POLYNOMIAL IDENTITIES AND NON-IDENTITIES OF SPLIT JORDAN PAIRS

ERHARD NEHER

Abstract. We show that split Jordan pairs over rings without 2-torsion can be distinguished by polynomial identities with integer coefficients. In particular, this holds for simple finite-dimensional Jordan pairs over algebraically closed fields of characteristic not 2. We also generalize results of Drensky-Racine and Rached-Racine on polynomial identities of Jordan algebras respectively Jordan triple systems.

0. Introduction. Identities are one of the key tools in Zel’manov’s description of prime Jordan algebras [Z1], [McZ] and Jordan triple systems [Z2], [D1] and [D2]. This led McCrimmon to a number of questions aimed at clarifying the structure of polynomial identities of Jordan triple systems [Mc], some of which were answered by Rached and Racine: simple finite-dimensional Jordan triple systems of degree ≤ 2 over algebraically closed fields of characteristic ≠ 2 can be separated by polynomial identities and non-identities [RR], and the same is true for the simple exceptional Jordan triple systems [RR2]. That the isomorphism classes of simple finite-dimensional Jordan algebras over algebraically closed fields of characteristic 0 are determined by the polynomial (non-)identities of the algebras had been shown before by Drensky and Racine [DR]. In this paper, we generalize these results to the setting of Jordan pairs.

One of the polynomials used in [RR] and [RR2] is the inner Capelli polynomial IC_n, which was shown to be a non-identity for several classes of simple Jordan triple systems (Prop. 16 - 19 of [RR] and Prop. 1 of [RR2]). These classes all have the property that the associated Jordan pair contains a connected grid of n idempotents. That Jordan pairs occur is not surprising. By its very definition, the inner Capelli polynomial is Jordan pair polynomial rather than a Jordan triple polynomial. It is therefore more natural to work with Jordan pairs. Our first theorem (see 2.) proves the obvious generalization of Rached’s and Racine’s results on the inner Capelli polynomial: over rings without 2-torsion, IC_n is not an identity for any Jordan pair containing a connected grid of n idempotents.

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It is an easy consequence of this theorem that the inner Capelli polynomials can be used to distinguish between split finite-dimensional Jordan pairs of different dimensions (see 3. for the definition of “split”). To separate non-isomorphic Jordan pairs of the same dimension, we use a new variant of the inner Capelli polynomial and other polynomials already introduced in [DR] and [RR]. This leads to our second theorem, proven in 9.: simple finite-dimensional Jordan pairs over algebraically closed fields of characteristic \( \neq 2 \) can be separated by polynomial identities. As a corollary we obtain the analogous result for Jordan algebras which generalizes the Drensky-Racine Theorem [DR]. That Jordan pairs of rectangular matrices can be distinguished by polynomial identities is also proven in [I].

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1. Inner Capelli Polynomials and Capelli sequences. Unless stated otherwise, Jordan pairs will be considered over arbitrary commutative rings of scalars. We will use the notation of [L].

Let \( X = (X^+, X^-) \) be a pair of non-empty sets. We denote by \( \text{FJP}(X) \) the free Jordan pair over \( \mathbb{Z} \) on \( X \), defined by the universal property that for every Jordan pair \( V \) (considered as Jordan pair over the integers) and every map \( \varphi : X \rightarrow V \) there exists a unique homomorphism \( \Phi : \text{FJP}(X) \rightarrow V \) extending \( \varphi \). Such a map (\( \varphi \) or \( \Phi \)) will be called a substitution in \( V \), while elements of \( \text{FJP}(X)^\sigma \), \( \sigma = \pm \), are called Jordan polynomials. If \( f = f(x, y) \in \text{FJP}(X)^\sigma \) is a Jordan polynomial in the generators \( x = (x_1, \ldots, x_m), x_i \in X^\sigma \), and \( y = (y_1, \ldots, y_n), y_j \in X^{-\sigma} \), and \( \varphi : (x, y) \mapsto (u, v) \in V^\sigma \times \cdots \times V^\sigma \times V^{-\sigma} \times \cdots \times V^{-\sigma} \) (\( m \) factors of \( V^\sigma \), \( n \) factors of \( V^{-\sigma} \)) is a substitution in \( V \) we put \( \Phi^\sigma(f) = f(u, v) = f(u_1, \ldots, u_m, v_1, \ldots, v_n) \), and say it results from \( f \) by the substitution \( \Phi \). We say that a Jordan polynomial \( f \in \text{FJP}(X)^\sigma \) is an identity of a Jordan pair \( V \) if \( \Phi^\sigma(f) = 0 \) for all substitutions in \( V \) and if \( f \) is monic in the sense that some leading monomial in \( f \) has coefficient 1.

Warning: Our concepts differ somewhat from the ones used in [D1], [DMc2] or [McZ]. We consider free Jordan pairs only over \( \mathbb{Z} \) and correspondingly Jordan polynomials and polynomial identities all have integer coefficients. We will remind the reader of this by speaking of integral Jordan polynomials and integral polynomial identities. Also, we do not require that a polynomial identity of a Jordan pair \( V \) holds in all scalar extensions of \( V \) (see however the criterion (3.2)).
For $X = \{x_1^+, x_2^+, \ldots, x_n^+\}, \{x_1^-, x_2^-, \ldots, x_n^-\}$ and $\sigma = \pm$ we put

$$DDQ_n^\sigma(x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma; x_1^-\sigma, x_2^-\sigma, \ldots, x_n^-\sigma) = D(x_1^\sigma, x_n^-\sigma) \cdots D(x_2^\sigma, x_2^-\sigma)Q(x_1^\sigma)x_1^-\sigma$$

$$= \{\cdots \{Q(x_1^\sigma)x_1^-\sigma, x_2^-\sigma \} x_3^-\sigma, x_3^\sigma \} \cdots \} x_n^-\sigma, x_n^\sigma\}.$$

The inner Capelli polynomial is then defined as $IC_n = (IC_n^+, IC_n^-) \in FJP(X)$,

$$IC_n^\sigma(x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma; x_1^-\sigma, x_2^-\sigma, \ldots, x_n^-\sigma) = \sum_{\tau \in S_n} (-1)^\tau DDQ_n^\sigma(x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma; x_{\tau(1)}^-\sigma, x_{\tau(2)}^-\sigma, \ldots, x_{\tau(n)}^-\sigma)$$

where $(-1)^\tau$ denotes the signature of the permutation $\tau$ in the symmetric group $S_n$, see [RR] and [RR2]. The Jordan polynomial $IC_n^\sigma$ is an alternating multilinear function in the generators $x_i^-\sigma$ and hence an identity of any Jordan pair for which $V^-\sigma$ is spanned by less than $n$ elements. On the other hand, the following theorem gives a criterion for $IC_n^\sigma$ not to be an identity.

To establish this result we prove the existence of a special substitution which is analogous to the concept of an Amitsur-Levitzki staircase sequence for the standard polynomial for matrix algebras. For a fixed $\sigma \in \{\pm\}$ we define an $n$th order Capelli sequence in a Jordan pair $V$ as a pair of sequences $(u; v) = (u_1, \ldots, u_n; v_1, \ldots, v_n)$, $(u_i, v_i) \in V^\sigma \times V^-\sigma$ such that

$$DDQ_n^\sigma(u_1, u_2, \ldots, u_n; v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n)}) = 0$$

for every permutation $\tau \in S_n$ with $\tau \neq 1$, and thus $IC_n^\sigma(u; v) = DDQ_n^\sigma(u; v)$. Obviously, the interest lies in those sequences with $IC_n^\sigma(u; v) \neq 0$. But because of the situation in a Jordan pair of hermitian matrices in characteristic 2 (see Theorem 2 below), we have not included this condition as part of the definition of a Capelli sequence.

We give an example of a Capelli sequence in the Jordan pair $I_{pq}, p \leq q$, of rectangular $p \times q$ matrices over a base ring $k$ with Jordan pair product given by $Q_{xy} = xy^T x$. Let $E_{ij}$ denote the usual rectangular matrix units: the $(ij)$-entry of $E_{ij}$ is 1 while all other entries of $E_{ij}$ are 0. The pairs $(u_j, v_j)$ of an $n$th order
Capelli sequence, \( n = pq \), and \( \sigma = + \) are

\[
(E_{11}, E_{11}), (E_{21}, E_{21}), \ldots , (E_{p-1,1}, E_{p-1,1}), (E_{p1}, E_{p1}),
\]
\[
(E_{1q}, E_{1q}), (E_{1,q-1}, E_{1,q-1}), \ldots , (E_{13}, E_{13}), (E_{22}, E_{12}),
\]
\[
(E_{2q}, E_{2q}), (E_{2,q-1}, E_{2,q-1}), \ldots , (E_{23}, E_{23}), (E_{32}, E_{22}),
\]
\[
(E_{3q}, E_{3q}), (E_{3,q-1}, E_{3,q-1}), \ldots , (E_{33}, E_{33}), (E_{42}, E_{32}),
\]
\[
\ldots , \ldots , \ldots , \ldots , \ldots , \ldots ,
\]
\[
(E_{p-1,q}, E_{p-1,q}), (E_{p-1,q-1}, E_{p-1,q-1}), \ldots , (E_{p-1,3}, E_{p-1,3}), (E_{p2}, E_{p-1,2}),
\]
\[
(E_{pq}, E_{pq}), (E_{p,q-1}, E_{p,q-1}), \ldots , (E_{p3}, E_{p3}), (E_{p2}, E_{p,2}).
\]

Note the “wrinkle” in the choice of the \( u_j \)'s in the last entry of the second to last row: the \( u_j \)'s miss \( E_{12} \) but repeat \( E_{p2} \). For this Capelli sequence one finds \( \text{IC}_n^{+}(u, v) = E_{p1} \). The reader may prove this now or specialize the proof of the following Theorem 2 which establishes a more general result.

This generalization arises from the observation that the Jordan pair \( I_{pq} \) is a Jordan pair containing a finite connected standard grid, namely the rectangular grid \( R(p, q) \) \( = \{ (E_{ij}, E_{ij}); 1 \leq i \leq p, 1 \leq j \leq q \} \), and that the Capelli sequence above is comprised of \( \pm \)-parts of idempotents in this grid (most but not all \( (u_j, v_j) \) are idempotents in \( R(p, q) \)). We claim that Capelli sequences always exist in Jordan pairs containing grids. In particular, we will give an inductive construction of Capelli sequences which works for all Jordan pairs containing ortho-collinear connected standard grids and which produces the sequence above when applied to the Jordan pair \( I_{pq} \) and the rectangular grid \( R(p, q) \). Regarding grids and standard grids in Jordan pairs the reader is referred to [N2] and [N3]. Some of the properties of standard grids are reviewed in the proof of Theorem 2, but we recall here that a connected standard grid is either an ortho-collinear grid (any two idempotents in the grid are either equal, orthogonal or collinear) or an odd quadratic form grid or a hermitian grid, see also 3. below.

2. Theorem. Suppose \( V \) contains a finite connected standard grid \( \mathcal{G} \) of size \( |\mathcal{G}| = n \). Then there exists a Capelli sequence \( (u; v) \) in \( V \) such that

(a) \( u = (u_1, \ldots , u_n) \subset \mathcal{G}^\sigma := \{ g^\sigma; g = (g^+, g^-) \in \mathcal{G} \} \),

(b) \( v = (v_1, \ldots , v_n) \subset \mathcal{G}^{-\sigma} \) is an enumeration of \( \mathcal{G}^{-\sigma} \),

(c) if \( \mathcal{G} \) is not a hermitian grid then \( \text{IC}_n(\sigma(u; v)) = \pm e^\sigma \) for some \( e \in \mathcal{G} \), while in the case of a hermitian grid \( \mathcal{G} = \{ h_{ij}; 1 \leq i \leq j \leq r \} \) of rank \( r \geq 2 \) we have \( \text{IC}_n^{\sigma}(u; v) = 2^{r-2}h_1^r \).
**Proof.** We will first consider ortho-collinear grids and construct the sequences \( u \) and \( v \) for all such grids at once. Thus, let \( G \) be a connected ortho-collinear standard grid. We will need the following known facts about \( G \):

Fact 1) For any \( g, h \in G \) the product \( Q(g^\sigma)h^{-\sigma} \) is zero unless \( g = h \) in which case \( Q(g^\sigma)h^{-\sigma} = g^\sigma \).

Fact 2) If \( g_1, g_2 \in G \) are collinear \((g_1 \in V_1(g_2) \text{ and } g_2 \in V_1(g_1))\), denoted \( g_1 \top g_2 \) then for any \( g \in G \) the product \( \{g_1^i g^{-\sigma} g_2^j\} \) is either zero or \( g = g_i, i = 1, 2, \) in which case \( \{g_1^i g^{-\sigma} g_2^j\} = g_2^j \) respectively \( \{g_1^i g^{-\sigma} g_2^j\} = g_1^i \).

Fact 3) If \( g_1, g_2 \in G \) are two orthogonal idempotents \((g_1 \perp g_2)\) then for any \( g \in G \) the product \( \{g_1^i g^{-\sigma} g_2^j\} \) either vanishes or \((g_1, g, g_2)\) is a hook, i.e., \( g_1 \top g \top g_2 = g_1 \perp g_1 \). In the latter case, there exists \( e \in \{\pm 1\} \) and \( e \in G \) such that \((g_1, g, g_2, e)\) is a quadrangle of idempotents. In particular, \( g \perp e \top g, i = 1, 2 \) and \( \{g_1^i g^{-\sigma} g_2^j\} = e e^\sigma \).

For \( F \subset G \) and \( g \in G \) we put \( F_i(g) = \{ f \in F; f \in V_i(g) \} \). We will use subfamilies \( F \) of \( G \) and elements \( e \in G \setminus F \) which are hooked up to \( F \): for any \( f \in F \) with \( f \perp e \) there exists \( \hat{f} \in F \) such that \( e \) is hooked to \( f \) via \( \hat{f} \), i.e., \((e, \hat{f}, f)\) is a hook. An example of such a configuration is:

Let \( F \subset G \) be a connected subgrid of \( G \).

\[
(1) \quad \text{Then any } e \in G \setminus F \text{ with } F_1(e) \neq \emptyset \text{ is hooked up to } F.
\]

Indeed, let \( f \in F \) with \( f \perp e \). By assumption we know that there exists \( \hat{f} \in F \) with \( e \top \hat{f} \). If \( \hat{f} \top f \) we are done: \( e \) is hooked to \( f \) via \( \hat{f} \in F \). Otherwise \( \hat{f} \perp f \), and by Fact 3) applied to the connected grid \( F \) there exist \( f_1, f_2 \in F \) such that \((\hat{f}, f_1, f, \pm f_2) \subset F \) is a quadrangle of idempotents. By ortho-collinearity of \( G \) we have \( f_1 \in V_i(e) \cap V_i(f) \) for \( i = 1 \) or \( 0 \). If \( i = 1 \) then \( e \) is hooked up to \( f \) via \( f_1 \in F \). If \( i = 0 \) then \( e \) is hooked up to \( f \) via \( f_2 \in F \): by the Peirce multiplication rules

\[
f_2 = \pm (\{\hat{f}^+ f_1^- f_1^+\}, \{\hat{f}^- f_1^+ f^-\}) \in V_1(e) \cap V_1(f).
\]

We also need:

\[
(2) \quad \text{If } F \subset G \text{ is a connected subgrid then so is } F_0(g) \text{ for any } g \in G.
\]

That \( F_0(g) \) is again a standard grid is immediate from the definitions. So it remains to prove connectivity, i.e. any orthogonal pair \((f_1, f_3) \subset F_0(g)\) imbeds in a hook in \( F_0(g) \). But since \( F \) is connected there exists an orthogonal pair \((f_2, f_4) \subset F \) such
that \((f_1, f_2, f_3, \pm f_4)\) is a quadrangle, and quadrangle is orthogonal to \(g\) as soon as two opposite corners are: since \(f_1 + f_3 \approx f_2 + f_4\) we have \(g \in V_0(f_1 + f_3) = V_0(f_2 + f_4)\), and thus \(f_2, f_4 \in \mathcal{F}_0(g)\).

We are now ready to construct a Capelli sequence with properties (a) – (c) for an ortho-collinear grid \(G\). If \(n = 1\) we are trivially done: \(G = \{e_1\}\) and \((u; v) = (e_1^\sigma; e_1^{-\sigma})\) is a Capelli sequence for \(IC_n^\sigma\). So in the following let \(n > 1\). The enumeration \((v_i)\) of \(G^\sigma\) and the elements \(u_i \in G^\sigma\) comprising the Capelli sequence will be constructed inductively. The \(i\)th induction step \((i \geq 1)\) will use the following data:

(i) a subset \(G^i \subset G\) which is a connected subgrid for \(i \geq 2\),

(ii) \(e_i \in G \setminus G^i\) which is hooked up to \(G^i\),

(iii) a choice of \((u_1, \ldots, u_{m_i}), u_j \in G^\sigma\), and an enumeration \((v_1, \ldots, v_{m_i})\) of \((G \setminus G^i)^{-\sigma}\) such that, regardless of the choice of the following \(u_{m_i+1}, \ldots, u_n\) and the enumeration \((v_{m_i+1}, \ldots, v_n)\) of \((G^i)^{-\sigma}\), we have

\[
\text{DDQ}_n^\sigma(u, \tau(v)) = 0 \quad \text{unless} \quad \tau(j) = j \quad \text{for} \quad 1 \leq j \leq m_i,
\]

where \(\text{DDQ}_n^\sigma(u, \tau(v)) := \text{DDQ}_n^\sigma(u_1, u_2, \ldots, u_n; v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n)})\), and

\[
\{\{\ldots \{Q(u_1)v_1, v_2, u_2\} \ldots \}v_{m_i}, u_{m_i}\} = \pm e_i^\sigma.
\]

Thus, for \(\tau \in S_n\) with \(\tau(j) = j\) for \(1 \leq j \leq m_i\) we have

\[
\text{DDQ}_n^\sigma(u, \tau(v)) = \pm \{\ldots \{e_i^\sigma, v_{\tau(m_i+1)}^\sigma, u_{m_i+1}^\sigma\} \ldots \}v_{\tau(n)}^\sigma u_n^\sigma\}.
\]

In the \(i\)th induction step we will construct the data (i)–(iii) for \(i + 1\) such that \(G^{i+1}\) is a proper subset of \(G^i\). After a finite number of steps this process stops, producing a Capelli sequence with properties (a) – (c). The reader may want to keep the example above in mind which arises from the general construction in \(p\) steps by taking \(e_i = (E_{i1}, E_{i1}), 1 \leq i \leq p\) and \(G^i = \{(E_{lj}, E_{lj}; i \leq l \leq p, 2 \leq j \leq q)\} \quad \text{for} \quad i \geq 2\).

Beginning of induction \((i = 1)\): We start by choosing arbitrarily some \(e_1 \in G\). By Fact 3), \(e_1\) is hooked up to \(G^1 := G \setminus \{e_1\}\). We let \((u_1, v_1) = (e_1^\sigma, e_1^{-\sigma})\). Then \(Q(u_1)v_1 = e_1^\sigma\) and, by Fact 1), for any enumeration \((v_2, \ldots, v_n)\) of the remaining \(g^{-\sigma}\) in \(G^{1,\sigma}\) the term \(Q(u_1)v_{\tau(1)}\) vanishes if \(\tau(1) \neq 1\). Thus (i) – (iii) hold for \(i = 1\) with \(m_1 = 1\).
Induction step: We suppose that we are given the data described in (i) – (iii) for some \( i \geq 1 \). We will distinguish two cases A) and B) depending on whether or not all idempotents of \( G^i \) are collinear to \( e_i \).

Case A): not all idempotents of \( G^i \) are collinear to \( e_i \). Let \( h \in G^i \) with \( h \perp e_i \).

Since \( e_i \) is hooked up to \( G^i \), there exists \( f \in G^i \) such that \((e_i, f, h)\) is a hook. By Fact 3), it can be completed to a quadrangle: there exists \( e_{i+1} \in G \) such that \((e_i, f, h, +e_{i+1})\) is a quadrangle of idempotents. We put \( G^{i+1} = G_0^i(e_i) \). This is a proper subset of \( G^i \) since \( f \in G^i \setminus G^{i+1} \).

It is also a connected subgrid of \( G \): for \( i = 1 \) we have \( G^2 = G_0^1(e_1) \) so that connectivity follows from (2); for \( i \geq 2 \) we know by induction that \( G^i \) is connected and hence again by (2) that \( G^{i+1} \) is connected. Thus (i) holds for \( i + 1 \). We also have (ii). Indeed, \( e_i \cup e_{i+1} \) implies that \( e_{i+1} \not\in G^{i+1} \) and since \( h \in G_0^i(e_i) \cap G_1^i(e_{i+1}) = (G^{i+1})_1(e_{i+1}) \) it follows from (1) that \( e_{i+1} \) is hooked up to \( G^{i+1} \). We now enumerate \( G^i_1(e_i) = (g_{m_i+1}, . . . , g_l = f), l = m_i+1, \) and choose

\[
(u_{m_i+1}, . . . , u_{l-1}, u_l) = (g_{m_i+1}^\sigma, . . . , g_{l-1}^\sigma, h^\sigma),
\]

\[
(v_{m_i+1}, . . . , v_{l-1}, v_l) = (g_{m_i+1}^-\sigma, . . . , g_{l-1}^-\sigma, g_l^-\sigma = f^-\sigma).
\]

Note the choice of \((u_l, v_l)\)!

By construction, all \( u_j \in G, 1 \leq j \leq l \), and \((v_1, . . . , v_l)\) is an enumeration of \((G \setminus G^{i+1})^{-\sigma}\). To show the remaining parts of (iii) we suppose that the sequence \((u_1, . . . , u_l; v_1, . . . , v_l)\) has been completed to a sequence \((u; v)\) satisfying (a) and (b) of the theorem, and we let \( \tau \in S_n \). We can assume \( \tau(j) = j \) for \( 1 \leq j \leq m_i \). For \( m_i + 1 < l = m_{i+1} \) we have \( g_{m_i+1} \cup v_i \) and hence, by Fact 2), \( \{e_i^\sigma, v_{\tau(m_i+1)}, u_{m_i+1}\} = 0 \) unless \( \tau(m_i+1) = m_i+1 \) in which case \( \{e_i^\sigma, v_{m_i+1}, u_{m_i+1}\} = e_i \). Analogously, for any \( j < l \) we have

\[
\{\{\ldots \{e_i^\sigma, v_{\tau(m_i+1)}, u_{m_i+1}\}\ldots \}v_{\tau(j)}, u_j\} = 0
\]

unless \( \tau(k) = k \) for \( m_i \leq k \leq j \), and in this case the product equals \( e_i \). Finally, we consider the product \( \{e_i^\sigma, v_{\tau(l)}, u_l\} = \{e_i^\sigma, v_{\tau(l)}, h^\sigma\} \). We can 0 that \( \tau(j) = j \)

for \( 1 \leq j < l \). Then \( v_{\tau(l)} = g^-\sigma \) for some \( g \in \{f\} \cup G_0^i(e_i) \). Hence, by Fact 3), this product vanishes unless \( g = f \), i.e. \( \tau(l) = l \), and in this case we obtain \( \{\{\ldots \{e_i^\sigma, v_l, h^\sigma\}\ldots \} = \pm e_{i+1} \). This finishes the induction step in case A.

Case B): all idempotents in \( G^i \) are collinear to \( e_i \). In this case the induction stops: we enumerate \( G^i = (g_{m_i+1}, . . . , g_n) \) arbitrarily and put \((u_j, v_j) = (g_j^\sigma, g_j^-\sigma)\) for \( m_i < j \leq n \). Then \((u; v) = (u_1, . . . , u_n; v_1, . . . v_n)\) satisfies (a) and (b) of the theorem, and we claim that it is also a Capelli sequence with property (c). Indeed, it is enough to consider \( \tau \in S_n \) with \( \tau(j) = j \) for \( 1 \leq j \leq m_i \). Since then
\( v_{\tau(m+1)} \neq e_i^* \) while \( g_{m+1} = e_i \). Fact 2) shows that \( \{e_i, v_{\tau(m+1)}, u_{m+1}\} = 0 \) unless \( \tau(m+1) = m+1 \), and in this case \( \{e_i^*, v_{m+1}, u_{m+1}\} = e_i^* \), so

\[
DDQ_n^\sigma(u, \tau(v)) = \pm \{ \ldots \{ e_i^* , \ v_{\tau(m+2)}, \ u_{m+2}^\sigma \} \ldots \} v_{\tau(n)}^\sigma u_n^\sigma \].

Repeating this argument shows \( DDQ_n^\sigma(u, \tau(v)) = 0 \) unless \( \tau = 1 \), in which case \( DDQ_n^\sigma(u, v) = \pm e_i^* \).

Assume now that \( \mathcal{G} = \mathcal{Q}_o \) is an odd quadratic form grid ([N2,II.1.1]): \( \mathcal{Q}_o = \{ g_0 \} \sqcup \mathcal{Q}_e \), where \( g_0 \) governs every \( g \in \mathcal{Q}_e \) (\( g \in V_2(g_0) \) and \( g_0 \in V_1(g) \), denoted \( g_0 \vdash g \)) and \( \mathcal{Q}_e = \{ g_{\pm i}; 1 \leq i \leq m \} \), \( m = \frac{n-1}{2} \), is an even quadratic from grid, i.e., \( g_{+i} \perp g_{-i} \) and \( g_{\pm i} \perp g_{\pm j} \) for \( i \neq j \). In this case we can explicitly list a Capelli sequence. Our choice is analogous to the ortho-collinear case: we choose \( e_1 = g_{+1} \), let \( (u_1; v_1) = (g_{+1}; g_{+1}^*) \), enumerate \( \mathcal{G}_1(g_{+1}) = \mathcal{Q}_o \setminus \{ g_{\pm 1} \} \) and build in a wrinkle at the end of the sequence. In precise terms, we let

\[
\begin{align*}
u_1 &= g_{+1}^*, \ldots, u_m = g_{+m}^*, u_{m+1} = g_{-m}^*, \ldots, u_{n-2} = g_{-2}^*, u_{n-1} = g_{-1}^* = u_n \\
v_1 &= g_{-1}^*, \ldots, v_m = g_{-m}^*, v_{m+1} = g_{-m}^*, \ldots, v_{n-2} = g_{-2}^*, v_{n-1} = g_0^*, v_n = g_{-1}^*.
\end{align*}
\]

In \( DDQ_n^\sigma(u, \tau(v)) \) the product \( Q(u_1)v_{\tau(1)} \) is non-zero only if \( \tau(1) = 1 \). For \( \tau(1) = 1 \) we have \( Q(u_1)v_1 = u_1 \), and for \( 2 \leq j \leq n-2 \) a product \( \{ u_1 v_{\tau(j)} u_j \} \) is non-zero only if \( \tau(j) = j \) since \( \{ g_{\pm i}^*, g_{0}^* g_{\pm j}^* \} = 0 \) for \( i \neq j \). Therefore \( DDQ_n^\sigma(u, \tau(v)) \) vanishes unless \( \tau(j) = j \) for \( 1 \leq j \leq n-2 \), and in this case \( DDQ_n^\sigma(u, \tau(v)) = \{ \{ g_{+1}^*, v_{\tau(n-1)}, g_{-1}^* \} v_{\tau(n)} g_{-1}^* \} \) where \( \{ v_{\tau(n-1)}, v_{\tau(n)} \} = \{ g_{-1}^*, g_0^* \} \) (equality of sets). Since \( g_1 \perp g_{-1} \) we must have \( v_{\tau(n-1)} = g_0^* \) and \( v_{\tau(n)} = g_{-1}^* \) for \( DDQ_n^\sigma(u, \tau(v)) \) to be non-zero. Thus \( \tau = 1 \) and \( DDQ_n^\sigma(u, v) = \{ g_0^* g_{-1}^* g_{-1}^* \} = g_0^* \).

Finally we consider a hermitian grid \( \mathcal{H} = \{ h_{ij} ; 1 \leq i \leq j \leq r \} \) of rank \( r \geq 2 \) ([N2,II.1.2]). But since for \( r = 2 \), \( \mathcal{H} \) is an odd quadratic form grid we can assume \( r \geq 3 \). We recall that the relations and multiplication rules of the idempotents \( h_{ij} \in \mathcal{H} \) are an axiomatization of the relations and multiplication rules satisfied by the “hermitian matrix units” \( h_{ii} = (E_{ii}, E_{i}) \) and \( h_{ij} = (E_{ij} + E_{ji}, E_{ij} + E_{ji}) \), \( i \neq j \), in the Jordan pair of hermitian matrices. In particular, if we put \( h_{ij} = h_{ji} \), we have the following relations for distinct \( i, j, k, l \):

\[
(3) \quad h_{ij} \vdash h_{ii} \perp h_{jj}, \quad h_{ij} \perp h_{ik}, \quad h_{ij} \perp h_{kl}.
\]

Also in this case we can list a Capelli sequences \( (u; v) \) all at once. The pairs \( (u_j, v_j) \)
are:

\[(h_{11}^\sigma, h_{11}^{-\sigma}), (h_{1r}^\sigma, h_{1r}^{-\sigma}), (h_{1,r-1}^\sigma, h_{1,r-1}^{-\sigma}), \ldots, (h_{13}^\sigma, h_{13}^{-\sigma}), (h_{22}^\sigma, h_{12}^{-\sigma}), \]

\[\ldots\]

\[(h_{ii}^\sigma, h_{ii}^{-\sigma}), (h_{ir}^\sigma, h_{ir}^{-\sigma}), (h_{i,r-1}^\sigma, h_{i,r-1}^{-\sigma}), \ldots, (h_{i,i+2}^\sigma, h_{i,i+2}^{-\sigma}), (h_{i+1,i+1}^\sigma, h_{i,i+1}^{-\sigma}), \]

\[\ldots\]

\[(h_{r-2,r-2}^\sigma, h_{r-2,r-2}^{-\sigma}), (h_{r-2,r}^\sigma, h_{r-2,r}^{-\sigma}), (h_{r-1,r-1}^\sigma, h_{r-2,r-1}^{-\sigma}), \]

\[(h_{r-1,r}^\sigma, h_{r-1,r}^{-\sigma}), (h_{rr}^\sigma, h_{rr}^{-\sigma})\]

The reader will notice that this sequence is constructed in a way similar to the ortho-collinear case. One proceeds in r steps, with auxiliary idempotents \(e_i = h_i\) and subgrids \(\mathcal{H}^i = \{h_{pq}; i \leq p \leq q \leq r\}, 1 \leq i \leq r\). The \(i^{th}\) step for \(1 \leq i < r\) corresponds to the ortho-collinear case A). After having chosen \(l = (i - 1)(r - \frac{1}{2})\) elements \(u_j\) and an enumeration of \(\mathcal{H} \setminus \mathcal{H}^i\) one puts \((u_{i+1}, v_{i+1}) = (e_i^\sigma, e_i^{-\sigma})\), chooses an enumeration \((g_{l+2}, \ldots, g_m), m = i(r - \frac{i+1}{2})\) of \(\mathcal{H}^i(\epsilon)\), puts \((u_j, v_j) = (g_j^\sigma, g_j^{-\sigma}), l + 2 \leq j \leq m - 1\) and builds in a wrinkle at the end by putting \((u_m, v_m) = (e_{i+1}, g_m)\), where \((g_m = \epsilon, e_i, e_i+1)\) is a triangle in the sense of [N2;I.2.1]. One then continues with the \((i+1)^{th}\) step for which \(\mathcal{H}^{i+1} = \mathcal{H}_0^i(\epsilon)\). In the final \(r^{th}\) step one has \(\mathcal{H}^r = \{e_r\}\) which corresponds to the ortho-collinear case B).

It remains to prove that the sequence \((u; v)\) above is a Capelli sequence with \(\text{IC}_n^\sigma(u; v) = 2^{r-2}h_{1r}^\sigma\). By (3) we have \(\mathcal{H}_2(h_{11}) = \{h_{11}\}\), hence \(Q(h_{11}^\sigma)h_{pq}^{-\sigma} = 0\) unless \(pq = 11\) in which case we get \(h_{11}^\sigma\). One now has to consider products \(\{h_{1j}^\sigma, h_{pq}^{-\sigma}, h_{1j}^\sigma\}\) for \(j > 1\). Since \(h_{11} \in \mathcal{H}_2(h_{1j})\) such a product vanishes unless also \(h_{pq} \in \mathcal{H}_2(h_{1j})\). But by (3), \(\mathcal{H}_2(h_{1j}) = \{h_{11}, h_{1j}, h_{jj}\}\) and \(\{h_{11}^\sigma, h_{jj}^{-\sigma}, h_{1j}^\sigma\} = 0\). Since \(pq = 11\) was already chosen we must have \(pq = 1j\) in which case \(\{h_{11}^\sigma, h_{jj}^{-\sigma}, h_{1j}^\sigma\} = 2h_{11}^\sigma\). At the end of the first row we have

\[\{(\ldots Q(u_1)v_1, v_2, u_2) \ldots \}v_{(r)}u_r = 2^{r-2}\{h_{11}^\sigma, h_{pq}^{-\sigma}, h_{22}^\sigma\}\]

which vanishes unless \(h_{pq} \in \mathcal{H}_2(h_{11} + h_{22}) = \{h_{11}, h_{12}, h_{22}\}\). The only possible choice left is therefore \(pq = 12\) in which case we obtain \(\{h_{11}^\sigma, h_{12}^{-\sigma}, h_{22}^\sigma\} = h_{12}^\sigma\) by the multiplication rules for hermitian grids. At the end of the \((i - 1)^{th}\) row we arrive at a product \(\{(\ldots Q(u_1)v_1, v_2, u_2) \ldots \}v_{(r)}u_r = 2^{r-2}h_{11}^\sigma\): each term \(\text{DDQ}_n^\sigma(u, \tau(v))\) vanishes unless \(\tau\) fixes the first \(i - 1\) rows of \(h\)’s. The same continues to hold in the \(i^{th}\) row: \(\{h_{11}^\sigma, h_{pq}^{-\sigma}, h_{pq}^\sigma\}\) for \(i \leq p, q\) is zero unless \(pq = ii\) in which case it
reproduces \( h_{1i}^\sigma \), and \( \{ h_{1i}^\sigma, h_{pq}^-, h_{ij}^\sigma \} \) for \( i \leq p, q \) and \( j \geq i + 2 \) vanishes by rigid-collinearity of \( h_{1i} \) and \( h_{ij} \) unless \( pq = 1i \) or \( pq = ij \). Therefore \( pq = ij \) and in this case we get \( \{ h_{1i}^\sigma, h_{pq}^-, h_{ij}^\sigma \} = h_{1i}^\sigma \). At the end of the \( i^{th} \) row we have to consider \( \{ h_{1i}^\sigma, h_{pq}^-, h_{i+1,i+1}^\sigma \}, i \leq p, q \). Since \( h_{1i} \perp h_{i+1,i+1} \) this product is 0 except for \( (pq) = (i, i + 1) \) when it produces \( h_{1,i+1}^\sigma \). Continuing in this way proves that \((u; v)\) is a Capelli sequence with \( \text{IC}_n^\sigma(u; v) = 2r-2h_{1r}^\sigma \).

**Remark.** The proof above is inspired by the proofs of Propositions 16 - 19 of [RR] and of Proposition 1 of [RR2], where special cases of the theorem were proven. Indeed, suppose \( T = V^+ \oplus V^- \) is the polarized Jordan triple system of a Jordan pair \( V = (V^+, V^-) \). Any Jordan triple polynomial \( f \) on \( T \) has the form \( f = f^+ \oplus f^- \) for a Jordan pair polynomial \( (f^+, f^-) \) of \( V \). Hence, the theorem also holds for \( T \) and \( \text{IC}_n = \text{IC}_n^+ \oplus \text{IC}_n^- \). Interpreted in this way, it yields Propositions 16, 17 and 19 of [RR]. The theorem can also be interpreted for the Jordan pair \((T, T)\) associated with a Jordan triple system \( T \). In this way, one obtains Proposition 18 of [RR] and Proposition 1 of [RR2].

### 3. Split Jordan pairs.

For the purpose of this paper, it is appropriate to call a Jordan pair \( V \) over some base ring \( k \) split, or split of type \( \mathcal{G} \) in case we need to be more precise, if \( V \) is freely spanned by a finite connected grid \( \mathcal{G} \): \( V^\sigma = \bigoplus_{g \in \mathcal{G}} k \cdot g^\sigma \) for \( \sigma = \pm \). In this case, we can assume that \( \mathcal{G} \) is a finite connected standard grid [N3, 3.8], and hence \( V \) is obtained by base ring extension from the Jordan pair \( \langle \mathcal{G} \rangle = \bigoplus_{g \in \mathcal{G}} (Zg^+, Zg^-) \) over \( Z \): \( V = \langle \mathcal{G} \rangle \otimes_Z k \). All base ring extensions of \( V \) are then again split of type \( \mathcal{G} \), i.e., for any commutative unital \( k \)-algebra \( K \) we have \( V \otimes_k K \approx \langle \mathcal{G} \rangle \otimes_Z K \).

Any finite-dimensional simple Jordan pair over an algebraically closed field is split. This is obvious from the classification [L, 17.12], and is proven without classification in [N3, 3.11]. The classification of standard grids [N3] shows that over any given base ring \( k \) there are the following six types of split Jordan pairs which we describe using the notation of [L, 17.12]. We also give the dimension and indicate
if the Jordan pair has invertible elements.

<table>
<thead>
<tr>
<th>Type</th>
<th>Grid</th>
<th>Dimension</th>
<th>Invertible Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>I_{pq} (1 ≤ p ≤ q)</td>
<td>rectangular grid $R(p, q)$</td>
<td>pq if $p = q$</td>
<td></td>
</tr>
<tr>
<td>II_{n} (n ≥ 5)</td>
<td>symplectic grid $S(n)$</td>
<td>$\frac{n(n-1)}{2}$ if $n \equiv 0(2)$</td>
<td></td>
</tr>
<tr>
<td>III_{n} (n ≥ 2)</td>
<td>hermitian grid $H(n)$</td>
<td>$\frac{n(n+1)}{2}$ yes</td>
<td></td>
</tr>
<tr>
<td>IV_{n} (n ≥ 5)</td>
<td>quadratic form grid $Q_e(n)$ or $Q_0(n)$</td>
<td>$n$ yes</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>Bi-Cayley grid $B$</td>
<td>16 no</td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>Albert grid $A$</td>
<td>27 yes</td>
<td></td>
</tr>
</tbody>
</table>

The types IV_{n} can of course be defined for every $n$, but become isomorphic to other types for small $n$. In particular, we have

(1) $III_2 \approx IV_3, \quad I_{22} \approx IV_4.$

An integral Jordan polynomial $f \in FJP(X)^\sigma$ is a strict identity of some Jordan pair $V$ over $k$ if $f$ is an identity for all base ring extensions $V \otimes_k K$ of $V$. For split Jordan pairs of type $\mathcal{G}$ the following conditions are equivalent:

(2.a) $f$ is a strict identity of the integral Jordan pair $\langle \mathcal{G} \rangle$, i.e., an identity for all split Jordan pairs of type $\mathcal{G}$,

(2.b) $f$ is an identity of the complex Jordan pair $\langle \mathcal{G} \rangle \otimes \mathbb{C}$.

Indeed, there exist polynomials $f_g$ over $\mathbb{Z}$ in a finite number of variables (depending on $f$ and $\mathcal{G}$) such that for every $k$ and every evaluation of $f$ on $V = \langle \mathcal{G} \rangle \otimes k$ we have

$f(u, v) = \sum_{g \in \mathcal{G}} g^\sigma \otimes f_g(u, v)$. If $f$ vanishes identically on $\langle \mathcal{G} \rangle \otimes \mathbb{C}$, then because $\mathbb{C}$ is infinite the polynomials $f_g$ are the zero polynomials hence $f = 0$ on $V$.

4. Proposition. Let $ICQ_l(x_1, \ldots, x_l, z_1, \ldots, z_l; y) \in FJP(\{x_1, \ldots, x_l, z_1, \ldots, z_l\}, \{y\})^+$ be defined by

$ICQ_l(x_1, \ldots, x_l, z_1, \ldots, z_l; y) = IC_l^+(x_1, \ldots, x_l, Q_y z_1, \ldots, Q_y z_l)$.

Then $ICQ_l$ is an identity for all split Jordan pairs of type $\mathcal{G}$ in the following cases:

(a) $\mathcal{G} = R(p, q)$, $p^2 < l$;

(b) $\mathcal{G} = S(n)$, $n \equiv 1(2)$, $(n - 1)(n - 2) < 2l$. 
However, ICQ\(_l\) is not an identity of a split Jordan pair of dimension \(l\) containing invertible elements. In particular, if \(T(V)\) denotes the \(T\)-ideal of identities of a Jordan pair \(V\) we have

\[
(1) \quad p \leq p' \text{ and } q \leq q' \iff I_{pq} \subset I_{p'q'} \iff T(I_{pq}) \supset T(I_{p'q'}).
\]

**Proof.** (a) and (b) By (3.2) it suffices to show that ICQ\(_l\) is an identity for the Jordan pair \(V = \langle \mathcal{G} \rangle \otimes \mathbb{C}\). Since IC\(_l^+\)(\(x_1, \ldots, x_l, Q_y z_1, \ldots, Q_y z_l\)) is alternating multilinear in \(Q_y z_1, \ldots, Q_y z_l\), it is enough to prove that every inner ideal \(Q_y V^+, v \in V^-,\) has dimension \(< l\).

But \(V\) is simple nondegenerate and hence regular, \(v\) is part of an idempotent \(c = (c_+, v)\) of \(V\). Therefore \(Q_y V^+ = V_2^-(c) \subset V_2^-(e)\) for a maximal idempotent \(e\) of \(V\). By the conjugacy theorem [L, 17.1], \(V_2(e)\) has dimension \(p^2\) in case (a) and dimension \(\frac{1}{2}(n - 1)(n - 2)\) in case (b).

If \(V\) contains invertible elements there exists \(v \in V^-\) such that \(Q_v V^+ = V^-\) and hence by 2. a substitution for which ICQ\(_l\) does not vanish. Finally, with respect to (1), it is clear that \(p \leq p'\) and \(q \leq q'\) implies \(I_{pq} \subset I_{p'q'}\) which in turn implies \(T(I_{pq}) \supset T(I_{p'q'})\).

Assuming \(T(I_{pq}) \supset T(I_{p'q'})\) we will show \(p \leq p'\) and \(q \leq q'\): if \(p > p'\) then ICQ\(_{p^2}\) \(\in T(I_{p'q'})\) but ICQ\(_{p^2}\) \(\not\in T(I_{pq})\) since it is not an identity of \(I_{pp} \subset I_{pq}\). Therefore \(p \leq q,\) and because \(I_{pq} \approx I_{q'p'}\) \(I_{p'q'} \approx I_{q'p'}\) we then also have \(q \leq q'\).

**Remark.** A different proof for the equivalences (1) is given in [I, Theorem 1].

5. Jordan pair polynomials obtained from Jordan algebra polynomials. Let \(X\) be a set and let FJA\((X)\) be the free non-unital Jordan algebra on \(X\) over \(\mathbb{Z}\). We put \(X = (X, \{y\})\) for some \(y \not\in X\) and denote by FJP\((X)\)\(_y^+\) the \(y\)-homotope of FJP\((X)\) [L, 1.9]. By the universal property of FJA\((X)\) there exists a unique (non-unital) Jordan algebra homomorphism \(\psi : \text{FJA}(X) \rightarrow \text{FJP}(X)\)\(_y^+\) mapping every \(x \in X \subset \text{FJA}(X)\) onto \(x \in X \subset \text{FJP}(X)\). (It is easily seen that \(\psi\) is an isomorphism but we do not need this.) We define \(g^{\text{JP}} := \psi(g)\) for \(g \in \text{FJA}(X)\) and call it the Jordan homotope polynomial associated to \(g\).

Intuitively, \(g^{\text{JP}}\) is obtained as follows: write \(g\) as a sum of monomials where each monomial is a composition of maps \(U_x\), squaring operators \(x \mapsto x^2\) and left multiplications \(V_{x,z}\) defined by \(V_{x,z} u = \{xzu\} = Q_{x+u}z - Q_x z - Q_u z\) for \(x, z, u \in \text{FJA}(X)\); then replace each factor \(U_x\) by \(U_x^{(y)} = Q_x Q_y\), squaring operators \(x^2\) by \(x^{(2,y)} = Q(x)y\) and the left multiplications \(V_{x,z}\) by \(V_{x,z}^{(y)} = D(x, Q_y z)\). For example,
the polynomial ICQ(x, z; y) of Proposition 4 is a Jordan homotope polynomial
since the term DDQ(x, z; y) = D(x_n, Q_y z_n) · · · D(x_2, Q_y z_2) Q_{x_1} Q_y z_1 is the image
under ψ of the Jordan algebra polynomial VVU(x, z) = V_{x_n, z_n} · · · V_{x_2, z_2} U_{x_1} z_1. We
will later use the following examples of homotope polynomials:

(a) (Racine’s central polynomials) Let n ≥ 3. By [R, Theorem 2] there exist
homogeneous integral polynomials R_n(x_1, x_2) ∈ FJA({x_1, x_2}) which are central
polynomials of the Jordan algebra J = H_n(C) of hermitian matrices over an associ-
vative composition algebra C over a field k, i.e., R_n(J, J) ⊂ k · 1 where 1 is the
identity element of J. Since the map V_{x_3, x_4} − V_{x_4, x_3} is a derivation, the derived
version of R_n(x_1, x_2),

\[ DR_n(x_1, x_2, x_3, x_4) = \{ x_3, x_4, R_n(x_1, x_2) \} - \{ x_4, x_3, R_n(x_1, x_2) \} \]

is then a homogeneous integral polynomial identity of J. The associated inte-
gral homotope polynomials will be denoted R_n(x_1, x_2; y) := R_n(x_1, x_2)^{JP} and

\[ DR_n(x_1, x_2, x_3, x_4; y) := DR(x_1, x_2, x_3, x_4; y)^{JP} \]

called the Racine homotope polynomial respectively the derived Racine homotope polynomial.

(b) We recall from [McZ; (0.25) and (7.6)] that in a Jordan algebra the commu-
tator square is defined as C(x_1, x_2) = x_1 ◦ U_{x_2} x_1 − U_{x_1} x_2^2 − U_{x_2} x_1^2, and the standard
Clifford polynomial is

\[ SC(x_1, x_2, x_3, x_4) = \{ C(x_1, x_2), x_3, x_4 \} - \{ x_3, C(x_1, x_2), x_4 \}. \]

In the setting of Jordan triple systems the associated homotope polynomials were
introduced in [RR]. We denote them by

\[ C(x_1, x_2; y) := C(x_1, x_2)^{JP} = \{ x_1, y, Q_{x_2} Q_y x_1 \} - Q_{x_1} Q_y Q_{x_2} y - Q_{x_2} Q_y Q_{x_1} y, \]
\[ SC(x_1, \ldots, x_4; y) := SC(x_1, x_2, x_3, x_4)^{ JP} \]
\[ = \{ C(x_1, x_2; y), Q_y x_3, x_4 \} - \{ x_3, Q_y C(x_1, x_2; y), x_4 \}, \]

and call SC(x_1, x_2, x_3, x_4; y) the standard Clifford homotope polynomial. As we will
show in the following lemma, SC is in fact a Clifford homotope polynomial in the
spirit of [DMc2]: it vanishes on quadratic form pairs but not on the Jordan pair of
hermitian matrices of rank ≥ 3.

6. Lemma [RR]. The standard Clifford homotope polynomial SC does not vanish
on any Jordan pair containing a rectangular grid R(2, 3) or a hermitian grid H(3).
On the other hand, SC is an identity of rectangular matrix pairs of size $1 \times q$ and of quadratic form pairs.

In particular, SC divides the split Jordan pairs into two groups: it does not vanish on $I_{pq}(2 \leq p \leq q, 3 \leq q), II_n(5 \leq n), III_n(3 \leq n), V$ and VI, but it vanishes on $I_{q1}(1 \leq q)$ and $IV_n(3 \leq n)$.

Proof. The first part follows essentially from [RR, Prop. 2] and the remark on page 976/8 of [RR]. Indeed, for any triangle $(g; e_1, e_2)$ of idempotents in a Jordan pair $V$ one easily calculates $C(e_{\sigma 1}, e_{\sigma 2}; g_{\sigma}) = g_{\sigma}$. If $V$ contains a rectangular grid $R(2,3) = (e_{ij}; 1 \leq i \leq 2, 1 \leq j \leq 3)$ then $(e_{12} + e_{21}; e_{11}, e_{22})$ is a triangle, and since $Q(e_{12} - e_{21})e_{13} = 0$ and $e_{13} \perp e_{21}$ we obtain

$$SC(e_{11}, e_{22}, e_{13}, e_{22}; e_{12} + e_{21}) = 0 - \{e_{13}, e_{12}, e_{22}\} = -e_{23}.$$

Similarly, if $H(3) = (h_{ij}; 1 \leq i \leq j \leq 3) \subset V$ is a hermitian grid then $(h_{12}; h_{11}, h_{22})$ is a triangle and one verifies, using $Q(h_{12})h_{13} = 0$ by (2.3),

$$SC(h_{11}, h_{22}, h_{13}, h_{23}; h_{12}) = 0 - \{h_{13}, h_{12}, h_{22}\} = -h_{23}.$$

That SC vanishes on rectangular matrix pairs of size $1 \times q$ and on quadratic form pairs is shown in [RR, Prop. 1 and Prop. 2].

Concerning split Jordan pairs, one only has to observe that the first group contains $R(2,3)$ or $H(3)$ while in view of the isomorphisms (3.1) the second group is made up of special types of rectangular matrix pairs and quadratic form pairs.

7. Lemma. Let $k$ be an infinite field of characteristic $\neq 2$, and let $J$ be a finite-dimensional unital Jordan algebra over $k$. Then a homogeneous integral polynomial $g \in FJA(X)$ is a polynomial identity of $J$ if and only if the integral homotope polynomial $g^{IP}$ is an identity of the Jordan pair $V = (J, J)$.

Proof. Under a specialization $X^+ = X \rightarrow V^+ = J$ and $y \mapsto v \in V^- = J$, the polynomial $g^{IP}$ becomes the polynomial $g$ evaluated on the Jordan algebra $V_v^+$ which is nothing else but the $v$-homotope of $J$. Since $J = V_1^+$ for the unit element $1$ of $J = V^-$, it is clear that $g$ is an identity of $J$ if $g^{IP}$ is an identity of $V$. To prove the converse we may after a base field extension assume that $k$ is algebraically closed. It suffices to prove that $g^{IP}$ vanishes under any specialization of type $y \mapsto v$ where $v$ belongs to the Zariski-dense subset of invertible elements of $J$. But for such a $v$ the Jordan algebra $V_v^+$ is isomorphic to $J$ since $v$ is a square in $J$ [J, p. 60] and [J, VI.7 Lemma, p.242]. Therefore $g$ vanishes on $V_v^+$. 
8. Proposition. Let $n \geq 3$ and let $k$ be a ring without 2-torsion. The derived Racine homotope polynomial $\text{DR}_n(x_1, x_2, x_3, x_4; y)$, see 5(a), is an identity of the Jordan pairs $I_{nn}$, $II_{2n}$ and $III_{n}$ over $k$, but not of $I_{mm}$, $II_{2m}$ and $III_{m}$ for any $m > n$.

Proof. Let $C = k, k \oplus k$ and $\text{Mat}_2(k)$. The canonical involution of $C$ considered as composition algebra, i.e., $\text{Id}_k$ for $C = k$, the exchange involution for $C = k \oplus k$ and the symplectic involution for $C = \text{Mat}_2(k)$, extends naturally to an involution on the associative algebra of $m \times m$ matrices over $C$. Let $H_m(C)$ be the Jordan algebra of hermitian matrices with respect to this involution. The Jordan pairs $III_n, I_{nn}$ and $II_{2n}$ are the Jordan pairs of the Jordan algebras $H_m(C)$. Since $\text{DR}_n(x_1, \ldots, x_4) \in \text{FJA}$ vanishes on the Jordan algebra $H_n(C)$ over fields, the corresponding homotope polynomial $\text{DR}_n(x_1, \ldots, x_4; y)$ vanishes on the Jordan pair $(H_n(C), H_n(C))$ for $k = \mathbb{C}$ by Lemma 7 and then for arbitrary $k$ by (3.2).

To prove that the homotope polynomial $\text{DR}_n(x_1, x_2, x_3, x_4; y) = \text{DR}_n^{JP}$ does not vanish on the Jordan pairs $(H_m(C), H_m(C))$ for $m > n$, it suffices to establish non-vanishing of the Jordan algebra polynomial $\text{DR}_n$ on $H_m(C)$ since $\text{DR}_n^{JP}$ evaluated for $y \mapsto 1$ yields $\text{DR}_n$. This in turn will follow from

(1) $\text{DR}_n$ does not vanish on $H_m(C), m > n$, for fields of characteristic $\neq 2$.

Indeed, if (1) holds we can proceed as in [RR2, p. 2691]: since $k$ has no 2-torsion, there exists a prime ideal $\mathfrak{p}$ of $k$ not containing 2. The quotient field $F$ of $R = k/\mathfrak{p}$ has characteristic $\neq 2$. By (1), $\text{DR}_n$ is not an identity of $H_m(C)$ over $F$, clearing denominators by homogeneity then shows that $\text{DR}_n$ is not an identity of $H_m(C)$ over $R$ and hence also not over $k$.

It remains to prove (1). We use an argument from [DR, p.312]. By [R, Theorem 2] one knows $R_n(H_n(C), H_n(C)) = k \cdot 1$ if $k$ is a field of characteristic $\neq 2$. Hence $R_n(u_1, u_2) = 1$ for some $u_1, u_2 \in H_n(C)$. Viewing then $u_1, u_2 \in H_m(C)$ gives $R_n(u_1, u_2) = \text{diag}(1, \ldots, 1, 0, \ldots, 0) = c \in H_m(C)$, and taking $u_3 = h_{n+1,n+1} = E_{n+1,n+1}$ and $u_4 = h_{n,n+1} = E_{n,n+1} + E_{n+1,n}$ gives $\{u_3 u_4 c\} - \{u_4 u_3 c\} = \frac{1}{4} u_4$. This finishes the proof of the proposition.

9. Theorem. Let $k$ be a ring without 2-torsion. Then the split Jordan pairs over $k$ can be distinguished by the following integral polynomial identities:

(1) inner Capelli polynomials $\text{IC}$,
(2) inner Capelli homotope polynomials $\text{ICQ}$,
(3) derived Racine homotope polynomials DR,
(4) the standard Clifford homotope polynomial SC, and
(5) the Jordan pair analogue of the Glennie polynomial.

In particular, this is so for simple finite-dimensional Jordan pairs over algebraically-closed fields of characteristic $\neq 2$.

To be distinguishable by integral polynomial identities means that if $V$ and $W$ are split Jordan pairs of types $\mathcal{G}$ and $\mathcal{G}'$ respectively with $\mathcal{G} \neq \mathcal{G}'$ then one of the five Jordan polynomials listed above is an identity of one of them but not of the other. We will denote this by $V \leftrightarrow W$. That split Jordan pairs of type I can be distinguished by polynomial identities is also proven in [I].

Proof. We will distinguish between split Jordan pairs of different dimensions by an appropriate inner Capelli polynomial. In particular, $V \leftrightarrow VI$. By [LMc, Theorem 3.10] the Jordan pair version of the Glennie identity does not vanish on the exceptional Jordan pair $V$, hence neither on VI. On the other hand, it vanishes on all a-special Jordan pairs, in particular on the first four types I - IV. Hence we can distinguish between $V, VI$ and the a-special types so that in the following it is sufficient to consider only the types I - IV.

Within the classes II, III and IV we can distinguish by dimensions via IC-$l$’s. If $pq = p'q'$ with $p \neq p'$, say $p < p'$, we have $I_{pq} \leftrightarrow I_{p'q'}$ by evaluating IC-$l$ for $l = p'^2$: by Proposition 4, it vanishes on $I_{pq}$ but does not vanish on $I_{p'q'} \subset I_{p'q'}$. Thus, in the following we only need to distinguish between Jordan pairs belonging to different classes I, II, III or IV.

By Lemma 6, non-vanishing respectively vanishing of the standard Clifford homotope polynomial SC will divide the special split Jordan pairs into the following two disjoint sets:

$$\{I_{pq}(2 \leq p \leq q, 3 \leq q), II_n(5 \leq n), III_n(3 \leq n)\} \leftrightarrow \{I_{1q}(1 \leq q), IV_n(3 \leq n)\}.$$

Within the second set we can distinguish $I_{1n} \leftrightarrow IV_n$ by IC-$n$, see Proposition 4. We are therefore left with distinguishing Jordan pairs in different classes $I_{pq}(2 \leq p \leq q, 3 \leq q), II_n(5 \leq n)$ and $III_n(3 \leq n)$ which have the same dimension.

$I_{pq} \leftrightarrow III_n$: We have $pq = \frac{n(n+1)}{2}$. If $p < q$ the polynomial IC-$pq$ will distinguish between $I_{pq}$ and $III_n$ since $l = pq > p^2$. In case $p = q$ we have $p < n$ and hence $I_{pp} \leftrightarrow III_n$ by Proposition 8.
$\Pi_m \leftrightarrow \Pi_n$: We have $\frac{m(m-1)}{2} = \frac{n(n+1)}{2} =: l$, i.e., $m = n + 1$. If $m \equiv 1(2)$ we can use ICQ$_l$ to distinguish between $\Pi_m$ and $\Pi_n$ since $\frac{(m-1)(m-2)}{2} < l$. If $m \equiv 0(2)$ we have $\frac{n}{2} < n$ and hence $\Pi_m \leftrightarrow \Pi_n$ by Proposition 8.

$I_{pq} \leftrightarrow \Pi_n$: Equality of dimensions leads to

$$2pq = n(n-1) \tag{1}$$

For $p = q$ we will distinguish between $I_{pp}$ and $\Pi_n$ by applying Proposition 8. Namely, DR$_{p-1}$ does not vanish on $I_{pp}$ while it will vanish on any $\Pi_n$ which is imbeddable in a $\Pi_{2(p-1)}$, i.e., if $n \leq 2p - 2$. These two inequalities are fulfilled if (*) $n + 2 \leq 2p$. But since $2p^2 = n(n+1)$ the inequality (*) is true for any $n \geq 5$.

We can now assume $p < q$. In this case the polynomial ICQ$_l$ vanishes on $I_{pq}$ for any $l > p^2$ but does not vanish on $\Pi_{2m}$ if $l = \dim \Pi_{2m} = m(2m-1)$. Hence it does not vanish on $\Pi_n$ for which $\Pi_{2m} \subset \Pi_n$, i.e., for $2m \leq n$. This yields the inequality

$$p^2 < m(2m-1) \tag{2}$$

For even $n$ we can choose $n = 2m$ and then (2) follows from (1) and $p < q$. Hence in the following we can assume that $n$ is odd. We take $2m = n - 1$, and then (2) is equivalent to $2p^2 < (n-1)(n-2)$. Observe that we can distinguish between $I_{pq}$ and $\Pi_n$ using the polynomial ICQ$_l$ in another way: it does not vanish on $I_{pp} \subset I_{pq}$ for $l = p^2$ but it vanishes on $\Pi_n$ whenever $(n-1)(n-2) < 2l$ using Proposition 4(b) for $n$ odd, i.e., whenever $2p^2 > (n-1)(n-2)$. Therefore $I_{pq} \leftrightarrow \Pi_n$ if we show that the following system does not have an integral solution:

$$2p^2 = (n-1)(n-2) \ , \ 2pq = n(n-1) \ , \ 2 \leq p < q , \ n \equiv 1(2).$$

Since $n - 1$ and $n - 2$ have different prime divisors, the first equation implies that there exist relatively prime $a, b \in \mathbb{N}$ such that $n - 1 = 2a^2$, $n - 2 = b^2$, $p = ab$. Substituting this into the second equation yields $2abq = (b^2 + 2)2a^2$, thus $bq = a(b^2 + 2)$. Since $b$ is odd it is relatively prime to $b^2 + 2$, hence $b$ divides $a$. On the other hand, $n - 1 = 2a^2 = b^2 + 1$ implies that $a < b$ or that $a = b = 1$, which contradicts that $b$ divides $a$ or that $p \geq 2$.

**10. Corollary.** Over algebraically closed fields of characteristic $\neq 2$, integral polynomial identities distinguish the isomorphism classes of the following simple finite-dimensional Jordan structures:

(a) Jordan algebras;

(b) polarized Jordan triple systems.
For fields of characteristic 0, (a) is proven in [DR, Thm. 1].

Proof. (a) A Jordan pair polynomial $f \in \text{FJP}(X)^{\sigma}$, $X = (X^+, X^-)$, is also a Jordan algebra polynomial in the free Jordan algebra over $\mathbb{Z}$ on the generating set $X = X^+ \cup X^-$. If $J$ is a Jordan algebra, $f$ vanishes on the Jordan pair $(J, J)$ if and only if $f$ vanishes on $J$. Over the algebraic closure, isotopy of Jordan algebras is the same as isomorphism. Hence two simple Jordan algebras are isomorphic if and only if the corresponding Jordan pairs are isomorphic.

(b) Let $S = S^+ \oplus S^-$ and $T = T^+ \oplus T^-$ be two simple polarized Jordan triple systems. By [N1, Lemma A.1] and [N1, Thm. A.3], the Jordan pairs $S = (S^+, S^-)$ and $T = (T^+, T^-)$ are simple, and $S$ and $T$ are isomorphic as Jordan triple systems if and only if $S$ is isomorphic to $T = (T^+, T^-)$ or to $T^{op} = (T^-, T^+)$ Since simple finite-dimensional Jordan pairs always have an involution, we have $S \approx T \iff S \approx T$. As in (a), a Jordan pair polynomial $f \in \text{FJP}(X)^{\sigma}$ can be interpreted as a Jordan triple polynomial in the free Jordan triple system over $\mathbb{Z}$ on the generating set $X = X^+ \cup X^-$. Moreover, if $f$ vanishes on $S$ but not on $T$, then $f$ vanishes on $S$ and not on $T$.

References


Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario K1N 6N5, Canada. Email: neher@uottawa.ca