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ÉTALE DESCENT OF DERIVATIONS

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Abstract. We study étale descent of derivations of algebras with values in a module. The algebras under consideration are twisted forms of algebras over rings, and apply to all classes of algebras, notably associative and Lie algebras, such as the multiloop algebras that appear in the construction of extended affine Lie algebras. The main result is Theorem 2.7.

Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over an algebraically closed field k of characteristic 0. The celebrated affine Kac-Moody Lie algebras are of the form

$$\mathcal{E} = \mathcal{L} \oplus kc \oplus kd$$

where $\mathcal{L}$ is a (twisted) loop algebra of the form $L(\mathfrak{g}, \pi)$ for some diagram automorphism π of $\mathfrak{g}$. The element c is central and d is a degree derivation for a natural grading of $\mathcal{L}$.

It is thus natural to study the derivations of loop, or more generally multiloop, algebras. This is a problem with a long history going back to [BM], [Bl]. In some sense there is a prescient aspect to [BM], which seems to sense the existence of Lie algebras that would have multiloop algebras play the same role that the loop algebras play in the affine case. These algebras would come into being a decade and a half later in the form of extended affine Lie algebras (EALAs) and Lie tori (see [AABGP], [N2], [N3]).

The first author has shown that any EALA is built from a Lie torus at the "bottom" in a way reminiscent of the affine construction. Loosely speaking an

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EALA is always of the form

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{C} \oplus \mathcal{D}$$

where $\mathcal{L}$ is a Lie torus, $\mathcal{C}$ is central in $\mathcal{L} \oplus \mathcal{C}$ and $\mathcal{D}$ is a space of derivations of the bottom Lie torus $\mathcal{L}$. It is known, save for perfectly understood exceptions in absolute type A, that Lie tori are always multiloop algebras [ABFP] (but not conversely, except for nullity 1 as shown in [P2]). We begin to see the central importance that the understanding of the Lie algebra of k-linear derivations of multiloop algebras has for EALA theory. They are also important for the structure of universal central extensions [N1].

Exploiting the fact that a centreless Lie torus $\mathcal{L}$ which is finitely generated over its centroid R is étale (even Galois) locally isomorphic to the R-algebra $\mathfrak{g} \otimes_k R$, there is a very transparent way of understanding the derivations of $\mathcal{L}$. The main idea (see [P1] for details) is disarmingly simple and we outline it here since it will serve as the blueprint for our work: Let S/R be an étale covering that trivializes $\mathcal{L}$.

(a) As shown in [BM], upstairs, namely for $\mathfrak{g} \otimes_k S$, the picture is perfectly understood: Besides the inner derivations we have $\mathrm{Der}_k(S)$ acting naturally as derivations of $\mathfrak{g} \otimes_k S$.

(b) One can recover S from $\mathfrak{g} \otimes_k S$ as its centroid. Thus (a) says that the derivations of the centroid of $\mathfrak{g} \otimes_k S$ extend naturally to $\mathfrak{g} \otimes_k S$. More precisely, $\mathrm{Der}_k(\mathfrak{g} \otimes_k S) \simeq \mathrm{IDer}(\mathfrak{g} \otimes_k S) \rtimes \mathrm{Der}_k(S)$ with $\mathrm{IDer}(\mathfrak{g} \otimes_k S) \simeq \mathrm{Der}_k(\mathfrak{g}) \otimes_k S$.

(c) By descent considerations ([GP, Lemma 4.6]) the centroid of $\mathcal{L}$ is R. Every derivation of $\mathcal{L}$ naturally induces a derivation of R, but it is not obvious that the derivations of R can be lifted to $\mathcal{L}$ (as they are in the trivial case (a) above).

(d) Because S/R is étale, every k-linear derivation of R extends uniquely to a k-linear derivation of S, hence to a derivation of $\mathfrak{g} \otimes_k S$ by (a).

(e) The derivation of $\mathfrak{g} \otimes_k S$ defined in (d) descends to $\mathcal{L}$.

The resulting picture is thus completely analogous to the one of the trivial case:

$$\mathrm{Der}_k(\mathcal{L}) \simeq \mathrm{IDer}_k(\mathcal{L}) \rtimes \mathrm{Der}_k(R).$$

A close inspection shows that more important than the structure of $\mathrm{Der}_k(\mathcal{L})$ to the theory of EALAs of central extensions is the structure of $\mathrm{Der}_k(\mathcal{L}, N)$ where N is the graded dual of $\mathcal{L}$. Rather than studying this particular case, we set out to see if the descent formalism will shed information about the structure of $\mathrm{Der}_k(\mathcal{L}, N)$ for an *arbitrary* $\mathcal{L}$-module N and S/R-form $\mathcal{L}$. The answer is a resounding yes. There are, however, subtle technical and philosophical difficulties to overcome before one can even state a structure result. This is already quite evident in (b) and (c) above. While a derivation of $\mathcal{L}$ induces a derivation of its centroid, what does a derivation in $\mathrm{Der}_k(\mathcal{L}, N)$ induce, and on what? As we shall see, proceeding in a natural way will guide us — as it always seems to do — towards the correct answer: Theorem 2.7.

Although in this introduction we have emphasized Lie algebras, the main results of this paper will be established for arbitrary algebras. This substantially broadens the applications of our work. For example, in 3.3 we use our main theorem to derive a new characterization of the first Hochschild cohomology group of separable associative algebras. Our approach is new even when specialized to the previously known cases.

Throughout, k will be a commutative associative unital ring. We denote by k-**alg** the category of commutative associative unital k-algebras with unital algebra homomorphisms as morphisms. If $S \in k$-**alg** and N is a k-module, we put $N_S = N \otimes_k S$. The term *k-algebra* will mean a k-module A together with a k-bilinear map $A \times A \to A$, $(a_1, a_2) \mapsto a_1 a_2$. In particular, we do not require any further identities (even though our interest is mostly with associative or Lie algebras). Also note that the left and right k-module structures of A are assumed to coincide: $ca = ac$ for all $c \in k$ and $a \in A$.

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1. Dimodules

We want to define the concept of a derivation of an *arbitrary* k-algebra A with coefficients in a k-module M. Our guiding principle is the Leibnitz rule. To make sense of it in the most general way we introduce the concept of dimodule. Proceeding in this way, we can apply our results to various classes of algebras (associative, Lie, Jordan, ...).

Let A be a k-algebra. An (A, k)-*dimodule* is a k-module M together with a pair of k-bilinear maps $A \times M \to M, (a, m) \mapsto a \cdot m$ and $M \times A \to M, (m, a) \mapsto m \cdot a$, which we call the *left* and *right* actions of A on M.[1]

Remark 1.1. The notion of an (A, k)-dimodule is stable under base change. Indeed, if $K \in k$-**alg** and M is a (A, k)-dimodule, then M_K is naturally a (A_K, K)-dimodule by defining

$$(a \otimes s_1) \cdot (m \otimes s_2) = (a \cdot m) \otimes s_1 s_2 \quad \text{and} \quad (m \otimes s_1) \cdot (a \otimes s_2) = (m \cdot a) \otimes s_1 s_2 \quad (1.1)$$

for all $a \in A$, $m \in M$ and $s_1, s_2 \in S$.

Example 1.2. The algebra A is in a natural way (via left and right multiplication) an (A, k)-dimodule. Similarly, $A^* = \mathrm{Hom}_k(A, k)$ is canonically an (A, k)-dimodule by defining $a \cdot \varphi$ and $\varphi \cdot a$ for $a \in A$, $\varphi \in A^*$ as follows: $(a \cdot \varphi)(a') = \varphi(a'a)$ and $(\varphi \cdot a)(a') = \varphi(aa')$ for $a' \in A$.

Suppose that A is in fact an R-algebra for some $R \in k$-**alg**. Then A^* is naturally an R-module via $(r \cdot \varphi)(a) = \varphi(ra)$ for $\varphi \in A^*$, $a \in A$ and $r \in R$. The A-action on A^* is compatible with the R-action in the following sense:

$$r \cdot (a \cdot \varphi) = (ra) \cdot \varphi = a \cdot (r \cdot \varphi) \quad \text{and} \quad r \cdot (\varphi \cdot a) = (r \cdot \varphi) \cdot a = \varphi \cdot (ra). \quad (1.2)$$

[1] We intentionally put no compatibility assumptions between the two actions. The authors are aware that this rather general concept does not agree with the usual notion of a bimodule if A is, for example, an associative algebra or a Lie algebra. We hope that the use of different terminology will avoid any possible confusion and that the following results will convince the reader of the usefulness of this new more general concept.

Thus A^* is an (A, R)-dimodule.

Given two (A, k)-dimodules M and N, an (A, k)-dimodule morphism $f : M \to N$ is a k-linear map satisfying $f(a \cdot m) = a \cdot f(m)$ and $f(m \cdot a) = f(m) \cdot a$ for $a \in A$ and $m \in M$. We thus have an obvious category of (A, k)-dimodules for any given algebra A. We leave it to the reader to define in the (A, k)-dimodule setting the concepts of submodule, kernel, quotient ...

For the remainder of this section A will denote a k-algebra and M an (A, k)-dimodule.

1.1. Derivations

A *derivation* of A with values in M is a k-linear map $d : A \to M$ satisfying $d(a_1 a_2) = d(a_1) \cdot a_2 + a_1 \cdot d(a_2)$ for $a_i \in A$. We denote by $\mathrm{Der}_k(A, M)$ the k-module of derivations of A with values in M. Note that $\mathrm{Der}_k(A, A) =: \mathrm{Der}_k(A)$ has a natural k-Lie algebra structure: The commutator of two derivations of A is a derivation of A.

Proposition 1.3. *Let A and M be as above and let $K \in k$-**alg**. Then:*

(a) *The canonical map $\omega : \mathrm{Hom}_k(A, M) \otimes_k K \to \mathrm{Hom}_K(A_K, M_K)$, given by $f \otimes x \mapsto f \otimes' x$ where $(f \otimes' x)(a \otimes y) = f(a) \otimes xy$ for $a \in A$ and $x, y \in K$, maps $\mathrm{Der}_k(A, M) \otimes_k K$ to $\mathrm{Der}_K(A_K, M_K)$.*

(b) *If K is a flat k-module and A is finitely presented as a k-module, then ω is an isomorphism for the K-modules considered in* (a)*, in particular,*

$$\mathrm{Der}_k(A, M)_K \simeq \mathrm{Der}_K(A_K, M_K). \tag{1.3}$$

Note. To simplify notation we will often write $f \otimes x$ for $f \otimes' x$. This is certainly justified in the setting of (b).

Proof. (a) is immediate. (b) Define $\delta_A : \mathrm{Hom}_k(A, M) \to \mathrm{Hom}_k(A \otimes_k A, M)$ as the unique k-linear map satisfying

$$\delta_A(f)\,(a_1 \otimes a_2) = f(a_1 a_2) - f(a_1) \cdot a_2 - a_1 \cdot f(a_2)$$

and observe $\mathrm{Der}_k(A, M) = \mathrm{Ker}\,\delta_A$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_k(A, M)_K \arrow[r] \arrow[d, "\omega|_{\mathrm{Der}}"] & \mathrm{Hom}_k(A, M)_K \arrow[r, "\delta_A \otimes \mathrm{Id}"] \arrow[d, "\omega"] & \mathrm{Hom}_k(A \otimes_k A, M) \otimes_k K \arrow[d, "\varepsilon"] \\
0 \arrow[r] & \mathrm{Der}_K(A_K, M_K) \arrow[r] & \mathrm{Hom}_K(A_K, M_K) \arrow[r, "\delta_{A \otimes K}"] & \mathrm{Hom}_K(A_K \otimes_K A_K, M_K)
\end{tikzcd}$$

where the top row is exact because K is flat and the bottom row is exact by definition of $\mathrm{Der}_K(A_K, M_K)$. Bijectivity of $\omega|_{\mathrm{Der}}$ now easily follows since ω is an isomorphism and ε is injective [B:AC, I, §2.10, Prop. 11]. $\square$

Example 1.4 (Lie algebras)**.** If L is a Lie algebra, an L-module M (in the usual sense) has an (L,k)-dimodule structure (that we call *canonical*) as follows. The left action is the given module action. The right action is defined by $m \cdot l = -l \cdot m$ for $l \in L$, $m \in M$. We leave it to the reader to check that the definition of $\mathrm{Der}_k(L,M)$ coincides with the usual definition of derivations of L with values in M. We will also consider the subdimodule of $\mathrm{Der}_k(L,M)$ consisting of *inner derivations* of M defined as usual by

$$\mathrm{IDer}_k(L,M) = \{\partial_m : m \in M\}, \quad \partial_m(l) = l \cdot m.$$

Corollary 1.5. *Let L be a Lie k-algebra and M an L-module.*

(a) *The canonical map*

$$\mathrm{IDer}_k(L,M) \otimes_k K \to \mathrm{IDer}_K(L_K, M_K), \quad \partial_m \otimes s \mapsto \partial_{m \otimes s}$$

is a well-defined epimorphism of k-modules.

(b) *The map in* (a) *is injective, whence an isomorphism, if K is a flat k-module and L is finitely generated as a k-module.*

(c) *Assume that K is a faithfully flat k-module and that L is finitely presented as a k-module. Then*

$$\mathrm{IDer}_k(L,M) = \mathrm{Der}_k(L,M) \quad \Longleftrightarrow \quad \mathrm{IDer}_k(L_K, M_K) = \mathrm{Der}_K(L_K, M_K).$$

Proof. (a) is immediate. Under the hypothesis of (b), the map ω of Proposition 1.3 is injective. (c) follows from (b) and the proposition. $\square$

Example 1.6 (Rings with twisted derivations)**.**[2] Even in the setting of associative algebras, derivations into dimodules rather than bimodules arise naturally in invariant theory. Recall [KP, §1] that a *ring with twisted derivations* is a quadruple $(A, I, (\varphi_i)_{i\in I}, (d_i)_{i\in I})$ consisting of a k-algebra A, an index set I, a family $(\varphi_i)_{i\in I}$ of automorphisms $\varphi_i \in \mathrm{Aut}_k(A)$ and a family $(d_i)_{i\in I}$ of endomorphisms $d_i \in \mathrm{End}_k(A)$ satisfying

$$d_i(a_1\, a_2) = d_i(a_1)\, a_2 + \varphi_i(a_1)\, d_i(a_2)$$

for all $a_1, a_2 \in A$. Given $(A, I, (\varphi_i)_{i\in I})$, let $M = \prod_{i\in I} M_i$ where M_i is the A-dimodule with $M_i = A$ as k-module and A-actions given by $a \cdot m_i = \varphi_i(a) m_i$ and $m_i \cdot a = m_i a$ (in both equations we use the multiplication of A on the right-hand side). The canonical isomorphism $\mathrm{Hom}_k(A,M) \simeq \prod_{i\in I} \mathrm{Hom}_k(A, M_i)$ induces a bijection between $\mathrm{Der}_k(A,M)$ and the set of all rings of twisted derivations of the form $(A, I, (\varphi_i)_{i\in I}, (d_i)_{i\in I})$.

1.2. Centroids

We remind the reader that throughout this section A denotes a k-algebra and M an (A,k)-dimodule.

[2] We thank Kirill Zainoulline for pointing out this example.

Definition 1.7. The *centroid of A with values in M* is defined as

$$\begin{aligned}\mathrm{Ctd}_k(A,M) = \{\chi \in \mathrm{Hom}_k(A,M) : \chi(a_1a_2) &= \chi(a_1)\cdot a_2 \\ &= a_1 \cdot \chi(a_2) \text{ for all } a_1, a_2 \in A\}.\end{aligned}$$

Observe that $\mathrm{Ctd}_k(A,A)$ is the standard centroid of the k-algebra A, which justifies our terminology and notation. For any k-algebra A there is always a canonical ring homomorphism

$$\chi\colon k \to \mathrm{Ctd}_k(A) \tag{1.4}$$

which sends $s \in k$ to χ_s defined by $\chi_s(a) = sa$ for $a \in A$. We call A *central* if χ is an isomorphism.

The next results collect some basic but important results about centroids. The mostly elementary proofs will be left to the reader.

Lemma 1.8. *Let $R \in k$-**alg**, and suppose that M has an R-module structure which is compatible with the (A,k)-dimodule structure in the following sense,*

$$r(m\cdot a) = (rm)\cdot a \quad \textit{and} \quad r(a\cdot m) = a\cdot(rm) \tag{1.5}$$

for $r \in R$, $a \in A$ and $m \in M$.

Assume now that A is also an R-algebra. Then the following hold:

(a) $\mathrm{Hom}_k(A,M)$ *is an R-bimodule via*

$$(r\cdot f)(a) = r\big(f(a)\big) \quad \textit{and} \quad (f\cdot r)(a) = f(ra) \tag{1.6}$$

for $r \in R$, $f \in \mathrm{Hom}_k(A,M)$ and $a \in A$.

(b) $\mathrm{Ctd}_k(A,M)$ *is a subbimodule of* $\mathrm{Hom}_k(A,M)$.

(c) $\mathrm{Der}_k(A,M)$ *is a submodule with respect to the left R-module structure of* $\mathrm{Hom}_k(A,M)$. □

Recall that an algebra B over some $R \in k$-**alg** is *perfect* if it equals its derived algebra B', defined to consist of the sums of products b_1b_2 with $b_i \in B$. It is immediate that the computation of the derived algebra commutes with base ring extensions: $(B')_S = (B_S)'$. In particular, if B is perfect then so is B_S for any $S \in R$-**alg**. The converse holds in the following situation.

Lemma 1.9. *Let B be an algebra over $R \in k$-**alg** and let $S \in R$-**alg** be faithfully flat. Then B is perfect if and only if B_S is.*

Proof. If B_S is perfect, then $0 = B_S/(B_S)' \simeq (B/B') \otimes_R S$ by flatness, whence $B/B' = 0$ by faithful flatness. □

Lemma 1.10. *Let A be a perfect R-algebra for some $R \in k$-**alg**, and suppose that M is also an R-module such that the R-module and the (A,k)-dimodule structures are related by*

$$r(a\cdot m) = (ra)\cdot m \tag{1.7}$$

for $r \in R$, $a \in A$ and $m \in M$. Then every k-linear centroidal transformation is already R-linear:

$$\mathrm{Ctd}_k(A,M) = \mathrm{Ctd}_R(A,M).$$

Furthermore, if also (1.5) *holds, the two R-module structures of* $\mathrm{Ctd}_k(A,M)$ *defined in* (1.6) *coincide.* □

We note that the conditions (1.5) and (1.7) are always fulfilled if M is an (A,R)-dimodule, where M is viewed as an (A,k)-dimodule by restriction of scalars.

Lemma 1.11. *Suppose that $R \in k$-**alg**, B is an R-algebra whose underlying R-module is finitely presented, N is a (B,R)-dimodule and $S \in R$-**alg** is a flat extension. Then the canonical map*

$$\mathrm{Ctd}_R(B,N) \otimes_R S \to \mathrm{Ctd}_S(B_S,N_S)$$

is an isomorphism of S-modules, where N_S is the (B_S,S)-dimodule obtained from the (B,R)-dimodule N by the base change S/R; see (1.1).

Proof. The proof is similar to that of [P1, Lemma 3.1], which deals with the case $N = B$. □

Corollary 1.12. *Let A be a central k-algebra which is finitely presented as k-module, let $R \in k$-**alg** be a flat extension, S/R a faithfully flat extension, and let B be an S/R-form of A, i.e., B is an R-algebra such that $B \otimes_R S$ is isomorphic as an S-algebra to $(A \otimes_k R) \otimes_R S \simeq A \otimes_k S$. Then B is a finitely presented R-algebra, and the canonical map χ of* (1.4) *is an isomorphism. In particular, B is a central R-algebra.*

Proof. Since S/k is flat by transitivity of flatness, Lemma 1.11 shows $\mathrm{Ctd}_S(A_S) \simeq \mathrm{Ctd}_k(A) \otimes_k S \simeq S$. Next we observe that B is a finitely presented R-algebra since finite presentation is preserved by faithfully flat descent [B:AC, I, §3.6, Prop. 11]. Hence, a second application of Lemma 1.11 yields $\mathrm{Ctd}_R(B) \otimes_R S \simeq \mathrm{Ctd}_S(B_S) \simeq \mathrm{Ctd}_S(A_S) \simeq S$. Thus χ is an isomorphism by [B:AC, I, §3.1, Prop. 2]. □

2. Derivations of twisted forms with values in a dimodule

2.1. The natural map $\eta : \mathrm{Der}_k(B,N) \to \mathrm{Der}_k(R, \mathrm{Ctd}_k(B,N))$

In this section B is an R-algebra for some $R \in k$-**alg** and N is a (B,R)-dimodule.

Proposition 2.1.

(a) *For $d \in \mathrm{Der}_k(B,N)$ and $r \in R$ the map $\eta_{B,N}(d)(r) \colon B \to N$, defined by $\big(\eta_{B,N}(d)(r)\big)(b) = d(rb) - rd(b)$, lies in* $\mathrm{Ctd}_k(B,N)$.

(b) *The map*

$$\eta_{B,N} \colon \mathrm{Der}_k(B,N) \to \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B,N)\big), \quad d \mapsto \eta_{B,N}(d) \tag{2.1}$$

is a well-defined k-linear map. It gives rise to the exact sequence

$$0 \to \mathrm{Der}_R(B,N) \to \mathrm{Der}_k(B,N) \xrightarrow{\eta_{B,N}} \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B,N)\big). \tag{2.2}$$

Proof. We denote $\eta_{B,N}$ by η. (a) Let $r \in R$ and put $\bar{d} = \eta(d)(r)$. Then for all $b_1, b_2 \in B$ we have

$$\begin{aligned}\bar{d}(b_1b_2) &= d(rb_1b_2) - rd(b_1b_2) = d(b_1(rb_2)) - rd(b_1b_2)\\ &= b_1 \cdot d(rb_2) + d(b_1)\cdot(rb_2) - r(d(b_1)\cdot b_2) - r\big(b_1 \cdot d(b_2)\big)\\ &= b_1 \cdot d(rb_2) - b_1 \cdot \big(rd(b_2)\big) = b_1 \cdot \big(\bar{d}(b_2)\big)\end{aligned}$$

where we used $d(b_1)\cdot(rb_2) = r(d(b_1)\cdot b_2)$ since N is a (B,R)-dimodule. One can similarly show $\bar{d}(b_1b_2) = \big(\bar{d}(b_1)\big)\cdot b_2$, thus proving that $\bar{d} \in \mathrm{Ctd}_k(B,N)$.

In (b) we first verify that $\eta(d) =: \tilde{d}$ is a derivation, which means $\tilde{d}(r_1r_2) = \tilde{d}(r_1)\cdot r_2 + r_1 \cdot \tilde{d}(r_2)$ in $\mathrm{Ctd}_k(B,N)$. For $b \in B$ we get in view of Lemma 1.8

$$\begin{aligned}\big(\tilde{d}(r_1r_2)\big)(b) &= d(r_1r_2b) - r_1r_2d(b) = d(r_1r_2b) - r_1d(r_2b) + r_1d(r_2b) - r_1r_2d(b)\\ &= \big(\tilde{d}(r_1)\cdot r_2\big)(b) + \big(r_1 \cdot \tilde{d}(r_2)\big)(b),\end{aligned}$$

which proves our claim. We therefore have a well-defined k-linear map

$$\eta : \mathrm{Der}_k(B,N) \to \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B,N)\big).$$

That η also has the other property stated in (b) is then immediate. $\square$

2.2. A section of η: Untwisted case

The map η of Proposition 2.1 admits a natural section whenever the algebra B comes from k. More precisely:[3]

Lemma 2.2. *Let A be a perfect k-algebra, $S \in k$-**alg** and M an (A_S, S)-dimodule. Then*

$$\sigma_{A_S,M}\colon \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S,M)\big) \to \mathrm{Der}_k(A_S,M), \quad \sigma_{A_S,M}(d)(a\otimes s) = d(s)(a\otimes 1_S)$$

is a well-defined k-linear map and a section of the map $\eta_{A_S,M}$ of (2.1)*; in particular, $\eta_{A_S,M}$ is surjective and*

$$\mathrm{Der}_k(A_S,M) \simeq \mathrm{Der}_S(A_S,M) \oplus \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S,M)\big) \tag{2.3}$$

as S-modules.

Proof. The map $A \times S \to M$, $(a,s) \mapsto d(s)(a\otimes 1_S)$ is well-defined and k-balanced, hence gives rise to a well-defined k-linear map $\sigma_{A_S,M}(d)$ which we denote by $\sigma(d)$. Next we check that $\sigma(d)$ is a derivation. With the obvious notation we have

$$\begin{aligned}\sigma(d)\big((a_1\otimes s_1)(a_2\otimes s_2)\big) &= \sigma(d)(a_1a_2\otimes s_1s_2) = d(s_1s_2)(a_1a_2\otimes 1_S)\\ &= \big(d(s_1)\cdot s_2 + s_1\cdot d(s_2)\big)(a_1a_2\otimes 1_S)\\ &= d(s_1)(a_1a_2\otimes s_2) + s_1\cdot d(s_2)(a_1a_2\otimes 1_S)\end{aligned}$$

[3]The change of notation from section 2.1 replacing B by A, R by k and N by M is put into place to match future references.

$$
\begin{aligned}
&= d(s_1)(a_1a_2 \otimes s_2) + d(s_2)(a_1a_2 \otimes s_1)\\
&\qquad \text{(since } s_1 \cdot d(s_2) = d(s_2)\cdot s_1 \text{ by Lemma 1.10)}\\
&= d(s_1)\big((a_1\otimes 1_S)(a_2\otimes s_2)\big) + d(s_2)\big((a_1\otimes s_1)(a_2\otimes 1_S)\big)\\
&= \big(d(s_1)(a_1\otimes 1_S)\big)\cdot(a_2\otimes s_2) + (a_1\otimes s_1)\cdot\big(d(s_2)(a_2\otimes 1_S)\big)\\
&= \big(\sigma(d)(a_1\otimes s_1)\big)\cdot(a_2\otimes s_2) + (a_1\otimes s_1)\cdot\big(\sigma(d)(a_2\otimes s_2)\big).
\end{aligned}
$$

Let $\eta = \eta_{A_S,M}$. Next we verify that σ is a section of η, i.e., $\big((\eta\circ\sigma)(d)\big)(s_1) = d(s_1)$ when evaluated on $a\otimes s_2 \in A\otimes_k S$:

$$
\begin{aligned}
\big(((\eta\circ\sigma)(d))(s_1)\big)(a\otimes s_2) &= \sigma(d)\big(s_1(a\otimes s_2)\big) - s_1\big(\sigma(d)(a\otimes s_2)\big)\\
&\qquad \text{(by the definition of } \eta)\\
&= \sigma(d)(a\otimes s_1s_2) - s_1\big(d(s_2)(a\otimes 1_S)\big)\\
&= d(s_1s_2)(a\otimes 1_S) - (s_1\cdot d(s_2))(a\otimes 1_S)\\
&= (d(s_1)\cdot s_2)(a\otimes 1_S) = d(s_1)(a\otimes s_2). \qquad \square
\end{aligned}
$$

Remark 2.3. We describe how this lemma relates to previously obtained results in the case when $M = A_S$, considered as a natural (A_S, S)-dimodule as in Example 1.2. In this case (2.3) becomes

$$\mathrm{Der}_k(A_S) \simeq \mathrm{Der}_S(A_S) \oplus \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S)\big). \tag{2.4}$$

In [Az1] S. Azam considers perfect algebras over a field k and $S \in k$-**alg** for which the canonical map $\mathrm{Ctd}_k(A)\otimes_k S \to \mathrm{Ctd}_k(A\otimes_k S)$ is an isomorphism of S-algebras. Assuming this, we get

$$\mathrm{Der}_k(A_S) \simeq \mathrm{Der}_S(A_S) \oplus \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A)\otimes_k S\big), \tag{2.5}$$

a decomposition which coincides with [Az1, Thm. 2.9]. In particular, if A is finite-dimensional, we have $\mathrm{Der}_k(S, \mathrm{Ctd}_k(A)\otimes_k S) \simeq \mathrm{Ctd}_k(A)\otimes_k \mathrm{Der}_k(S)$, thus using Proposition 1.3:

$$
\begin{aligned}
\mathrm{Der}_k(A_S) &\simeq \mathrm{Der}_S(A_S) \rtimes \big(\mathrm{Ctd}_k(A)\otimes_k \mathrm{Der}_k(S)\big)\\
&\simeq \big(\mathrm{Der}_k(A)\otimes_k S\big) \rtimes \big(\mathrm{Ctd}_k(A)\otimes_k \mathrm{Der}_k(S)\big).
\end{aligned} \tag{2.6}
$$

The isomorphism (2.6) had been previously established by Benkart–Moody [BM, Thm. 1.1] and Block [Bl, Thm. 7.1].

2.3. A section of η : Twisted case

Our goal is to extend (2.3) to the setting of étale forms of $A\otimes_k R$, namely algebras B over R such that $B\otimes_R S$ is isomorphic as an S-algebra to $(A\otimes_k R)\otimes_R S \simeq A\otimes_k S$ for some *étale covering* S/R, by which we mean that S/R is an étale extension, i.e., flat and unramified, which is also faithfully flat.[4] We pause to remind the reader that up to R-isomorphism we may assume that B is an R-subalgebra of $A\otimes_k S$: The algebra B can be defined in terms of cocycles, just as one does in usual Galois cohomology (see [KO, II] and [P1] for details and references).

[4] It would be more correct that Spec(S) is a covering of Spec(R) on the étale site of Spec(R).

2.4. Recall that k denotes our base ring. We will make the following natural assumptions, analogous in spirit to those made in [P1, Thm. 4.2].

(i) A is a perfect k-algebra which is finitely presented (in particular of finite type) as a k-module.
(ii) $R \in k$-**alg** is a flat extension of k.
(iii) $S \in R$-**alg** is an étale covering of R.
(iv) $B \subset A_S = A \otimes_k S$ is an (S/R)-form of A_R.
(v) N is a (B, R)-dimodule.

Our goal is to describe the nature of $\mathrm{Der}_k(B, N)$. To do this we need the following additional crucial assumption. As we see later in Lemma 2.8, the assumption is fulfilled in the most important case of a finite étale covering, in other words when B is isotrivial.

(vi) There exists an R-linear map $\pi : N_S = N \otimes_R S \to N$ which satisfies
$$\pi(b \cdot m) = b \cdot \pi(m) \quad \text{and} \quad \pi(m \cdot b) = \pi(m) \cdot b \quad \text{for } b \in B,\ m \in N_S \tag{2.7}$$
where the (B, R)-dimodule structure of N_S is given by (1.1).

We will immediately put assumption (iii) to good use to show how the description of $\mathrm{Der}_k(B, N)$ becomes a descent problem.

Proposition 2.5. *Let P be an R-module. Then there exists a canonical injective map*
$$\varepsilon_P : \mathrm{Der}_k(R, P) \to \mathrm{Der}_k(S, P_S). \tag{2.8}$$
This map is such that after identifying R and P with subsets of S and P_S, respectively, we have
$$\big(\varepsilon_P(d)\big)(r) = d(r) \tag{2.9}$$
for $d \in \mathrm{Der}_k(R, P)$ and $r \in R$.

Proof. Since S/R is faithfully flat (being a covering), we can canonically identify R (respectively P) with a subset of S (respectively P_S). Since S/R is also étale, the result is well-known; see [EGA IV, Chap. 0, §20]. ☐

By Lemma 1.9, B is perfect. It is also finitely presented as explained in the proof of Corollary 1.12. From Lemma 1.10 and Lemma 1.11 together with our assumptions it follows that
$$\mathrm{Ctd}_k(B, N) \otimes_R S = \mathrm{Ctd}_R(B, N) \otimes_R S \simeq \mathrm{Ctd}_S(B_S, N_S) \simeq \mathrm{Ctd}_k(A_S, N_S).$$
Hence, using Proposition 2.5, we have an injective map
$$\varepsilon_N = \varepsilon_{\mathrm{Ctd}_k(B,N)} : \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big) \hookrightarrow \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S, N_S)\big). \tag{2.10}$$
The situation that we are presently at can be summarized by the following diagram:[5]

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_S(A_S, N_S) \arrow[r] & \mathrm{Der}_k(A_S, N_S) \arrow[r, "\eta_S"'] \arrow[d, squiggly] & \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S, N_S)\big) \arrow[r] \arrow[l, bend right, "\sigma_S"'] & 0 \\
0 \arrow[r] & \mathrm{Der}_R(B, N) \arrow[r] & \mathrm{Der}_k(B, N) \arrow[r, "\eta_B"'] & \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big) \arrow[u, "\varepsilon_N"'] \arrow[l, dotted, bend right, "\sigma"'] &
\end{tikzcd} \tag{2.11}$$

[5] "What can be shown cannot be said." (L. Wittgenstein)

where we have abbreviated $\eta_S = \eta_{A_S,N_S}$, $\eta_B = \eta_{B,N}$ and $\sigma_S = \sigma_{A_S,N_S}$. The exactness of the rows follows from Proposition 2.1, and the splitting of η_S from Lemma 2.2. Recall that by the faithfull flatness of S/R there is no harm to assume $R \subset S$ and $N \subset N_S$.

Our immediate goal is to construct the dotted arrow σ. We will do this by going to $\mathrm{Der}_k(A_S, N_S)$ using $\sigma_S \circ \varepsilon_N$ and then require that the derivations obtained in this way map B to N as indicated by the arrow $\rightsquigarrow$ (recall that we know $B \subset A_S$ and $N \subset N_S$). Thus we need the condition

$$\big((\sigma_S \circ \varepsilon_N)(d)\big)(B) \subset N \quad \text{for all } d \in \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big) \tag{2.12}$$

which we will establish in our main theorem below.

Remark 2.6. In the present situation the diagram (2.11) can be thought of as descent data, while (2.12) is the condition for the descent data to be effective.

The R-linear map $\pi : N_S \to N$ satisfying (2.7) leads to two very important k-linear maps:

(1) The *restriction map*

$$\rho : \mathrm{Hom}_k(A_S, N_S) \to \mathrm{Hom}_k(B, N), \quad \rho(f) : b \mapsto (\pi \circ f)(b). \tag{2.13}$$

(2) The *double restriction map* (shown to be well-defined in the next theorem)

$$\tilde\rho : \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S, N_S)\big) \to \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big),\ \tilde\rho(d) : r \mapsto \rho(d(r)). \tag{2.14}$$

Theorem 2.7. *We assume* (i)–(iv) *of* 2.4. *Then*

(a) *The restriction map* (2.13) *preserves derivations and centroidal transformations:*

$$\rho\big(\mathrm{Der}_k(A_S, N_S)\big) \subset \mathrm{Der}_k(B, N) \quad \text{and} \quad \rho\big(\mathrm{Ctd}_k(A_S, N_S)\big) \subset \mathrm{Ctd}_k(B, N).$$

(b) *The restriction of ρ to* $\mathrm{Ctd}_k(A_S, N_S)$ *is an R-bimodule homomorphism with respect to the R-bimodule structures of* $\mathrm{Ctd}_k(A_S, N_S)$ *and* $\mathrm{Ctd}_k(B, N)$ *of Lemma* 1.8:[6]

$$\rho(r \cdot \chi) = r \cdot \rho(\chi) \quad \text{and} \quad \rho(\chi \cdot r) = \rho(\chi) \cdot r \tag{2.15}$$

for $r \in R$ and $\chi \in$ $\mathrm{Ctd}_k(A_S, N_S)$.

(c) *The double restriction map* (2.14) *is well-defined and satisfies*

$$\eta_B \circ \rho \circ \sigma_S = \tilde\rho \tag{2.16}$$

for η_B and σ_S as in (2.11).

(d) *With ϵ_N as in* (2.10) *the map $\sigma = \rho \circ \sigma_S \circ \varepsilon_N$ is a section of η_B, whence*

$$\mathrm{Der}_k(B, N) \simeq \mathrm{Der}_R(B, N) \oplus \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big). \tag{2.17}$$

[6]Since B is perfect, Lemma 1.10 shows that the two R-module structures coincide.

(e) *Summarizing, we have the following diagram:*

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_S(A_S, N_S) \arrow[r] & \mathrm{Der}_k(A_S, N_S) \arrow[r, "\eta_S"'] \arrow[d, "\rho"'] & \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S, N_S)\big) \arrow[l, bend right, "\sigma_S"'] \arrow[d, shift left, "\tilde\rho"] \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Der}_R(B, N) \arrow[r] & \mathrm{Der}_k(B, N) \arrow[r, "\eta_B"'] & \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big). \arrow[l, bend right, "\sigma"'] \arrow[u, shift left, "\varepsilon_N"]
\end{tikzcd}
\tag{2.18}
$$

Proof. (a) Let $d \in \mathrm{Der}_k(A_S, N_S)$ and put $\tilde d = \rho(d)$. For $b_i \in B$ we then get, using (2.7),

$$
\begin{aligned}
\tilde d(b_1 b_2) &= \pi\big(d(b_1 b_2)\big) = \pi\big(d(b_1)\cdot b_2\big) + \pi\big(b_1 \cdot d(b_2)\big) \\
&= \pi\big(d(b_1)\big)\cdot b_2 + b_1 \cdot \pi\big(d(b_2)\big) = \tilde d(b_1)\cdot b_2 + b_1 \cdot \tilde d(b_2).
\end{aligned}
$$

This shows $\rho\big(\mathrm{Der}_k(A_S, N_S)\big) \subset \mathrm{Der}_k(B, N)$. The second inclusion can be proven in a similar fashion: Let $\chi \in \mathrm{Ctd}_k(A_S, N_S)$ and put $\tilde\chi = \rho(\chi)$. Then

$$
\tilde\chi(b_1 b_2) = \pi\big(\chi(b_1 b_2)\big) = \pi\big(\chi(b_1)\cdot b_2\big) = \pi\big(\chi(b_1)\big)\cdot b_2 = \tilde\chi(b_1)\cdot b_2.
$$

The equation $\tilde\chi(b_1 b_2) = b_1 \cdot \tilde\chi(b_2)$ follows in the same way.

(b) For the proof of (2.15) we use that π is R-linear. We have for $\chi \in \mathrm{Ctd}_k(A_S, N_S)$ and $b \in B$

$$
\begin{aligned}
\big(\rho(r\cdot\chi)\big)(b) &= \pi\big((r\cdot\chi)(b)\big) = \pi\big(r\cdot(\chi(b))\big) = r\cdot\pi\big(\chi(b)\big) = r\cdot\big(\rho(\chi)(b)\big) \\
&= \big(r\cdot\rho(\chi)\big)(b), \\
\big(\rho(\chi\cdot r)\big)(b) &= \pi\big((\chi\cdot r)(b)\big) = \pi\big(\chi(rb)\big) = \rho(\chi)(rb) = \big(\rho(\chi)\cdot r\big)(b).
\end{aligned}
$$

(c) We need to show that $\tilde\rho$ is well-defined, i.e., that $\tilde\rho(d)$ is a derivation (it is clearly a k-linear map $R \to \mathrm{Ctd}_k(B, N)$). Thus, for $\bar d = \tilde\rho(d)$ and $r_i \in R$ we need to prove that $\bar d(r_1 r_2) = \bar d(r_1)\cdot r_2 + r_1 \cdot \bar d(r_2)$. To this end we use (2.15):

$$
\begin{aligned}
\bar d(r_1 r_2) &= (\rho \circ d)(r_1 r_2) = \rho\big(d(r_1 r_2)\big) = \rho\big(d(r_1)\cdot r_2 + r_1 \cdot d(r_2)\big) \\
&= \big(\rho(d(r_1))\big)\cdot r_2 + r_1 \cdot \big(\rho(d(r_2))\big) = \bar d(r_1)\cdot r_2 + r_1 \cdot \bar d(r_2).
\end{aligned}
$$

Finally, for the proof of (2.16) let $d \in \mathrm{Der}_k\big(S, \mathrm{Ctd}_k(A_S, N_S)\big)$, $r \in R$, $b \in B$ and put $d' = \sigma_S(d) \in \mathrm{Der}_k(A_S, N_S)$. Then, using that σ_S is a section of η_S, we get

$$
\begin{aligned}
\big(\big(\eta_B(\rho(d'))\big)(r)\big)(b) &= \rho(d')(rb) - r\big(\rho(d')(b)\big) = \pi\big(d'(rb)\big) - r\pi\big(d'(b)\big) \\
&= \pi\big(d'(rb) - rd'(b)\big) = \pi\big(\eta_S(d')(r)(b)\big) \\
&= \pi\big(d(r)(b)\big) = \big(\big(\tilde\rho(d)\big)(r)\big)(b).
\end{aligned}
$$

(d) follows from $\eta_B \circ \sigma = \tilde\rho \circ \varepsilon_N = \mathrm{Id}$ in view of (2.9) and (2.16). (e) follows from the diagram (2.11) and (d). $\square$

We finish this section by discussing important situations in which the assumptions of the theorem hold. Assumptions (i) through (v) are quite mild and natural within the theory of forms. The key to effective descent is (vi). As we shall presently see, it holds in one of the most important cases, namely in the case of a finite étale covering S/R. In particular, it holds when the trivializing extension S/R is Galois or when S/R is a finite separable extension of fields.

Lemma 2.8.

(a) *Assume B is an R-algebra for $R \in k$-**alg**, N is a (B, R)-dimodule and $S \in R$-**alg** is a faithfully flat extension. After canonically identifying R with a subring of S we further suppose that R is a direct summand of the R-module S, so that we have*

$$S = R \oplus S', \tag{2.19}$$

as a direct sum of R-submodules. Define $\pi : N_S \to N$ as the projection onto N with respect to the decomposition $N_S = N \oplus (N \otimes_R S')$. Then π is R-linear and satisfies (2.7)*.*

(b) *Suppose S/R is a faithful, finitely generated and projective R-module, e.g., a finite étale covering, then* (2.19) *holds.*

Proof. (a) Since $N_S = N \oplus (N \otimes_R S')$ is a decomposition of N_S as R-module, the map π is R-linear. From the (B, R)-dimodule structure of N_S given in (1.1) it follows that $N = N \otimes_R R$ and $N \otimes_R S'$ are subdimodules, which easily implies that (2.7) holds.

(b) This is [KO, III, Lemme 1.9]. □

Remark 2.9. We point out that the assumption in Lemma 2.8(b) is not necessary for (2.19) to hold. For example, let $R = k[t]$, $S = k[t] \times k[t^{\pm 1}]$ viewed as R-algebra by embedding R diagonally into S. Then S is an étale covering of R which is not finite. But (2.19) holds, for example by taking $S' = 0 \oplus k[t^{\pm 1}]$.

Example 2.10. The decomposition (2.19) takes place naturally whenever a finite group Γ acts completely reducibly on $S \in k$-**alg** by k-algebra automorphisms. One then knows that $S = S^\Gamma \oplus S'$ where

$$S^\Gamma = \{s \in S : \gamma \cdot s = s \text{ for all } \gamma \in \Gamma\},$$
$$S' = \operatorname{Span}_k\{\gamma \cdot s - s : \gamma \in \Gamma, s \in S\}.$$

It is immediate that $R = S^\Gamma$ is a k-algebra and that S' is an R-submodule: $r(\gamma \cdot s - s) = \gamma \cdot (rs) - rs \in S'$ where $r \in R$ and $s \in S$. Such a situation occurs, for example, in equivariant map algebras [NSS].

Remark 2.11. Again let Γ be a finite group of automorphisms of $S \in k$-**alg** and let $R = S^\Gamma$. Then [SGA1, V.2] gives sufficient conditions for S/R to be a finite étale covering. The most relevant case is that of a finite Galois extension S/R with Galois group Γ. The quintessential example is that of a multiloop algebra [P1].

3. Applications

In this section we discuss several special cases of our main result as well as generalizations and applications.

3.1. Algebra derivations

We assume the conditions (i)–(vi) of 2.4 where $N = B$ with its natural (B, k)-dimodule structure of Example 1.2. Moreover, we suppose that A is central. By definition, $\mathrm{Ctd}_k(B, B) = \mathrm{Ctd}_k(B)$ is the usual centroid of B. From Corollary 1.12 and Lemma 1.10 we get

$$R \simeq \mathrm{Ctd}_R(B) \simeq \mathrm{Ctd}_k(B).$$

The k-module $\mathrm{Der}_k(B)$ has a natural Lie k-algebra structure. Our Theorem 2.7 states

$$\mathrm{Der}_k(B) \simeq \mathrm{Der}_R(B) \rtimes \mathrm{Der}_k(R). \tag{3.1}$$

If S/R is a Galois extension, 2.4(vi) holds and (3.1) generalizes the main theorem of [P1] (in the Galois case).

Of course, the most important case is that in which k is a field and A is a finite-dimensional (perfect and central) k-algebra. For example, if B is a multiloop algebra, say $B = M(A, \sigma_1, \dots, \sigma_n)$ for commuting finite order automorphisms σ_i of the k-algebra A, it can be shown that $\mathrm{Der}_R(B) \simeq M(\mathrm{Der}_k(A), \sigma_1^*, \dots, \sigma_n^*)$ where $\sigma_i^*(d) = \sigma_i \circ d \circ \sigma_i^{-1}$. We thus recover [Az2, Thm. 3.7] in the case of a central A. We point out that in case A is not necessarily central, we can also get the description of $\mathrm{Der}_k(B)$ of [Az2, (3.8)] by interpreting the second sum loc. cit. as $\mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B)\big)$.

3.2. Lie algebras

We specialize Theorem 2.7 to the setting of forms of Lie algebras. We will change the notation to abide by standard conventions. Recall that a module M of a Lie algebra L over a field is called *locally finite* if every $m \in M$ belongs to a finite-dimensional L-submodule of M.

Lemma 3.1. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field k of characteristic zero, let $K \in k$-**alg** and let M be a $\mathfrak{g}_K$-module which is locally finite as a $\mathfrak{g}$-module. Then $\mathrm{Der}_K(\mathfrak{g}_K, M) = \mathrm{IDer}(\mathfrak{g}_K, M)$.*

Proof. Let $d \in \mathrm{Der}_K(\mathfrak{g}_K, M)$. Then $d(\mathfrak{g} \otimes 1_K)$ is a finite-dimensional subspace of M. Hence there exists a finite-dimensional $\mathfrak{g}$-submodule N of the $\mathfrak{g}$-module M such that $d(\mathfrak{g} \otimes 1_K) \subset N$. By the first Whitehead Lemma, there exists $n \in N$ such that $d(x \otimes 1_K) = (x \otimes 1_K) \cdot n$ for all $x \in \mathfrak{g}$. But then for $s \in K$ we get $d(x \otimes s) = sd(x \otimes 1_K) = s\big((x \otimes 1_K) \cdot n\big) = (x \otimes s) \cdot n$, which shows that d is the inner derivation given by n. $\square$

Proposition 3.2. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra defined over a field k of characteristic zero, let $R \in k$-**alg**, $S \in R$-**alg** an étale covering, $\mathcal{L}$ an (S/R)-form of $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$ and N an $\mathcal{L}$-module such that the canonical $\mathfrak{g}_S$-action on N_S is locally-finite. Also suppose that we have an R-linear map $\pi : N_S \to N$ satisfying* (2.7)*. Then*

$$\mathrm{Der}_k(\mathcal{L}, N) \simeq \mathrm{IDer}(\mathcal{L}, N) \oplus \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(\mathcal{L}, N)\big).$$

In particular, the first cohomology group of $\mathcal{L}$ with coefficients in N is

$$H^1(\mathcal{L}, N) \simeq \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(\mathcal{L}, N)\big).$$

Proof. All assumptions of Theorem 2.7 are fulfilled. The result then follows from (2.17) as soon as we have shown that $\mathrm{Der}_R(\mathcal{L}, N) = \mathrm{IDer}(\mathcal{L}, N)$. Since $\mathcal{L}$ is finitely presented, an application of Corollary 1.5 shows that this holds if and only if $\mathrm{Der}_S(\mathcal{L}_S, N_S) = \mathrm{IDer}(\mathcal{L}_S, N_S)$, equivalently, $\mathrm{Der}_S(\mathfrak{g}_S, N_S) = \mathrm{IDer}(\mathfrak{g}_S, N_S)$. But the latter equality is a consequence of Lemma 3.1. $\square$

3.3. Associative algebras

Theorem 2.7 applies also to associative algebras. For them, it is natural to assume that N is a B-bimodule. Suppose that this is the case and that A is a separable k-algebra. It then follows that B is separable too [KO, III, §2]. Hence ([KO, III, Thm. 1.4]) all R-linear derivations $d\colon B \to N$ are inner, i.e., $\mathrm{Der}_R(B, N) = \mathrm{IDer}(B, N)$, and so the first Hochschild cohomology group of the associative k-algebra B with values in the k-module N is

$$HH^1(B, N) \simeq \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(B, N)\big).$$

For example, for $B = N$ we have $\mathrm{Ctd}_k(B, N) = \mathrm{Ctd}_k(B) = Z(B)$, the centre of B. We therefore get

$$HH^1(B) \simeq \mathrm{Der}_k\big(R, Z(B)\big).$$

In particular, for a central B we get $HH^1(B) \simeq \mathrm{Der}_k(R)$.

3.4. Jordan algebras

We will leave the interpretation of Theorem 2.7 for general Jordan algebras to the reader, and only consider the most interesting special case here.

The analogue of Lemma 3.1 for Jordan algebras remains true by replacing Whitehead's Lemma with Jacobson's Theorem [Ja, VIII, Thm. 10], which says that every derivation of a semisimple finite-dimensional Jordan algebra with values in a Jordan bimodule is inner. We therefore also have the analogue of Proposition 3.2 which we state here in simplified form.

Proposition 3.3. *Let J be a finite-dimensional semisimple Jordan algebra over a field k of characteristic 0, $R \in k$-**alg** an extension of k, $S \in R$-**alg** a finite étale covering, $\mathcal{J}$ an S/R-form of J_R and N a Jordan bimodule of $\mathcal{J}$ such that the canonical action of J_S on N_S is locally finite. Then*

$$\mathrm{Der}_k(\mathcal{J}, N) \simeq \mathrm{IDer}(\mathcal{J}, N) \oplus \mathrm{Der}_k\big(R, \mathrm{Ctd}_k(\mathcal{J}, N)\big).$$

We have assumed characteristic zero only to simplify the presentation. This can be generalized depending on the type of J. For example, if J is separable simple exceptional, then characteristic $\neq 2$ suffices (one must then however replace Jacobson's Theorem with Harris' Theorem [Ja, VII.7, Thm. 14]).

3.5. Work in preparation

Our main theorem is also at the heart of several works in preparation. We outline three of them.

(a) In this case $A = \mathfrak{g}$ is a finite-dimensional central-simple Lie algebra over a field k of characteristic 0. We take $N = B^*$. If V is a k-space, viewed as a trivial B-module, the extensions

$$0 \to V \to E \to B \to 0$$

are measured (up to isomorphisms) by $H^2(B,V)$. Our main theorem can be used to show that every cocycle in $Z^2(B,V)$ is cohomologous to a *unique* standard cocycle. Examples of standard cocycles are given by the universal cocycle of Kassel [Ka]. Details will be given in [PPS].

(b) In [NPPS] we will use the relation between invariant bilinear forms of an algebra A and $\mathrm{Ctd}_k(A, A^*)$ to describe invariant bilinear forms on algebras obtained by étale descent. Among other things we will recover [MSZ, Lemma 2.3], which considers this question for Lie algebras in the untwisted case. The need to consider graded invariant forms will require a graded version of our main theorem. Besides other applications, we will give a classification-free proof of Yoshii's Theorem [Y] for multiloop Lie tori stating that graded invariant bilinear forms are unique up to scalars.

(c) The results of this paper are also relevant for the description of the universal central extension $\mathfrak{uce}(L)$ of a perfect graded Lie algebra L. Namely, denote by $\mathrm{AD}(L) = \{d \in \mathrm{Der}_k^\Lambda(L, L^{\mathrm{gr}*}) : d(l)(l) = 0\}$, the alternating Λ-graded derivations of L into its graded dual. Then ([N1]; see also [N2, 5.1.3]) the homology $H_2(L)$, which is the kernel of the universal central extension, is canonically isomorphic to $\mathrm{AD}(L)/\,\mathrm{IDer}(L, L^{\mathrm{gr}*})$. Moreover, this approach naturally leads to a description of the universal central extension $\mathfrak{uce}(L^{\mathcal{F}})$ of a fixed point subalgebra of L under a group of automorphisms $\mathcal{F}$ as the fixed point subalgebra $\mathfrak{uce}(L)^{\mathcal{F}}$, where $\mathcal{F}$ is canonically extended to $\mathfrak{uce}(L)$. In particular, this applies to certain equivariant map algebras, like multiloop algebras, and to algebras obtained by Galois descent.

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