley superpair $\mathbb{B}(A)$. In this case, one can choose A to be the McCrimmon-Meyberg superalgebra of some collinear pair in $\mathcal{B}$. Moreover, $(V, \mathcal{B})$ has a refined root grading of type $(\mathrm{E}_6^{\mathrm{bi}}, \Lambda)$ if and only if A is Λ-graded, and then V is even graded-isomorphic to $\mathbb{B}(A)$.*

4.21. Albert grids. An *Albert grid* is a family $\mathcal{A}$ of 27 non-zero idempotents which we write in the form

$$\mathcal{A} = ([1], [2], [3]) \cup ([ij]_{\varepsilon r} : 1 \le i < j \le 3, \varepsilon = \pm, 1 \le r \le 4)$$

such that, putting $[ij]_{\varepsilon 1} = [ji]_{-\varepsilon 1}$ and $[ij]_{\varepsilon r} = -[ji]_{\varepsilon r}$ for $2 \le r \le 4$, the following properties hold:

- (i) for each $i \in \{1, 2, 3\}$ the family $(e_{\pm s}; 1 \le s \le 8)$ given by $e_{\varepsilon r} = [ij]_{\varepsilon r}$, $e_{\varepsilon(r+4)} = [ik]_{\varepsilon r}$, $1 \le r \le 4$, $i, j, k \ne$, is a Bi-Cayley grid;
- (ii) for each pair $(ij), 1 \le i < j \le 3$, the family $([ij]_{\pm 1}, [ij]_{\pm 2}, [ij]_{\pm 3}, [ij]_{\pm 4}, [i], -[j])$ is an even quadratic form grid (of size 10);
- (iii) $\mathcal{A}$ is pure.

An equivalent definition is given in [**31**, II§3.2]. An Albert grid is a connected standard grid. Its associated 3-graded root system is the Albert grading $\mathrm{E}_7^{\mathrm{alb}}$ of E_7, as defined in [**22**, 17.9].

Albert grids can be realized in 3×3-hermitian matrix superpairs. Namely, let A be a superextension of S and let $\mathbb{O}_A$ be the split Cayley superalgebra over A, 4.19. It is a unital alternative A-superalgebra. The A-extension of the canonical involution of $\mathbb{O}_k$ is an involution π of the superalgebra $\mathbb{O}_A$ for which $A.1 \subset \mathbb{O}_A$ is an ample subspace. Hence the hermitian matrix superpair

$$\mathbb{AB}(A) := \mathbb{H}_3(\mathbb{O}_A, A.1, \pi),$$

as defined in 4.11, is a Jordan superpair over A and by restriction of scalars over S. It will be called the *Albert superpair over A*. Note that $\mathbb{AB}(A)$ contains the Jordan pair $\mathbb{AB}(k)$ as a subpair. By [**35**, 7.3] we know that $\mathbb{AB}(k)$ is split of type $\mathcal{A}$, hence so is $\mathbb{AB}(A)$. In particular, $\mathbb{AB}(A)$ is covered by an Albert grid. Conversely, using the classical Albert Coordinatization Theorems ([**31**, III.3.5] and [**35**, (7.3.1)]) one can easily establish:

4.22. Albert Coordinatization. *A Jordan superpair V over S is covered by an Albert grid $\mathcal{A}$ if and only if V is isomorphic to an Albert superpair $\mathbb{AB}(A)$ for some superextension A of S. In this case, one can choose A to be the McCrimmon-Meyberg superalgebra of some collinear pair in $\mathcal{A}$.*

Moreover, $(V, \mathcal{A})$ has a refined root grading of type $(\mathrm{E}_7^{\mathrm{alb}}, \Lambda)$ if and only if A is Λ-graded, and in this case V is graded-isomorphic to $\mathbb{AB}(A)$.

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