Cohomological Invariants of degree 3 and Chow ring of a versal flag.

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Introduction: Cohomological invariants

- F - a base field. For simplicity let's restrict to the case *char* $F = 0$
- G - algebraic group over F.
- $\mathbf{Fields}_F$ - category of field extensions L/F.
- Functor $H^1(*, G)$

$$H^1(*, G) \colon \mathbf{Fields}_F \to \mathbf{Sets}$$

$$L/F \mapsto H^1(L, G)$$

Introduction: Cohomological invariants

Let H be another functor

$$H\colon \textbf{Fields}_F \to \textbf{Abelian Groups}$$

Definition

Cohomological invariant a with values in H is a natural transformation

$$a\colon H^1(*, G) \to H, \text{ i.e.}$$

For any L/F a map $a_L\colon H^1(L, G) \to H(L)$ such that

$$\begin{tikzcd}
H^1(L_1, G) \arrow[r, "a_{L_1}"] \arrow[d] & H(L_1) \arrow[d] \\
H^1(L_2, G) \arrow[r, "a_{L_2}"] & H(L_2)
\end{tikzcd}$$

commutes for any $L_1 \to L_2$.

Introduction: Cohomological invariants

Invariants with values in H form an abelian group

$$Inv(G, H)$$

Definition

Invariant a is normalized if $a_L(E) = 0$ for every trivial torsor E over L.

Normalized invariants form a subgroup

$$Inv(G, H)_{norm}$$

By functorial property,

$$Inv(G, H) = Inv(G, H)_{norm} \oplus H(F)$$

Introduction: Galois cohomology

- F^{sep} denotes a separable closure of F
- $\Gamma = Gal(F^{sep}/F)$ denotes the absolute Galois group
- A is a discrete Γ-module

Galois cohomology $H^n(F, A)$ is the cohomology of the profinite group Γ with coefficients in A, i.e. the homology group of the complex $C^\bullet(\Gamma, A)$ with $C^n(\Gamma, A) = Map_{cont}(\Gamma^n, A)$

$$\ldots \to C^{n-1}(\Gamma, A) \overset{d^{n-1}}{\to} C^n(\Gamma, A) \overset{d^n}{\to} C^{n+1}(\Gamma, A) \to \ldots$$

$$d^n(f)(g_1..g_{n+1}) = g_1 f(g_2..g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1..g_i g_{i+1}..g_{n+1}) +$$

$$+(-1)^{n+1} f(g_1..g_n)$$

Introduction: Galois modules $\mathbb{Q}/\mathbb{Z}(d)$

For a prime p

$$\mathbb{Q}_p/\mathbb{Z}_p(d) = \varinjlim_{m} (\mu_{p^m})^{\otimes d}$$

Set

$$H^{d+1}(F, \mathbb{Q}/\mathbb{Z}(d)) = \coprod_{p-prime} H^{d+1}(F, \mathbb{Q}_p/\mathbb{Z}_p(d))$$

Examples:

- $H^1(F, \mathbb{Q}/\mathbb{Z}(0)) = Hom_{cont}(\Gamma_F, \mathbb{Q}/\mathbb{Z})$
- $H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = Br(F)$

Introduction: Cup product and residue maps

Definition

For a field F with discrete valuation v and the residue field $F(v)$ there is the residue map homomorphism

$$\partial_v \colon H^{d+1}(F, \mathbb{Q}/\mathbb{Z}(d)) \to H^d(F(v), \mathbb{Q}/\mathbb{Z}(d-1))$$

Definition

The cup product

$$\cup \colon H^{p_1}(F, \mathbb{Q}/\mathbb{Z}(d_1)) \times H^{p_2}(F, \mathbb{Q}/\mathbb{Z}(d_2)) \to H^{p_1+p_2}(F, \mathbb{Q}/\mathbb{Z}(d_1+d_2))$$

In particular this gives rise to

$$F^{\times} \times H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)) \to H^{d+1}(F, \mathbb{Q}/\mathbb{Z}(d))$$

Introduction: Invariants of degree d

Definition

Degree d invariants are invariants with values in $H^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$

$$Inv^d(G, d-1) = Inv(G, H^d(F, \mathbb{Q}/\mathbb{Z}(d-1)))$$

For d=1 and a connected group G

$$Inv^1(G, 0)_{norm} = 0$$

Introduction: Invariants of degree 2

Let G be a semisimple algebraic group. Consider a simply-connected cover

$$1 \to C \to \widetilde{G} \to G \to 1$$

For every E/F this gives an exact sequence of pointed sets

$$H^1(E, \widetilde{G}) \to H^1(E, G) \xrightarrow{d} H^2(E, C)$$

for every character $\chi \colon C \to \mathbb{G}_m$ this gives composition

$$\beta_\chi \colon H^1(E, G) \xrightarrow{d} H^2(E, C) \xrightarrow{\chi_*} H^2(E, \mathbb{G}_m) = Br(E)$$

This gives rise to a group homomorphism (in fact an isomorphism)

$$C^* \to Inv^2(G, 1)_{norm}, \chi \mapsto \beta_\chi$$

Introduction: (Semi-)decomposable invariants

We consider two subgroups of $Inv^3(G,2)_{norm}$:

- The subgroup $Inv^3(G,2)_{dec}$ of decomposable invariants is generated by invariants a of the form

$$a_L(E) = (\alpha) \cup \beta_\chi(E) \text{ for } E \in H^1(G,L)$$

  where α is a fixed element in $F^\times$, $\chi \in C^*$

- The subgroup $Inv^3(G,2)_{semi}$ of semi-decomposable invariants consisting of invariants a such that for every L/F and $E \in H^1(L,G)$ there is $a_\chi \in L^\times$ such that

$$a(E) = \sum_{\chi \in C^*} (a_\chi) \cup \beta_\chi(E)$$

So, $Inv^3(G,2)_{dec} \subseteq Inv^3(G,2)_{semi} \subseteq Inv^3(G,2)_{norm}$

Introduction: (Semi-)decomposable invariants

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So, $Inv^3(G,2)_{dec} \subseteq Inv^3(G,2)_{semi} \subseteq Inv^3(G,2)_{norm}$

Introduction: Classifying variety

Definition

Let V be a G-representation, U an open subset of V such that *codim* $V \setminus U \geq 3$ and G acts freely on U. Then $U \to U/G$ is called a classifying torsor G-torsor.

Classifying property: For every field extension L/F and G-torsor $E \in H^1(G, L)$ there is a morphism $x\colon Spec\ L \to U/G$ such that $E = U \times_{U/G} Spec\ L$

$$\begin{array}{ccc} Y & \longrightarrow & U \\ \downarrow & & \downarrow \\ Spec\ L & \xrightarrow{x} & U/G \end{array}$$

Introduction: Versal torsor and versal flag

Let ξ be the generic point of U/G

Definition

The versal torsor Y_ξ is the generic fiber of $U \to U/G$:

$$Y_\xi = U \times_{U/G} \xi$$

The versal flag X_ξ is the generic fiber of $U/T \to U/G$:

$$X_\xi = U/T \times_{U/G} \xi$$

Main result: Statement

Theorem

There is a short exact sequence

$$0 \to Inv^3(G,2)_{semi} \to Inv^3(G,2)_{norm} \xrightarrow{\alpha} CH^2(X_\xi)_{tors} \to 0$$

Main result: Difference

Theorem

The difference between decomposable and semidecomposable invariants can be computed as

$$\frac{Inv^3(G,2)_{semi}}{Inv^3(G,2)_{dec}} = \frac{(\tilde{I}^W \cap \mathbb{Z}[T^*]) + I^3}{I^W + I^3}$$

- T is a maximal split torus of G
- Λ denotes the weight lattice, $T^* \subseteq \Lambda$
- W denotes the Weyl group of G.
- $\mathbb{Z}[T^*] \subseteq \mathbb{Z}[\Lambda]$ are the group rings of lattices T^* and Λ
- $\tilde{I}$ is the kernel of the augmentation map $aug\colon \mathbb{Z}[\Lambda] \to \mathbb{Z},\ e^\omega \mapsto 1$
- $I = \mathbb{Z}[T^*] \cap \tilde{I}$
- $\tilde{I}^W$ denotes the ideal in $\mathbb{Z}[\Lambda]$ generated by W-invariant elements of $\tilde{I}$
- I^W denotes the ideal in $\mathbb{Z}[T^*]$ generated by W-invariant elements of I

Main result: Some corollaries

By the classification of degree 3 invariants made by Merkurjev, $Inv^3(G,2) = Inv^3(G,2)_{dec}$ in the following cases:

- G of type A_n
- G of type B_n
- G of type C_n and $n \equiv 1,2,3 \ mod\ 4$
- G of type D_n and $n \equiv 1,2,3 \ mod\ 4$

Then immediately in all these cases $CH^2(X_\xi)$ is torsion-free.

Idea of the proof

- By the results of Rost, the assignment

$$\Theta\colon Inv^3(G,2) \to H^3(F(U/G),\mathbb{Q}/\mathbb{Z}(2)), a \mapsto a_{F(U/G)}(Y_\xi)$$

  is injective
- The image of Θ coincides with the kernel of the residue map

$$\partial =\colon H^3(F(U/G),\mathbb{Q}/\mathbb{Z}(2)) \to \coprod_{z\in U/G^{(1)}} H^2(F(z),\mathbb{Q}/\mathbb{Z}(1))$$

Idea of the proof: Rost cycle modules

For a scheme X Consider the cycle complex

$$\coprod_{z\in X^{(0)}} K_n(F(z)) \to \coprod_{z\in X^{(1)}} K_{n-1}(F(z)) \to \coprod_{z\in X^{(2)}} K_{n-2}(F(z)) \to \dots$$

Its p cohomology group $A^p(X, K_n)$ is called cycle module cohomology.
Note that $A^n(X, K_n) = CH^n(X)$.

Idea of the proof: Map ρ

Let $\widetilde{H}^3(F(U/G),2)$ denote the kernel of the pullback map

$$\widetilde{H}^3(F(U/G),2) = \ker[H^3(F(U/G),2) \to H^3(F(X_\xi),2)]$$

Note that $C^* = \Lambda/T^*$, for $\lambda \in \Lambda$ $\bar{\lambda}$ denotes the image of λ in C^*. Then there is the map

$$\rho\colon F(U/G) \otimes \Lambda \to \widetilde{H}^3(F(U/G),2)$$

$$\phi \otimes \lambda \mapsto (\phi) \cup \beta_{\bar{\lambda}}(Y_\xi)$$

Idea of the proof: Map α

For a prime p let

$$\widetilde{H}^3(X_\xi, \mu_{p^n}^{\otimes 2}) = \ker[H^3_{et}(X_\xi, \mu_{p^n}) \to H^3(F(X_\xi), \mu_{p^n}^{\otimes 2})]$$

By the Leray spectral sequence and Bloch-Ogus theorem it coincides with the homology of the cycle module complex

$$H^2(F(X_\xi), \mu_{p^n}^{\otimes 2}) \to \coprod_{z \in X_\xi^{(1)}} H^1(F(z), \mu_{p^n}) \to \coprod_{z \in X_\xi^{(2)}} \mathbb{Z}/p^n\mathbb{Z}$$

Idea of the proof: Map α

By Merkurjev-Suslin norm residue isomorphism theorem

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
K_2(F(X_\xi)) \arrow[r] \arrow[d, "\hat{\ }p^n"] & \coprod_{z \in X_\xi^{(1)}} F(z)^\times \arrow[r] \arrow[d, "\hat{\ }p^n"] & \coprod_{z \in X_\xi^{(1)}} \mathbb{Z} \arrow[d, "\cdot p^n"] \\
K_2(F(X_\xi)) \arrow[r] \arrow[d] & \coprod_{z \in X_\xi^{(1)}} F(z)^\times \arrow[r] \arrow[d] & \coprod_{z \in X_\xi^{(1)}} \mathbb{Z} \arrow[d] \\
H^2(F(X_\xi), \mu_{p^n}^{\otimes 2}) \arrow[r] \arrow[d] & \coprod_{z \in X_\xi^{(1)}} H^1(F(z), \mu_{p^n}) \arrow[r] \arrow[d] & \coprod_{z \in X_\xi^{(2)}} \mathbb{Z}/p^n\mathbb{Z} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
$$

$$\alpha_{p^n} \colon \widetilde{H}^3(F(U/G), \mu_{p^n}^{\otimes 2}) \to \widetilde{H}^3(X_\xi, \mu_{p^n}^{\otimes 2}) \to CH^2(X_\xi)_{p^n-tors}$$

Taking limit in p, n we get $\alpha \colon \widetilde{H}^3(F(U/G), \mathbb{Q}/\mathbb{Z}(2)) \to CH^2(X_\xi)_{tors}$

Idea of the proof: The key diagram

For any point z in U/G X_z denotes the fiber product $X_z = U/T \times_{U/G} z$. There is the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A^1(X_\xi, K_2) \arrow[r] \arrow[d, "\partial^X"] & F(U/G)^\times \otimes \Lambda \arrow[r, "\rho"] \arrow[d, "d"] & \widetilde{H}^3(F(U/G), 2) \arrow[r, "\alpha"] \arrow[d, "\partial"] & CH^2(X_\xi)_{tors} \arrow[r] & 0 \\
0 \arrow[r] & \coprod_{z \in U/G^{(1)}} Pic\, X_z \arrow[r] & \coprod_{z \in U/G^{(1)}} \Lambda \arrow[r] & \coprod_{z \in U/G^{(1)}} \widetilde{H}^2(F(z), 1) \arrow[r] & 0
\end{tikzcd}$$

$$d_z \colon \phi \otimes \chi \mapsto v_z(\phi)\chi \text{ for } \phi \in F(U/G)^\times, \chi \in \Lambda.$$

Note that d is surjective since $PicU/G = 0$

$$Inv^3(G,2)_{norm} = \ker \partial, \;\; im(\rho) \cap \ker \partial = Inv^3(G,2)_{semi}$$

Then the diagram chase gives the short exact sequence

$$0 \to Inv^3(G,2)_{semi} \to Inv^3(G,2)_{norm} \to CH^2(X_\xi)_{tors} \to 0$$

Idea of the proof: Decomposable and semidecomposable invariants

In the previous diagram $Inv^3(G,2)_{dec} = \rho(F \otimes \Lambda)$. Then the follwoing sequence is exact

$$0 \to Inv^3(G,2)_{dec} \to Inv^3(G,2)_{semi} \to coker(\partial^X) \to 0$$