Semiprime, prime and simple Jordan superpairs covered by grids

Dedicated to Kevin McCrimmon on the occasion of his sixtieth birthday

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Introduction

Summary. In this paper we continue the investigation of Jordan superpairs covered by a grid which have recently been described by the second author ([**21**]). For such a Jordan superpair we characterize semiprimeness, primeness and simplicity in terms of its associated supercoordinate system.

An important part of the present research activities in Jordan theory is devoted to simple Jordan superstructures, i.e., Jordan superalgebras, supertriples and superpairs. One now has a classification of finite dimensional simple Jordan superalgebras over algebraically closed fields of characteristic $\neq 2$, due to the work of Kac [**5**], Martínez-Zelmanov [**14**] and Racine-Zelmanov [**28**], [**27**] (this latter paper also considers superalgebras over non-algebraically closed fields). The (infinite dimensional) graded simple Jordan superalgebras of growth one were recently described by Kac-Martínez-Zelmanov [**6**], and one has a classification of simple finite-dimensional Jordan superpairs over fields of characteristic 0 by Krutelevich [**9**].

While these works are devoted to classifying simple Jordan superalgebras and superpairs, the second author's recent paper [**21**] developed a structure theory of Jordan superpairs over superrings covered by a grid. (Here and in the remainder of this introduction, the term "grid" means a connected standard grid in the sense of [**24**].) Roughly speaking, this condition means that one has a "nice" family of idempotents whose simultaneous

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Peirce spaces make up the whole Jordan superpair. This type of Jordan superpairs is in general not simple, not even semiprime, nevertheless the presence of a grid allows one to give a precise classification. It is the goal of this paper to connect the theory of Jordan superpairs covered by grids to the works quoted above by answering the following questions: *Among the Jordan superpairs covered by a grid, which are semiprime, prime or simple? Or, more generally, what are the ideals of Jordan superpairs covered by a grid?*

Our answers to these questions will be given in terms of the supercoordinate system $\mathcal{C}$ associated to any Jordan superpair V covered by a grid $\mathcal{G}$. That this can be done is, at least in principle, not surprising since the results of [**21**] together with §1 show that V is determined by $\mathcal{G}$ and $\mathcal{C}$ (for Jordan pairs this goes back to [**22**]). It is the interplay between V and $(\mathcal{C}, \mathcal{G})$ which lies at the heart of our paper.

The possible supercoordinate systems depend on the type of the covering grid $\mathcal{G}$. A supercoordinate system always consists of a unital coordinate superalgebra, which is a Jordan or alternative superalgebra, plus possibly some additional structure, like a pair of orthogonal strongly connected idempotents in a Jordan superalgebra (if the grid is a triangle), or an involution and ample subspace of an alternative superalgebra (hermitian grids), or a quadratic form with base point (odd quadratic form grids). More intricate and large grids will force the coordinate superalgebra to be associative (rectangular grids, hermitian grids) or even supercommutative associative (quadratic form grids, alternating grids, Bi-Cayley grids or Albert grids). Our answer to the problem above is summarized in the following.

Theorem. *Let V be a Jordan superpair covered by a grid and let $\mathcal{C}$ be the associated supercoordinate system with coordinate superalgebra A. Then V is semiprime, prime or simple if and only if $\mathcal{C}$ is respectively semiprime, prime or simple.*

This theorem will be proven in §3 where we will determine precisely what it means for a supercoordinate system to be semiprime, prime or simple (3.4, 3.5, 3.7, 3.8, 3.10 and 3.13). For this introduction suffice it to say that in case $\mathcal{C}$ just consists of a superalgebra A, i.e., the root system associated to the covering grid is simply-laced, $\mathcal{C}$ is semiprime, prime or simple if and only if the superalgebra A has this property. The following example indicates that one can interpret our result by saying that *semiprimeness, primeness and simplicity of Jordan superpairs covered by grids are Morita-invariant.*

Example. A Jordan superpair V is covered by a rectangular grid of size $J \times K$ with $|J| + |K| \geq 4$ if and only if V is isomorphic to a rectangular matrix superpair $\mathbb{M}_{JK}(A)$, consisting of $J \times K$- respectively $K \times J$-matrices over an associative superalgebra A with only finitely many non-zero entries. In this case, the supercoordinate system is $\mathcal{C} = A$. The ideals of $\mathbb{M}_{JK}(A)$ are $\mathbb{M}_{JK}(B)$ where B is an ideal of A, and $\mathbb{M}_{JK}(A)$ is semiprime, prime or simple if and only if the superalgebra A is so. In particular, using the structure of simple associative superalgebras ([**31**], see 2.5), $\mathbb{M}_{JK}(A)$ is a simple Jordan superpair if and only if either A is a simple algebra or A is the double of the simple algebra $A_{\bar{0}}$ (2.4), which means $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with $A_{\bar{1}} = uA_{\bar{0}}$ and u in the algebra centre of A satisfying $u^2 = 1$. In the second case, $\mathbb{M}_{JK}(A)$ is the double (3.2) of the Jordan pair $\mathbb{M}_{JK}(A_{\bar{0}})$: we have $\mathbb{M}_{JK}(A_{\bar{0}} \oplus uA_{\bar{0}}) = \mathbb{M}_{JK}(A_{\bar{0}}) \oplus u\mathbb{M}_{JK}(A_{\bar{0}})$.

A review of the structure of Jordan superpairs covered by grids is given in §1. In this section we will also formally introduce supercoordinate algebras A and systems $\mathcal{C}$. While their definition depends on some choices of idempotents from the covering grid, we will show that the coordinate superalgebra obtained from a different choice is either isomorphic to A or to its opposite algebra. The following section §2 contains a review and some new results on semiprime, prime and simple alternative or associative superalgebras, with or without involution. The theorem above is proven in §3 by first describing all the ideals of V, something which we feel is of independent interest. The results in §2 then allow us to work out the precise structure of the simple Jordan superpairs covered by a grid, at least in the case of finite-dimensional superpairs over algebraically closed fields (3.7, 3.9, 3.11 and 3.13).

In view of the close connection between Jordan superpairs V covered by grids and their supercoordinate systems $\mathcal{C}$, one can expect that many properties of V can be expressed in terms of $\mathcal{C}$ or the coordinate superalgebra A. For example, in a sequel to this paper ([**1**]) we will show that, over fields, the Gelfand-Kirillov dimensions of V, A and the Tits-Kantor-Koecher superalgebra $\mathfrak{K}(V)$ coincide.

In another sequel ([**2**]) we will describe the Tits-Kantor-Koecher superalgebras $\mathfrak{K}(V)$ of Jordan superpairs V covered by a grid in detail, and we will see how properties like (semi)primeness, simplicity or finite generation flow back and forth between V and $\mathfrak{K}(V)$. The Lie superalgebras $\mathfrak{K}(V)$ are of interest because they provide examples of Lie superalgebras graded by 3-graded root systems. Indeed, generalizing results from [**25**], we will see that a Lie superalgebra is graded by a 3-graded root system R if and only if it is a perfect central extension of the TKK-superalgebra of a Jordan superpair covered by a grid with associated root system R.

1. Jordan superpairs covered by grids.

1.1. Setting. In this paper we will consider quadratic Jordan superpairs as introduced in [**21**]. All unexplained notions used here can be found there, but we recall that Jordan superpairs are assumed to be defined over a base superring S, i.e., a (super)commutative associative unital superring.

More specifically, we will study Jordan superpairs V covered by a grid $\mathcal{G} = \{g_\alpha : \alpha \in R_1\}$ where (R, R_1) is the 3-graded root system associated to $\mathcal{G}$. We denote by $V = \bigoplus_{\alpha\in R_1} V_\alpha$ the Peirce decomposition of V with respect to $\mathcal{G}$. Recall $g_\alpha \in V_\alpha = (V_\alpha^+, V_\alpha^-)$. Also, we denote by G the Grassmann algebra in a countable number of generators and by $G(.)$ the Grassmann envelopes of algebraic structures.

Suppose $(R, R_1) = \bigoplus_{i\in I}(R^{(i)}, R_1^{(i)})$ is a direct sum of 3-graded root systems $(R^{(i)}, R_1^{(i)})$. Then, by [**21**, 3.5],

$$V = \bigoplus\nolimits_{i\in I} V^{(i)}, \quad V^{(i)} = \bigoplus\nolimits_{\alpha\in R_1^{(i)}} V_\alpha \triangleleft V \tag{1}$$

is a direct sum of ideals $V^{(i)}$ of V, each covered by the grids $\mathcal{G}^{(i)} = \{g_\alpha : \alpha \in R_1^{(i)}\}$. By [**11**], every 3-graded root system is a direct sum of irreducible 3-graded root systems. The decomposition (1) then reduces the classification of Jordan pairs covered by a grid to the case of an irreducible root system R or, equivalently, of a connected grid $\mathcal{G}$. It is no harm to assume that $\mathcal{G}$ is a standard grid, of which there are the following seven types:

(i) rectangular grid $\mathcal{R}(J,K), 1 \le |J| \le |K|$, (R, R_1) is the rectangular grading $\mathrm{A}_I^{J,K}$ where $J \dot\cup K = I \dot\cup \{0\}$ for some element $0 \notin I$ and R is a root system of type A and rank $|I|$ (notation of [**21**]);

(ii) hermitian grid $\mathcal{H}(I)$, $2 \le |I|$, (R, R_1) is the hermitian grading of $R = \mathrm{C}_I$;

(iii) even quadratic form grid $\mathcal{Q}_e(I)$, $3 \le |I|$, (R, R_1) is the even quadratic form grading of $R = \mathrm{D}_{I \dot\cup \{0\}}$;

(iv) odd quadratic form grid $\mathcal{Q}_o(I)$, $2 \le |I|$, (R, R_1) is the odd quadratic form grading of $R = \mathrm{B}_{I \dot\cup \{0\}}$;

(v) alternating (= symplectic) grid $\mathcal{A}(I)$, $5 \le |I|$, (R, R_1) is the alternating grading of $R = \mathrm{D}_I$;

(vi) Bi-Cayley grid $\mathcal{B}$, $R = \mathrm{E}_6$;

(vii) Albert grid $\mathcal{A}$, $R = \mathrm{E}_7$.

1.2. McCrimmon-Meyberg superalgebras, supercoordinate systems. Let V be a Jordan superpair over S covered by a connected standard grid $\mathcal{G}$ with associated 3-graded root system (R, R_1). For the classification (= coordinatization) of these Jordan superpairs the concept of a McCrimmon-Meyberg superalgebra [**21**, 3.2] is important. This is an alternative superalgebra over S defined for every collinear pair (g_α, g_β) on $V_\alpha^+ = V_2^+(g_\alpha) \cap V_1^+(g_\beta)$ by the product formula $ab = \{\{a\, g_\alpha^-\, g_\beta^+\}\, g_\beta^- b\}$.

We will associate to V a *supercoordinate system* $\mathcal{C}$. Its definition depends on the type of R. However, for a simply-laced R of rank $R \ge 2$, equivalently $\mathcal{G}$ is an ortho-collinear family of $|\mathcal{G}| \ge 2$, we have the following uniform description

$$\mathcal{C} = \textit{McCrimmon-Meyberg superalgebra of some collinear pair } g_\alpha, g_\beta \in \mathcal{G}. \tag{1}$$

This algebra is associative for rank $R \ge 3$ and even associative supercommutative, i.e. a *superextension of S*, for R of type D or E. For non-simply-laced root systems, $\mathcal{C}$ will have more structure and will be defined in the review of the coordinatization theorems below.

Modulo isomorphisms and taking the opposite algebra, the McCrimmon-Meyberg algebra does not depend on the chosen collinear pair g_α, g_β, see 1.4. To establish this, the following lemma will be needed.

1.3. Lemma. *Let $\mathcal{G} \subset V$ be a connected covering standard grid, and let $e, f \in \mathcal{G}$ be two collinear idempotents. Then the exchange automorphism $t_{e,f}$,* [**21**, 3.2]*, preserves $\mathcal{G}$ up to sign:*

$$t_{e,f}(\mathcal{G}) \subset \{\pm g\,;\, g \in \mathcal{G}\}. \tag{1}$$

Proof. Let $g \in \mathcal{G}$. Since $t := t_{e,f}$ exchanges e and f we can assume that g is distinct from e and f. The action of t depends on ij for $g \in V_i(e) \cap V_j(f)$. We therefore consider the various possibilities for ij:

$ij = 21$: This case cannot occur since $e \ne g \in V_2(e)$ implies $g \dashv e$, and then $g \dashv e \top f$ yields $g \dashv f$ or $g \perp f$, contradicting $g \in V_1(f)$. By symmetry, $ij = 12$ does not occur either.

$ij = 11$: In this case $t(g) = \{ef\{feg\}\} - g$, and hence (1) holds if $\{feg\} = 0$. If not, then $\{feg\}$ lies in a Peirce space V_γ with respect to $\mathcal{G}$ and also lies in $V_2(f) \cap V_0(e)$ by the Peirce multiplication rules. The idempotent $h \in \mathcal{G} \cap V_\gamma$ therefore has the property $f \vdash h \perp e$. It then follows that $e \top g \top f$, hence $h \in V_1(g)$ and therefore $h \dashv g$. Thus $(h; f, e, g)$ is a diamond of idempotents, hence $\{feg\} = 2h$ and so $\{ef\{feg\}\} = \{e\, f\, 2h\} = 2g$.

$ij = 01$: In this case $e \perp g$ and $f \dashv g$ or $f \top g$. Since $f \dashv g \perp e$ implies $f \perp e$ we must have $f \top g$. Because $\mathcal{G}$ is a grid, there exists $\varepsilon \in \{\pm\}$ and $h \in \mathcal{G}$ such that $(e, f, g, \varepsilon h)$ is a quadrangle of idempotents. But then $t(g) = -\{efg\} = -\varepsilon h$. The case $ij = 10$ follows by symmetry.

$ij = 02$: Here $t(g) = Q_{\bar{0}}(e) Q_{\bar{0}}(f) g$ and $f \dashv g \perp e$. Because $\mathcal{G}$ is a grid, f, g imbed into a triangle $(f; g, \tilde{g}) \subset \mathcal{G}$ of idempotents and therefore $Q_{\bar{0}}(f) g = \tilde{g}$. It follows that $\tilde{g} \in V_2(e)$. Hence $e \vdash \tilde{g}$, and by the same argument there exists $h \in \mathcal{G}$ such that $(e; \tilde{g}, h)$ is a triangle of idempotents. It then follows that $t(g) = h$. The case $ij = 20$ follows again by symmetry.

$ij = 00$ or $ij = 22$: In these cases $t = \mathrm{Id}$ on $V_i(e) \cap V_j(f)$, finishing the proof of the lemma.

1.4. Lemma. *In the setting of* 1.2 *let* (α', β') *be another collinear pair in* R_1 *and denote by* A' *the McCrimmon-Meyberg superalgebra defined on* $V_{\alpha'}^+$ *by* $(g_{\alpha'}, g_{\beta'})$. *Then* A' *is isomorphic to* A *or to its opposite algebra* A^{op}.

Proof. Our first goal is to see that, modulo an algebra isomorphism, we can assume $\alpha = \alpha'$. Because of [**22**, I, Thm. 4.9] the roots $\alpha, \beta, \alpha', \beta'$ all have the same length. Hence either $\alpha = \alpha'$ or $\alpha \top \alpha'$ or $\alpha \perp \alpha'$. We claim that there is an automorphism φ of V and $g_\gamma \in \mathcal{G}$ such that

$$\varphi(g_{\alpha'}) = g_\alpha \quad \text{and} \quad \varphi(g_{\beta'}) = \pm g_\gamma. \tag{1}$$

This is clear for $\alpha = \alpha'$ and follows from 1.3 in case $\alpha \top \alpha'$. If $\alpha \perp \alpha'$ we can take a product of two exchange automorphisms, thanks to:

$$\text{if } \alpha \perp \alpha' \text{ there exists } \delta \in R_1 \text{ such that } \alpha \top \delta \top \alpha'. \tag{2}$$

Indeed, since connectedness of $\mathcal{G}$ is equivalent to R being irreducible, we know, e.g. from [**11**], that there exists $\delta \in R_1$ with $\alpha \top \delta \top \alpha'$ or $\alpha \dashv \delta \vdash \alpha'$. But in the latter case, $\beta \top \alpha \dashv \delta$ implies $\beta \dashv \delta \vdash \beta'$ and hence $\alpha \top \beta \top \alpha'$ by properties of 3-graded root systems, see e.g. [**22**, I, Thm. 4.9] or [**11**].

We have now established (1). The definition of A and A' shows that $\varphi | A'$ is an algebra isomorphism from A' onto the McCrimmon-Meyberg superalgebra A'' defined on V_α^+ by the collinear pair g_α, g_γ. If $\gamma = \beta$ the lemma is proven. Otherwise, $\gamma \perp \beta$ or $\gamma \top \beta$. By [**21**, 3.4.(7)] we have $V_\alpha = V_2(g_\alpha) \cap V_1(g_\beta) \cap V_1(g_\gamma)$. Hence, by [**18**, 2.6], $A'' = A^{\mathrm{op}}$ in case $\gamma \perp \beta$ and $A'' = A$ in case $\gamma \top \beta$ and $\{V_\alpha^+\, e_\beta^-\, e_\gamma^+\} = 0$. But if $\gamma \top \beta$ and $\{V_\alpha^+\, e_\beta^-\, e_\gamma^+\} \neq 0$ then $\mathcal{G}$ has a non-zero Peirce space in $V_2(g_\alpha) \cap V_0(g_\beta) \cap V_2(g_\gamma)$ and therefore $g_\alpha, g_\beta, g_\gamma$ imbed in a diamond of idempotents $(g_\delta; g_\alpha, g_\beta, g_\gamma)$. We will show that $A'' = A^{\mathrm{op}}$ in this case.

By [**22**, I Thm. 2.11], the diamond of idempotents $(g_\delta; g_\alpha, g_\beta, g_\gamma)$ generates a hermitian grid $\{h_{ij}\, ;\, 1 \le i \le j \le 3\}$ in the sense of [**21**, 4.8]. We can arrange the notation so that $g_\alpha = h_{12}, g_\beta = h_{13}$ and $g_\gamma = h_{23}$. For $a, b \in V_\alpha^+$ let $a \cdot b$ be the McCrimmon-Meyberg superalgebra product with respect to h_{12}, h_{13} and $a * b$ the one with respect to h_{12}, h_{23}. Using $h_{13}^- = \{h_{12}^- \, h_{23}^+ \, h_{33}^-\}$ and the identity (JSP15) of [**21**, 2.2] together with the abbreviations $a_{23} = \{a \, h_{12}^- \, h_{13}^+\}$ and $b_{23} = \{h_{23}^+ \, h_{12}^- \, b\} \in V_\gamma^+$ we obtain

$$
\begin{aligned}
a \cdot b &= \{a_{23} \, h_{13}^- \, b\} = \{a_{23} \, \{h_{12}^- \, h_{23}^+ \, h_{33}^-\} \, b\} \\
&= -\{h_{23}^+ \, h_{12}^- \, \{a_{23} \, h_{33}^- \, b\}\} + \{\{h_{23}^+ \, h_{12}^- \, a_{23}\} \, h_{33}^- \, b\} + \{a_{23} \, h_{33}^- \{h_{23}^+ \, h_{12}^- \, b\}\} \\
&= \{a_{23} \, h_{33}^- \, \{h_{23}^+ \, h_{12}^- \, b\}\} \quad (\text{since } \{V_\alpha^+ \, h_{33}^- \, b\} = 0 \text{ because } h_{33} \perp h_{12}) \\
&= \{\{h_{13}^+ \, h_{12}^- \, a\} h_{33}^- \, b_{23}\} \\
&= \{h_{13}^+ \, h_{12}^- \, \{a \, h_{33}^- \, b_{23}\}\} + \{a \, \{h_{12}^- \, h_{13}^+ \, h_{33}^-\} \, b_{23}\} - \{a \, h_{33}^- \{h_{13}^+ \, h_{12}^- \, b_{23}\}\} \\
&= \{a \, h_{23}^- \, b_{23}\} \quad (\text{since } \{a \, h_{33}^- \, V_\alpha^+\} = 0 \text{ and } \{h_{12}^- \, h_{13}^+ \, h_{33}^-\} = h_{23}^-) \\
&= (-1)^{|a||b|} \, b * a
\end{aligned}
$$

by definition of $*$.

In order to establish the notation used in the following and to define the supercoordinate system in the other cases, we will now review the coordinatization theorems of Jordan superpairs covered by connected standard grids.

1.5. A_1-Coordinatization. A Jordan superpair V over S is covered by a single idempotent if and only if V is isomorphic to the superpair $\mathbb{J} = (J, J)$ of a unital Jordan superalgebra J over S. In this case we put

$$\mathcal{C} = J. \tag{1}$$

1.6. Rectangular grids and rectangular matrix superpairs. Let A be a unital alternative superalgebra over S and let J, K be arbitrary sets. We denote by $\mathrm{Mat}(J, K; A)$ the left A-module of all matrices over A of size $J \times K$ with only finitely many non-zero entries. By restriction of scalars, this becomes an S-supermodule with even part $\mathrm{Mat}(J, K; A_{\bar 0})$ and odd part $\mathrm{Mat}(J, K; A_{\bar 1})$. The rectangular matrix superpair of size $J \times K$ and with supercoordinate algebra A is the Jordan superpair $\mathbb{M}_{JK}(A) = (\mathrm{Mat}(J, K; A), \mathrm{Mat}(K, J; A))$ with product given by

$$Q_{\bar 0}^+(x_{\bar 0})y = x_{\bar 0}(y x_{\bar 0}); \ \{x \, y \, z\} = x(yz) + (-1)^{|x||y| + |x||z| + |y||z|} z(yx) \tag{1}$$

for homogeneous elements $x, z \in V^+$, $y \in V^-$ and $x_{\bar 0} \in V_{\bar 0}^+$ [**21**, 4.6]. One obtains $Q_{\bar 0}^-$ and the other triple product $\{...\}$ by shifting the brackets in the expressions above one position to the left. We require A to be associative if $|J| + |K| \geq 4$. In this case, the brackets in the definition of the Jordan product are of course not necessary.

Let E_{jk} be the usual matrix units. Then $e_{jk} = (E_{jk}, E_{kj}) \in \mathbb{M}_{JK}(A)$ is an idempotent and $\mathcal{R}(M, N) = \{e_{jk} : j \in J, k \in J\}$ is a rectangular grid of size $J \times K$ which covers $\mathbb{M}_{JK}(A)$. The joint Peirce spaces of $\mathcal{R}(J, K)$ are (AE_{jk}, AE_{kj}). Moreover, A is the McCrimmon-Meyberg superalgebra of any collinear pair (e_{ij}, e_{ik}) for distinct j, k. The

McCrimmon-Meyberg superalgebra of the collinear pairs (e_{ij}, e_{lj}) for distinct i, l is isomorphic to A^{op}.

By [**21**, 4.5, 4.7] a Jordan superpair V over S is covered by a rectangular grid $\mathcal{R}(J, K)$, $|J| + |K| \geq 3$, if and only if there exists a unital alternative superalgebra A over S, which is associative if $|J| + |K| \geq 4$, such that $V \cong \mathbb{M}_{JK}(A)$.

1.7. C_2**-coordinatization.** A grid with associated 3-graded root system (R, R_1) and $R = \mathrm{B}_2 = \mathrm{C}_2$ is a triangle of idempotents $(g_0; g_1, g_2)$. By [**21**, 4.9] a Jordan superpair V over S is covered by a such a $\mathcal{G}$ if and only if $V \cong \mathbb{J} = (J, J)$ where J is a Jordan superalgebra over S which contains two strongly connected supplementary orthogonal idempotents c_1, c_2. In fact, J is the $(g_1 + g_2)^-$-isotope of V and the two orthogonal idempotents $c_i = g_i^+$ are connected by g_0^+. We have a Peirce decomposition $\mathfrak{P}$ of J with respect to (c_1, c_2) in the form $\mathfrak{P} : J = J_{11} \oplus J_{12} \oplus J_{22}$. In this case, the supercoordinate system of V is

$$\mathcal{C} = (J, \mathfrak{P}). \tag{1}$$

1.8. Hermitian grids $\mathcal{H}(I)$, $|I| \geq 3$ and hermitian matrix superpairs. Let V be a Jordan superpair covered by a hermitian grid $\mathcal{H}(I) = \{h_{ij} : i, j \in I\}$. The joint Peirce spaces of $\mathcal{H}(I)$ coincide with the Peirce spaces V_{ij} of the orthogonal system $\{h_{ii} : i \in I\}$. For the purpose of coordinatization we need an additional assumption in this case: for all $i, j \in I, i \neq j$ the maps

$$D(h_{ij}^{\sigma}, h_{jj}^{-\sigma}) \ : V_{jj}^{\sigma} \to V_{ij}^{\sigma} \quad \textit{are injective} \tag{1}$$

(e.g. V has vanishing extreme radical or V has no 2-torsion). Assuming (1), we associate to V the supercoordinate system

$$\mathcal{C} = (A, A_0, \pi) \tag{2}$$

defined as follows:

(a) A is the McCrimmon-Meyberg superalgebra of a fixed collinear pair (h_{ij}, h_{ik}) which is necessarily associative for $|I| \geq 4$;
(b) π is the involution of A given by $a^{\pi} = Q_{\bar{0}}^{+}(h_{ij})\{h_{ii}^{-} \ a \ h_{jj}^{-}\}$;
(c) A^0 is the ample subspace $A^0 = D(h_{ij}^+, h_{jj}^-)V_{jj}^+$ of (A, π).

It is shown in [**21**, 4.12] that, assuming (1), V is isomorphic to a hermitian matrix superpair $\mathbb{H}_I(A, A^0, \pi) = (\mathrm{H}_I(A, A^0, \pi), \mathrm{H}_I(A, A^0, \pi))$, where

$$\mathrm{H}_I(A, A^0, \pi) = \{x = (x_{ij}) \in \mathrm{Mat}(I, I; A) : x = x^{\pi\mathrm{T}}, \text{ all } x_{ii} \in A^0\}. \tag{3}$$

Conversely, for data (A, A^0, π) as above, we can form the hermitian matrix superpair $\mathbb{H}_I(A, A^0, \pi)$. The S-module $\mathrm{H}_I(A, A^0, \pi)$ is spanned by elements of type $a[ij] = aE_{ij} + a^{\pi}E_{ji}$ $(a \in A, \ i \neq j)$ and $a_0[ii] = a_0 E_{ii}$ $(a_0 \in A^0)$. The elements $h_{ii} = (1[ii], 1[ii])$ and $h_{ij} = (1[ij], 1[ij]) = h_{ji}, i \neq j$, are idempotents of $\mathbb{H}_I(A, A^0, \pi)$ such that $\mathcal{H}(I) = \{h_{ij}; i, j \in I\}$ is a covering hermitian grid of $\mathbb{H}_I(A, A^0, \pi)$ satisfying (1). The joint Peirce spaces are $(A[ij], A[ij])$ for $i \neq j$ and $(A^0[ii], A^0[ii])$. For a suitable choice, the supercoordinate system of $\mathbb{H}_I(A, A^0, \pi)$ is (A, A^0, π).

From now on, we will assume that (1) *holds whenever we speak of the supercoordinate system of* $(V, \mathcal{G})$ *for a hermitian grid* $\mathcal{G}$.

1.9. Even quadratic form grids and even quadratic form superpairs. An even quadratic form superpair $\mathbb{EQ}_I(A)$, A a superextension of S, has the form $\mathbb{EQ}_I(A) = (H(I,A), H(I,A))$ where $H(I,A)$ is the free A-module with basis $\{h_{\pm i}; i \in I\}$ and $\mathbb{EQ}_I(A)$ is the quadratic form superpair associated to the hyperbolic form $q_I : H(I,A) \to A$. In $\mathbb{EQ}_I(A)$ the pairs $g_i = (h_{+i}, h_{-i})$ and $g_{-i} = (h_{-i}, h_{+i})$ are idempotents, and the family $\mathcal{Q}_e(I) = \{g_{\pm i}; i \in I\}$ is an even quadratic form grid which covers $\mathbb{EQ}_I(A)$.

It is shown in [**21**, 4.14] that a Jordan superpair V over S is covered by an even quadratic form grid $\mathcal{Q}_e(I)$, $|I| \geq 3$, if and only if V is isomorphic to the even quadratic form superpair $\mathbb{EQ}_I(A)$ where A is the McCrimmon-Meyberg superalgebra of $(V, \mathcal{Q}_e(I))$ which necessarily is a superextension of S.

1.10. Odd quadratic form grids $\mathcal{Q}_o(I)$,$|I| \geq 2$ and odd quadratic form superpairs. Let $0 \notin I$. An odd quadratic form grid can be written in the form $\mathcal{Q}_o(I) = \{g_0\}\dot\cup\{g_{\pm i} : i \in I\}$ with $g_0 \vdash g_{\pm i}$. The supercoordinate system associated to a Jordan superpair V covered by $\mathcal{Q}_o(I)$ is

$$\mathcal{C} = (A, X, q_X) \tag{1}$$

defined as follows:

(a) A is the McCrimmon-Meyberg superalgebra of $g_1, g_2 \in \mathcal{Q}_o(I)$ where $1, 2$ are two distinct elements of I (note that $A = V_2^+(g_1)$ as S-module), A necessarily is a superextension of S;

(b) X is the A-supermodule $X = \bigcap_{i\in I,\sigma=\pm} V_1^+(g_{\sigma i})$ with the canonical induced $\mathbb{Z}_2$-grading and the A-action $a.x = \{a\, g_1^-\, x\}$ for $a \in A, x \in X$.

(c) $q_X = (b_X, q_{X\bar 0})$ is the A-quadratic form on X given by $b_X(x, x') = \{x\, g_{-1}^-\, x'\}$ and $q_{X\bar 0}(x_{\bar 0}) = Q_{\bar 0}(x_{\bar 0})g_{-1}^-$, where $g_{-1} \in \mathcal{Q}_0(I)$ is the unique idempotent orthogonal to g_1. The element g_0^+ is a base point of q_X, i.e., $q_{X\bar 0}(g_0^+) = 1$.

The supercoordinate system $\mathcal{C}$ determines V: by [**21**, 4.16] V is isomorphic over S to the odd quadratic form superpair $\mathbb{OQ}_I(A, q_X)$. This Jordan superpair is the superpair of the A-quadratic form $q_I \oplus q_X$ where q_I is the hyperbolic form on the hyperbolic superspace over A of rank $2|I|$, see 1.9. Conversely, $\mathbb{OQ}_I(A, q_X)$ is covered by a odd quadratic form grid.

1.11. S-extensions. Let U be a Jordan pair over some ring k, and let A be a superextension of k. By [**21**, 2.6] there exists a unique Jordan superpair structure on $U_A = (A \otimes_k U^+, A \otimes_k U^-)$, called the *$A$-superextension of U*, such that its Grassmann envelope $G(U_A)$ is canonically isomorphic to the $G(A)$-extension $U_{G(A)}$ of U.

The Jordan superpairs arising in the coordinatization theorems for the remaining three types of grids, alternating, Bi-Cayley and Albert grids, are all A-superextensions of suitable split Jordan pairs over k ([**21**, 3.9]) — in fact, one can take $k = \mathbb{Z}$. The supercoordinate system is given by 1.2.1.

1.12. Alternating grids and alternating matrix superpairs. By [**21**, 4.18], a Jordan superpair is covered by an alternating grid $\mathcal{A}(I)$, $|I| \geq 4$, if and only if it is isomorphic to an alternating matrix superpair $\mathbb{A}_I(A)$ where A is the McCrimmon-Meyberg superalgebra of some collinear pair in $\mathcal{A}(I)$. Here $\mathbb{A}_I(A)$ is the A-extension of the split

Jordan pair $\mathbb{A}_I(k) = (\mathrm{Alt}(I,k), \mathrm{Alt}(I,k))$ of alternating matrices over k. Recall that in $\mathbb{A}_I(k)$ the family of all $e_{ij} = (E_{ij} - E_{ji}, E_{ji} - E_{ij}),\ i < j$, where $<$ is some total order on I, is an alternating covering grid.

1.13. Bi-Cayley grids and Bi-Cayley superpairs. By [**21**, 4.20] a Jordan superpair over S is covered by a Bi-Cayley grid $\mathcal{B}$ if and only if it is isomorphic to the Bi-Cayley superpair $\mathbb{B}(A) = A \otimes_k \mathbb{M}_{12}(\mathbb{O}_k)$, the A-extension of the rectangular matrix superpair $\mathbb{B}(k) = \mathbb{M}_{12}(\mathbb{O}_k)$ for $\mathbb{O}_k$ the split Cayley algebra over k. The corresponding McCrimmon-Meyberg superalgebra is A for a suitable collinear pair in $\mathcal{B}$.

1.14. Albert grids and Albert superpairs. By [**21**, 4.22] a Jordan superpair V over S is covered by an Albert grid $\mathcal{A}$ if and only if there exists a superextension A of S such that V is isomorphic to the Albert superpair $\mathbb{AB}(A) = A \otimes_k \mathbb{AB}(k)$, the A-extension of the split Jordan pair $\mathbb{AB}(k) = \mathbb{H}_3(\mathbb{O}_k, k \cdot 1, \pi)$ where $\mathbb{O}_k$ is the split Cayley algebra over k with canonical involution π. The corresponding McCrimmon-Meyberg superalgebra is A for a suitable collinear pair in $\mathcal{A}$.

1.15. Standard examples. It will be useful to have a common notation for the Jordan superpairs arising in the coordinatization theorems. We denote by $\mathbb{V}(\mathcal{G}, \mathcal{C})$ and call it a *standard example* the following Jordan superpairs over S:

- (i) a Jordan superpair $\mathbb{J} = (J, J)$ of a unital Jordan superalgebra J over S, 1.5, or of a unital Jordan superalgebra with a Peirce decomposition $\mathfrak{P}$ as defined in 1.7;
- (ii) a rectangular matrix superpair $\mathbb{M}_{JK}(A)$ for $|J| + |K| \geq 3$, 1.6;
- (iii) a hermitian Jordan superpair $\mathbb{H}_I(A, A^0, \pi)$, 1.8;
- (iv) an even or odd quadratic form superpair $\mathbb{EQ}_I(A)$, 1.9, or $\mathbb{OQ}_I(A, q_X)$, 1.10;
- (v) an alternating matrix superpair $\mathbb{A}_I(A)$, 1.12, a Bi-Cayley superpair $\mathbb{B}(A)$ or an Albert superpair $\mathbb{AB}(A)$;

The coordinatization theorems can then be summarized by saying that *V is a Jordan superpair over S covered by a connected standard grid $\mathcal{G}$ if and only if $V \cong \mathbb{V}(\mathcal{G}, \mathcal{C})$ where $\mathcal{C}$ is the supercoordinate system of V.* (Recall that we assume 1.8.1 in the hermitian case.)

2. Semiprime, prime and simple superalgebras.

In this section we will consider semiprime, prime and simple alternative superalgebras and superalgebras with involutions, with emphasis on associative superalgebras. It is not the goal to present a complete theory, rather the results of this section serve as preparation for the following section §3 where we will describe semiprime, prime and simple Jordan superpairs covered by a grid.

Unless stated otherwise, all superalgebras are defined over some base superring S. All alternative superalgebras are assumed to be unital, unless stated otherwise.

2.1. Definitions. By definition, any ideal in a superalgebra is $\mathbb{Z}_2$-graded. We recall that an alternative or Lie superalgebra A is called

(i) *semiprime* if $II \neq 0$ for any non-zero ideal I of A,
(ii) *prime* if $IJ \neq 0$ for any two non-zero ideals I, J of A, or
(iii) *simple* if every ideal I of A is trivial, $I = 0$ or $I = A$, and $A^2 \neq 0$.

For prime alternative superalgebras one has the fundamental result 2.2 below.

For an associative superalgebra A the usual proof in the non-supercase shows that semiprimeness and primeness have the following elementwise characterizations ([**26**, p. 595]):

(1) A is semiprime if and only if $a_\mu A a_\mu \neq 0$ for all $0 \neq a_\mu \in A_\mu$, $\mu \in \{\bar{0}, \bar{1}\}$.
(2) A is prime if and only if $a_\mu A b_\nu \neq 0$ for all $0 \neq a_\mu \in A_\mu$, $0 \neq b_\nu \in A_\nu$ and $\mu, \nu \in \{\bar{0}, \bar{1}\}$.

More results on prime associative superalgebras over rings containing $\frac{1}{2}$ are given in [**20**, 1].

Recall that an involution π of a superalgebra A is an S-linear map satisfying the conditions $A_\mu^\pi \subset A_\mu$ for $\mu = \bar{0}, \bar{1}$, $(ab)^\pi = (-1)^{|a||b|} b^\pi a^\pi$ and $(a^\pi)^\pi = a$ for homogeneous $a, b \in A$. One says A is π-*semiprime* or π-*prime* or π-*simple* if the conditions (i) or (ii) or (iii) above hold respectively for π-invariant ideals $I = I^\pi$.

2.2. Theorem. (Shestakov-Zelmanov [**32**, Thm. 2]) *A prime alternative superalgebra A over a field of characteristic $\neq 2, 3$ is either associative or $A = A_{\bar{0}}$ is a Cayley-Dickson ring.*

This result is no longer true in characteristic $2, 3$. For example, let $\mathbb{O} = \mathbb{H} \oplus v\mathbb{H}$ be a Cayley-Dickson algebra over a field of characteristic 2, obtained by the Cayley-Dickson process from a quaternion subalgebra $\mathbb{H}$. Then $\mathbb{O}$ with respect to the $\mathbb{Z}_2$-grading $\mathbb{O}_{\bar{0}} = \mathbb{H}$, $\mathbb{O}_{\bar{1}} = v\mathbb{H}$ is a simple alternative superalgebra. Nonassociative alternative superalgebras over arbitrary fields which are simple or which are prime and satisfy an additional assumption are classified in [**30**].

We next describe the structure of simple associative superalgebras. We start with two important classes of examples.

2.3. Matrices over associative superalgebras. The concept of matrices over an associative superalgebra A, introduced in [**12**, Ch. 3, 1.7] for matrices of finite size, can easily be extended to finite matrices of arbitrary size. As in 1.6 we denote by $\mathrm{Mat}(J, K; A)$ the A-bimodule of matrices of size $J \times K$, where J and K are arbitrary sets, with entries from A which are finite in the sense that only finitely many entries are non-zero. In 1.6 we have used the $\mathbb{Z}_2$-grading of $\mathrm{Mat}(J, K; A)$ given by the grading of A, i.e., $\mathrm{Mat}(J, K; A)_\mu = \mathrm{Mat}(J, K; A_\mu)$ for $\mu = \bar{0}, \bar{1}$. This can be generalized to a $\mathbb{Z}_2$-grading which, in addition to the grading of A, uses two partitions $J = M \dot{\cup} N$, $K = P \dot{\cup} Q$ and which will therefore be denoted $\mathrm{Mat}(M|N, P|Q; A)$. For this new grading, a matrix $x \in \mathrm{Mat}(J, K; A)$, written as

$$x = \begin{array}{cc} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} & \begin{array}{c} M \\ N \end{array} \\ \begin{array}{cc} P & \;\; Q \end{array} & \end{array}$$

with $x_1 \in \mathrm{Mat}(M, P; A)$, $x_2 \in \mathrm{Mat}(M, Q; A)$, $x_3 \in \mathrm{Mat}(N, P; A)$ and $x_4 \in \mathrm{Mat}(N, Q; A)$, is even (respectively odd) if all entries in x_1 and x_4 are even (respectively odd) and if all entries in x_2 and x_3 are odd (respectively even). Thus, symbolically,

$$\mathrm{Mat}(M|N, P|Q; A)_\mu = \begin{pmatrix} \mathrm{Mat}(M, P; A_\mu) & \mathrm{Mat}(M, Q; A_{\mu+\bar{1}}) \\ \mathrm{Mat}(N, P; A_{\mu+\bar{1}}) & \mathrm{Mat}(N, Q; A_\mu) \end{pmatrix} \tag{1}$$

for $\mu = \bar{0}, \bar{1}$. We note that ([**12**, III.1.7])

$$\mathrm{Mat}(M|N, P|Q; A) \cong \mathrm{Hom}_A(A^{(P|Q)}, A^{(M|N)}) \tag{2}$$

where $A^{(M|N)} = A^{(M)} \oplus (\mathit{\Pi} A)^{(N)}$ and $\mathit{\Pi} A$ denotes the regular module with changed parity. Thus, $A_\mu^{(M|N)} = (A_\mu)^{(M)} \oplus (A_{\mu+\bar{1}})^{(N)}$ for $\mu = \bar{0}, \bar{1}$ and $A^{(M|N)}$ has a basis composed of an even part of size M and an odd part of size N. With the usual matrix product

$$\mathrm{Mat}_{P|Q}(A) := \mathrm{Mat}(P|Q, P|Q; A) \tag{3}$$

becomes an associative (not necessarily unital) superalgebra. As in 3.8 one can show that the ideals of $\mathrm{Mat}_{P|Q}(A)$ are given by $\mathrm{Mat}_{P|Q}(B)$ where B is an ideal of A, and hence

$$\begin{aligned} &\mathrm{Mat}_{P|Q}(A) \textit{ is semiprime, prime or simple} \\ &\iff A \textit{ is semiprime, prime or simple.} \end{aligned} \tag{4}$$

Following standard practice, we will replace the sets M, N, P, Q in the notations above with their cardinality in case they are all finite. Thus, we will write $\mathrm{Mat}_{p|q}(A) = \mathrm{Mat}_{P|Q}(A)$ if $|P| = p < \infty$ and $q = |Q| < \infty$. Note that in this case $\mathrm{Mat}_{p|q}(A)$ is unital.

2.4. Doubles of associative algebras. Let B be an associative algebra. The superalgebra $\mathrm{Mat}_{1|1}(B)$ has a canonical subalgebra

$$\mathbb{D}(B) := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in B \right\} \tag{1}$$

$$= B \oplus Bu \quad \text{for } u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

which we call the *double of* B. For $B = \mathrm{Mat}(n, n; k)$ this superalgebra already occurs in [**5**] where it is denoted $Q_n(k)$ because of its relation to the simple Lie superalgebras $\mathbf{Q}_n$, see [**2**]. Note that u is a central element of the associative algebra $\mathbb{D}(B)$, i.e., $xu = ux$ for all $x \in \mathbb{D}(B)$, but does not supercommute with all elements, in fact $[u, u] = 2$. In terms of the representation (2), the superalgebra $\mathbb{D}(B)$ can equivalently be described as follows:

$$\mathbb{D}(B)_{\bar{0}} = B\,, \quad \mathbb{D}(B)_1 = Bu. \tag{3}$$

with associative product $\cdot$ given by $(a, b \in B)$

$$a \cdot b = ab, \quad a \cdot bu = (ab)u = au \cdot b, \quad au \cdot bu = ab. \tag{4}$$

These formulas show that this doubling process is closely related to the Kantor doubling process in which B is replaced by a superextension S of k and $au \cdot bu$ is given by a second product on S, see [**7**], [**8**], [**29**] or the recent paper [**13**].

In the following we will discuss some easily established properties of doubles. Any ideal of $\mathbb{D}(B)$ has the form $\mathbb{D}(I) = I \oplus Iu$ where I is an ideal of B. It then follows easily that $\mathbb{D}(B)$ *is semiprime, prime or simple if and only if B is respectively semiprime, prime or simple.* A double $\mathbb{D}(B)$ is a division superalgebra if and only if B is a division algebra.

Suppose π is an involution of $\mathbb{D}(B)$. Then there exists a unique element $z \in \mathrm{Z}(B)$, the centre of B, satisfying

$$z^2 = -1, \quad z^\pi = -z \quad \text{and} \quad (a \oplus bu)^\pi = a^\pi \oplus b^\pi zu \tag{5}$$

for $a, b \in B$. Conversely, given an associative algebra B with involution π and an element $z \in Z(B)$ with $z^2 = -1$ and $z^\pi = -z$ the extension of π to $\mathbb{D}(B)$, defined as in (5) above, is an involution. In particular, if $2B = 0$ then $\mathbb{D}(B)$ always has an involution given by $z = 1$, and this is the only one in case B is simple. If B is an algebra over a field of characteristic $\neq 2$ then a necessary condition for the existence of an involution on $\mathbb{D}(B)$ is that B has an involution of second kind, i.e., $\pi | Z(B) \neq \mathrm{Id}$. In particular, if B is central in the sense that $Z(B) = k$ then $\mathbb{D}(B)$ does not have an involution. The symmetric elements of $\mathbb{D}(B)$ with respect to the involution π are given by

$$\mathrm{H}(\mathbb{D}(B), \pi) = \mathrm{H}(B, \pi) \oplus u \begin{cases} \mathrm{H}(B, \pi) & \text{if } z = 1 \\ (1+z)\mathrm{H}(B, \pi) & \text{if } \frac{1}{2} \in S. \end{cases} \tag{6}$$

Any $\mathbb{D}(B)$-module X satisfies $X_{\bar{1}} = uX_{\bar{0}}$, and hence we can put $X = \mathbb{D}(X_{\bar{0}})$ with obvious meaning. Observe $\mathbb{D}(B)^{(P|Q)} \cong \mathbb{D}(B^{(P \dot\cup Q)})$ for free $\mathbb{D}(B)$-modules.

For two $\mathbb{D}(B)$-modules $X = \mathbb{D}(X_{\bar{0}}), Y = \mathbb{D}(Y_{\bar{0}})$ we have $\mathrm{Hom}_{\mathbb{D}(B)}(X, Y) = \mathrm{Hom}_B(X_{\bar{0}}, Y_{\bar{0}}) \oplus u \,\mathrm{Hom}_B(X_{\bar{0}}, Y_{\bar{0}}) = \mathbb{D}(\mathrm{Hom}_B(X_{\bar{0}}, Y_{\bar{0}}))$. For endomorphisms between free modules this leads to the following identification of matrix superalgebras:

$$\mathrm{Mat}_{P|Q}(\mathbb{D}(B)) \cong \mathbb{D}(\mathrm{Mat}(P \dot\cup Q, P \dot\cup Q; B)) \tag{7}$$

with respect to the map $\mathrm{Mat}_{P|Q}(\mathbb{D}(B)) \to \mathbb{D}(\mathrm{Mat}(P \dot\cup Q, P \dot\cup Q; B)$ given by

$$\begin{pmatrix} x_1 u^s & x_2 u^{s+1} \\ x_3 u^{s+1} & x_4 u^s \end{pmatrix} \mapsto u^s \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad s = 0, 1.$$

A double $\mathbb{D}(B)$ is supercommutative if and only if B is commutative and $2B = 0$. Assume this to be the case. Then $\mathbb{D}(B)$ is commutative as an algebra and a $\mathbb{D}(B)$-quadratic form $q = (b, q_{\bar{0}})$ on a $\mathbb{D}(B)$-module $X = \mathbb{D}(X_{\bar{0}})$ is uniquely determined by $q_{\bar{0}}$ and its polar $b_{\bar{0}} = b | X_{\bar{0}} \times X_{\bar{0}}$: b is the $\mathbb{D}(B)$-extension of $b_{\bar{0}}$.

2.5. Proposition. (Wall) *An associative superalgebra A is simple as superalgebra if and only if either A is simple as algebra or $A = \mathbb{D}(A_{\bar{0}})$ and $A_{\bar{0}}$ is simple.*

In particular, an associative superalgebra A over an algebraically closed field k is finite dimensional and simple if and only if either $A \cong \mathbb{D}(\mathrm{Mat}(m, m; k))$ or $A \cong \mathrm{Mat}_{p|q}(k)$ for finite numbers m, p and q.

Proof. The description of a simple associative superalgebra is given in [**31**, Lemma 3]. The two cases are mutually exclusive since in the second case $\{a_{\bar{0}} \oplus a_{\bar{0}} u : a_{\bar{0}} \in A_{\bar{0}}\}$ is a proper ideal.

The finite-dimensional result is implicit in [**31**]. It can also be derived from 2.7 and 2.4.7 using the easily established fact that a finite-dimensional associative division superalgebra D over k is either isomorphic to k or to $\mathbb{D}(k)$.

2.6. Corollary. *Let A be a superextension of S, and put* $\mathrm{Tor}_2(A) = \{a \in A : 2a = 0\}$*.*

(a) *A is semiprime if and only if A does not have non-zero homogeneous nilpotent elements. In this case, $A_{\bar 1} \neq 0 \Longrightarrow \mathrm{Tor}_2(A) \neq 0$.*

(b) *A is prime if and only if A does not contain non-zero homogeneous zero divisors. In this case, the algebra A is commutative and $A_{\bar 1} \neq 0 \Longrightarrow 2A = 0$, i.e., $\mathrm{Tor}_2(A) = A$.*

(c) *The following are equivalent:*

(i) *A is a simple superalgebra,*

(ii) *every non-zero homogeneous element is invertible,*

(iii) *either $A = A_{\bar 0}$ is a field or $A = \mathbb{D}(A_{\bar 0})$ where $A_{\bar 0}$ is a field of characteristic 2.*

(d) *In particular, if $\frac{1}{2} \in A$ then A is prime or simple if and only if $A = A_{\bar 0}$ is an integral domain or a field.*

Proof. (a) By 2.1(1) we have A is semiprime if and only if $a_\mu^2 \neq 0$ for all $0 \neq a_\mu \in A_\mu$, $\mu \in \{\bar 0, \bar 1\}$. This in turn is equivalent to the stated condition. We have $a_{\bar 1}^2 = -a_{\bar 1}^2$ whence $a_{\bar 1}^2 \in \mathrm{Tor}_2(A)$ for all $a_{\bar 1} \in A_{\bar 1}$.

(b) The characterization of prime superextensions A is immediate from 2.1(2). The algebra A is a commutative algebra if and only if $A_{\bar 1}^2 \subset \mathrm{Tor}_2(A)$, a condition trivially fulfilled if $A = A_{\bar 0}$. Also, $\mathrm{Tor}_2(A)$ and $2A$ are ideals of A with $\mathrm{Tor}_2(A)(2A) = 0$. Hence $\mathrm{Tor}_2(A) = 0$ or $2A = 0$ in any prime A. In the first case $A = A_{\bar 0}$ by (a), while in the second case we have $A = \mathrm{Tor}_2(A)$ and again commutativity follows.

(c) Suppose that A is simple, and let $0 \neq a_\mu \in A_\mu$. Then $A = a_\mu A$ by simplicity, and hence there exists $b_\mu \in A_\mu$ with $a_\mu b_\mu = 1$. If $\mu = \bar 1$ we have $2A = 0$ by (b), and so also $b_\mu a_\mu = 1$. Hence in all cases a_μ is invertible. The converse is obvious. That (i) $\Leftrightarrow$ (iii) follows from (b) and 2.5.

(d) is a special case of (b) and (c).

More can be said in the case of associative superalgebras which are *Artinian*, i.e., they satisfy the descending chain condition for right ideals ([**26**]). Let us also recall that a superalgebra is a *division superalgebra* if every non-zero homogeneous element is invertible. Clearly, division superalgebras are simple superalgebras. Associative division superalgebras are described in [**26**, p. 605].

2.7. Theorem. (Racine) *An associative superalgebra A is Artinian and simple if and only if there exists an associative division superalgebra D such that one of the following two alternatives holds:*

(i) *$D_{\bar 1} \neq 0$, and there exists a finite m such that $A \cong \mathrm{Mat}(m, m; D)$ where the $\mathbb{Z}_2$-grading is given by $\mathrm{Mat}(m, m; D)_\mu = \mathrm{Mat}(m, m; D_\mu)$ for $\mu = \bar 0, \bar 1$;*

(ii) *$D_{\bar 1} = 0$, and there exist finite p, q such that $A \cong \mathrm{Mat}_{p|q}(D)$.*

Proof. It is shown in [**26**, Thm. 3] that an Artinian simple associative superalgebra is isomorphic, as superalgebra, to $\mathrm{End}_D V$ where D is a division superalgebra and V is supermodule over D which is finite-dimensional as $D_{\bar 0}$-module. As is mentioned in the discussion before [**26**, Thm. 3] this implies the two cases above. Indeed, if $D_{\bar 1} \neq 0$ then $D_{\bar 1} = xD_{\bar 0}$ for any non-zero $x \in D_{\bar 1}$ and $V_{\bar 1} = xV_{\bar 0}$ follows, implying that $V \cong D^m$ as supermodules and $A \cong \mathrm{Mat}(m, m; D)$. The converse is clear.

We now turn to superalgebras with involutions. The following characterization of π-simple superalgebras (2.1) can be proven in the same way as the corresponding result for algebras ([**26**, Lemma 11] for A associative).

2.8. Lemma. *A superalgebra A with involution π is π-simple if and only if either A is simple as superalgebra or $A = B \oplus B^\pi$ for a simple ideal B of A.*

We note that in the second case we have $A \cong B \oplus B^{\mathrm{op}}$ with exchange involution π. More precise information can be obtained for alternative superalgebras containing an ample subspace.

2.9. Lemma. *Let A be an alternative superalgebra which is π-simple with respect to an involution π of A, and let A^0 be an ample subspace of (A, π). Then there are the following two possibilities:*

- (i) *$A \cong B \oplus B^{\mathrm{op}}$ with π the exchange involution, B a simple associative superalgebra and $A^0 = \mathrm{H}(A, \pi)$, or*
- (ii) *A is a simple alternative superalgebra (see 2.2).*

Proof. We can assume that $A \cong B \oplus B^{\mathrm{op}}$ where B is a simple superalgebra and π is the exchange involution. Then $\mathrm{H}(A, \pi) = \{b \oplus b : b \in B\} = \{a + a^\pi : a \in A\} \subset A^0 \subset \mathrm{H}(A, \pi)$, whence $\mathrm{H}(A, \pi) = A^0$. Since $A^0 \subset \mathrm{N}(A)$, the nucleus of A, and since $\mathrm{N}(A) = \mathrm{N}(B) \oplus \mathrm{N}(B^{\mathrm{op}})$, we obtain $A = \mathrm{N}(A)$, i.e., A is associative.

2.10. Theorem. *Let A be a π-simple associative superalgebra where π is an involution π of A. Then precisely one of the following cases holds:*

- (i) *A is not a simple superalgebra: there exists a simple ideal B of A such that $A = B \oplus B^\pi$.*
- (ii) *$A = \mathbb{D}(A_{\bar{0}})$ for a simple $A_{\bar{0}}$ and there exists $z \in Z(A_{\bar{0}})$ such that 2.4.5 holds.*
- (iii) *A is simple as algebra, $A_{\bar{0}} = B_1 \oplus B_2$ is a direct sum of non-zero simple ideals B_i and $A_{\bar{1}} = C_1 \oplus C_2$ is a direct sum of irreducible $A_{\bar{0}}$-bimodules $C_i = e_i A e_j$ where e_i is the unit element of B_i and $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$. Moreover,*

$$\text{either } B_i^\pi = B_j, C_i^\pi = C_i \text{ or } B_i^\pi = B_i, C_i^\pi = C_j. \tag{1}$$

- (iv) *A and $A_{\bar{0}}$ are simple as algebras.*

Proof. By 2.8 we either have (i) or A is a simple superalgebra with involution. In the second case we can apply 2.5: either we have (ii) or A is simple as an algebra. If A is a simple algebra we either have (iv) or $A_{\bar{0}}$ is not simple. In the following we can therefore assume that A is a simple algebra but $A_{\bar{0}}$ is not. We will see that this leads to (iii).

Let then B be a non-trivial ideal of $A_{\bar{0}}$. By [**31**, Lemma 1], we have

$$A_{\bar{0}} = A_{\bar{1}}^2 = B + A_{\bar{1}} B A_{\bar{1}} \quad \text{and} \quad A_{\bar{1}} = A_{\bar{1}} B + B A_{\bar{1}}. \tag{2}$$

In particular, this implies that $A_{\bar{1}} B A_{\bar{1}}$ is a non-zero ideal. If the ideal $I = B \cap A_{\bar{1}} B A_{\bar{1}}$ is non-zero, it is proper and hence (2) also holds for I leading to the contradiction $A_{\bar{1}} I A_{\bar{1}} \subset A_{\bar{1}}^2 B A_{\bar{1}}^2 \subset B$ and $A_{\bar{0}} = B$. Therefore $A_{\bar{0}} = B \oplus A_{\bar{1}} B A_{\bar{1}}$ is a direct sum of ideals. Since this holds for any proper ideal it is immediate that $B_1 := B$ and $B_2 := A_{\bar{1}} B A_{\bar{1}}$ are simple. It follows that either $B_i^\pi = B_i$ or $B_i^\pi = B_j$ for i,j as in (iii). Any $x \in A_{\bar{1}} B \cap B A_{\bar{1}}$ has the property $x A_{\bar{1}} \in B A_{\bar{1}}^2 \cap A_{\bar{1}} B A_{\bar{1}} = B_1 \cap B_2 = 0$, whence $A_{\bar{1}} = B A_{\bar{1}} \oplus A_{\bar{1}} B$. Also observe $A_{\bar{1}} B_1 = A_{\bar{1}} B A_{\bar{1}}^2 = B_2 A_{\bar{1}}$ and then $B_1 A_{\bar{1}} = A_{\bar{1}} B_2$ by symmetry. We can now show the missing part of (1), putting $C_1 = B A_{\bar{1}} = B_1 A_{\bar{1}}$ and $C_2 = A_{\bar{1}} B = B_2 A_{\bar{1}}$: if $B_i = B_i^\pi$ then $C_i^\pi = (B_i A_{\bar{1}})^\pi = A_{\bar{1}} B_i = C_j$ while for $B_i^\pi = B_j$ we get $C_i^\pi = A_{\bar{1}} B_j = C_i$. Let $e_i \in B_i$ such that $1 = e_1 \oplus e_2 \in A^0 = B_1 \oplus B_2$. Then e_1, e_2 are orthogonal idempotents. Their Peirce spaces $A_{ij} = e_i A e_j$ are $A_{11} = B_1$, $A_{12} = C_1$, $A_{21} = C_2$ and $A_{22} = B_2$. Therefore we have the following multiplication rules

$$B_i C_i = C_i = C_i B_j, \quad C_i C_j = B_i, \quad 0 = B_i C_j = C_i B_i = C_i C_i \tag{3}$$

for (i,j) as in (iii). Using these, it is straightforward to show that any proper $A_{\bar{0}}$-sub-bimodule D of C_i gives rise to an ideal $DC_j \oplus C_j D \oplus D \oplus C_j D C_j$ of A, proving that C_i is irreducible.

Remarks. **(a)** This theorem generalizes [**26**, Thm. 12] which describes π-simple associative superalgebras A for which $A_{\bar{0}}$ is not π-simple, i.e., the cases (i) and (iii) with the second alternative in (1). Involutions of associative division superalgebras are studied in Prop. 9 and 10 of [**26**]. These are special cases of (ii) and (iv) above. We point out that case (iv) can only occur if A is infinite-dimensional over the even part of the centre of the associative algebra A ([**26**, Lemma 8 and Remark p. 607]). Involutions of simple associative superalgebras are also studied in [**4**, 3] and [**3**, 2].

(b) It is convenient to view A in case (iii) as a Morita context,

$$A = \begin{pmatrix} B_1 & C_1 \\ C_2 & B_2 \end{pmatrix} \tag{4}$$

with involutions given by

$$\begin{pmatrix} b_1 & c_1 \\ c_2 & b_2 \end{pmatrix} \mapsto \begin{pmatrix} b_2^\pi & c_1^\pi \\ c_2^\pi & b_1^\pi \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b_1 & c_1 \\ c_2 & b_2 \end{pmatrix} \mapsto \begin{pmatrix} b_1^\pi & c_2^\pi \\ c_1^\pi & b_2^\pi \end{pmatrix} \tag{5}$$

For $A = \mathrm{Mat}_{p|q}(D)$ with $D = D_{\bar{0}}$ a division algebra and $p, q > 0$ more precise information on these two types of involutions is given in Prop. 13 and Prop. 14 of [**26**]. In particular, both types do occur, see also [**26**, Example p. 602] for the first type. Note that by 2.4 and 2.5 any finite-dimensional simple associative superalgebra over k of characteristic $\neq 2$ which has an involution is isomorphic to some $\mathrm{Mat}_{p|q}(k)$.

(c) The theorem above is similar to the description of (π, φ)-simple associative algebras with commuting involution π and involutorical automorphism φ ([**19**, Prop. 2.8]).

3. Semiprime, prime and simple Jordan superpairs covered by grids.

In this section we will describe semiprime, prime and simple Jordan superpairs covered by a grid. Our analysis will first reduce the problem to the case of a connected grid, whence one of the standard examples introduced in §1. We will then describe all ideals in the standard examples in terms of the associated supercoordinate system. Their knowledge will easily allow us to determine the conditions for semiprimeness, primeness and simplicity.

Unless stated otherwise, all Jordan superpairs are defined over some base superring S.

3.1. Semiprime and prime Jordan superpairs. The definition of an ideal in a Jordan superpair is given in [**21**, 3.3]. In particular, any ideal in a Jordan superpair is $\mathbb{Z}_2$-graded. For ideals I, J of a Jordan superpair V we define their *Jordan product* $I \diamond J = ((I \diamond J)^+, (I \diamond J)^-))$ by

$$(I \diamond J)^\sigma = Q_{\bar 0}(I^\sigma_{\bar 0})J^{-\sigma} + \{I^\sigma\,,\,J^{-\sigma}\,,\,I^\sigma\}.$$

We note that $I \diamond J$ is in general not an ideal. We will say V is

(i) *semiprime* if the *Jordan cube* $I \diamond I \neq 0$ for any non-zero ideal I of V,

(ii) *prime* if $I \diamond J \neq 0$ for any two non-zero ideals I and J of V.

Primenness can also be defined in terms of annihilators, where for an ideal I in a Jordan superpair V the *annihilator*

$$\mathrm{Ann}_V(I) = (\mathrm{Ann}(I)^+_{\bar 0} \oplus \mathrm{Ann}(I)^+_{\bar 1}, \mathrm{Ann}(I)^-_{\bar 0} \oplus \mathrm{Ann}(I)^-_{\bar 1})$$

is given by $z \in \mathrm{Ann}(I)^\sigma_\mu$ for $\sigma = \pm$ and $\mu \in \{\bar 0, \bar 1\}$ if and only if

$$\begin{aligned} 0 \;&=\; D(z^\sigma, V^{-\sigma}) \;=\; D(V^{-\sigma}, z^\sigma) \;=\; Q(z^\sigma, V^\sigma) \\ &=\; Q_{\bar 0}(I^{-\sigma}_{\bar 0})z = Q_{\bar 0}(I^\sigma_{\bar 0})Q_{\bar 0}(V^{-\sigma}_{\bar 0})z\,, \quad \text{and in addition for } \mu = \bar 0: \\ 0 \;&=\; Q_{\bar 0}(z)I^{-\sigma} = Q_{\bar 0}(I^{-\sigma}_{\bar 0})Q_{\bar 0}(z)\,. \end{aligned}$$

The Grassmann envelope of $\mathrm{Ann}(I)$ coincides with the annihilator of the ideal $G(I)$ in $G(V)$. Since the annihilator of an ideal in a Jordan pair is again an ideal, [**17**, §4], $\mathrm{Ann}_V(I)$ is an ideal of the Jordan superpair V. As in [**17**, Prop. 1.6] one can show that the following conditions are equivalent for a Jordan superpair V:

(i) V is prime,

(ii) V is semiprime and any two non-zero ideals of V have non-zero intersection, and

(iii) V is semiprime and the annihilator of any non-zero ideal of V vanishes.

3.2. Doubles of Jordan pairs. The double $\mathbb{D}(B)$ of an associative algebra B as defined in 2.4 can be viewed as the tensor product algebra $\mathbb{D}(B) = \mathbb{D}(k) \otimes B$. This motivates the following definition. For a Jordan pair U over k, the *double of* U is defined as the superextension

$$\mathbb{D}(U) := U_{\mathbb{D}(k)} = \mathbb{D}(k) \otimes_k U \tag{1}$$

of U in the sense of [**21**, 2.6]. We recall that $\mathbb{D}(U)$ is a Jordan superpair over $\mathbb{D}(k)$ with $\mathbb{D}(U)_{\bar 0} = U$, $\mathbb{D}(U)_{\bar 1} = Uu \cong U$ and whose Jordan triple product satisfies $\{x_1 \otimes v_1, x_2 \otimes v_2, x_3 \otimes v_3\} = (x_1x_2x_3) \otimes \{v_1, v_2, v_3\}$.

The relation between ideals of U and $\mathbb{D}(U)$ is not as simple as in the case of superalgebras (2.4). A submodule $I_0 \oplus I_1 u$ is an ideal of $\mathbb{D}(U)$ if and only if I_0 is an ideal and I_1 is an outer ideal of U, related to each other by $\{U\,U\,I_1\} + \{U\,I_1\,U\} \subset I_0$ and $Q(I_0)U + \{U\,U\,I_0\} + \{U\,I_0\,U\} \subset I_1$. In particular, if I is an ideal of U then $\mathbb{D}(I) = I \oplus I u$ is an ideal of $\mathbb{D}(U)$. We have $\mathbb{D}(I) \diamond \mathbb{D}(J) = \mathbb{D}(I \diamond J)$ for two ideals I, J of U. It is then easy to see that *U is semiprime, prime or simple if $\mathbb{D}(U)$ is so, and the converse is true if $\frac{1}{2} \in k$.* We note that for any field F of characteristic 2, the Jordan pair $U = (F, F)$ is simple but Uu is a nontrivial ideal of $\mathbb{D}(U)$ with $Uu \diamond Uu = 0$, showing that $\mathbb{D}(U)$ is not even semiprime.

Let B be an extension of k, i.e., a unital associative commutative k-algebra. Then $B \otimes_k U$ is a Jordan pair over B and we have a canonical isomorphism of Jordan superpairs over $\mathbb{D}(B)$

$$\mathbb{D}(B \otimes_k U) = \mathbb{D}(B) \otimes_B (B \otimes_k U) \cong \mathbb{D}(B) \otimes_k U \tag{2}$$

where $\mathbb{D}(B) \otimes_k U$ is the $\mathbb{D}(B)$-extension of U. Similarly, taking doubles commutes with taking rectangular matrices, in the following sense. Let $\mathbb{M}_{JK}(B)$ be a rectangular matrix pair with an associative coordinate algebra B. Then

$$\mathbb{D}(\mathbb{M}_{JK}(B)) \cong \mathbb{M}_{JK}(\mathbb{D}(B)). \tag{3}$$

Recall 2.4 that a supercommutative superalgebra $\mathbb{D}(B)$ necessarily has $2B = 0$ and hence is commutative. In this case, any $\mathbb{D}(B)$-quadratic form q on $X = \mathbb{D}(X_{\bar{0}})$ is uniquely given by the quadratic form $q_{\bar{0}}$ on $X_{\bar{0}}$. This easily implies that

$$\mathbb{OQ}_I(\mathbb{D}(B), q_X) = \mathbb{D}(\mathbb{OQ}_I(B, q_{\bar{0}})). \tag{4}$$

After these preparations we now consider Jordan superpairs covered by a grid. Our first result will reduce our study to the case of connected grids.

3.3. Proposition. *Let V be a Jordan superpair covered by a grid $\mathcal{G} = \{g_\alpha : \alpha \in R_1\}$ with associated 3-graded root system (R, R_1) and joint Peirce spaces V_α, whence $V = \bigoplus_{\alpha \in R_1} V_\alpha$. Then any ideal I of V splits in the sense that*

$$I = \textstyle\bigoplus_{\alpha \in R_1} (I \cap V_\alpha), \quad I \cap V_\alpha = (I^+ \cap V_\alpha^+, I^- \cap V_\alpha^-). \tag{1}$$

(a) *Suppose $(R, R_1) = \bigoplus_{c \in C} (R^{(c)}, R_1^{(c)})$ is a direct sum of 3-graded root systems $(R^{(c)}, R_1^{(c)})$ so that*

$$V = \textstyle\bigoplus_{c \in C} V^{(c)}, \quad V^{(c)} = \bigoplus_{\alpha \in R_1^{(c)}} V_\alpha \vartriangleleft V$$

is a direct sum of ideals $V^{(c)}$ of V, covered by the grids $\mathcal{G}^{(c)} = \{g_\alpha : \alpha \in R_1^{(c)}\}$, [**21**, 3.5]. *Then $I = \bigoplus_{c \in C} (I \cap V^{(c)})$ and each $I \cap V^{(c)}$ is an ideal of the Jordan superpair $V^{(c)}$. In particular,*

$$V \text{ semiprime} \iff \text{every } V^{(c)} \text{ is semiprime, and} \tag{2}$$
$$V \text{ prime} \implies R \text{ is irreducible.} \tag{3}$$

(b) *Let* (α,β) *be a collinear pair and let* A_α *be the associated McCrimmon-Meyberg superalgebra algebra on* V_α^σ. *Then* $I\cap V_\alpha$ *is an ideal of* A_α.

Proof. Any $x\in I^\sigma$ is a finite sum $x=\sum_{\alpha\in R_1}x_\alpha$, $x_\alpha\in V_\alpha^\sigma$. Suppose there exists $x\in I$ for which $x_\alpha\notin I^\sigma$ for some $\alpha\in R_1$. Among all such x we choose one which has a minimal number of non-zero components x_α, whence all non-zero $x_\alpha\notin I^\sigma$. Let $E_i^\sigma(\alpha)$ be the projection operator onto the Peirce space $V_i^\sigma(g_\alpha)$, $i=0,1,2$, [**10**, 5.4]. Since all $E_i^\sigma(\alpha)$ are multiplication operators we have $E_i^\sigma(\alpha)I^\sigma\subset I^\sigma$. In particular, if $V_2(g_\alpha)=V_\alpha$, i.e., α is a long root, $x_\alpha=E_2^\sigma(\alpha)x_\alpha=E_2^\sigma(\alpha)x\in I^\sigma$. Thus, for all non-zero components x_α we have $V_\alpha^\sigma\subsetneqq V_2^\sigma(g_\alpha)$. The connected component of α is therefore either a hermitian grading or an odd quadratic form grading. In the first case, α imbeds in a triangle $(\alpha;\beta,\gamma)$ such that $V_2(g_\alpha)=V_\alpha\oplus V_\beta\oplus V_\gamma$, and $V_\alpha^\sigma=E_1^\sigma(\beta)E_1^\sigma(\gamma)V^\sigma$. As before this implies $x_\alpha\in I^\sigma$, contradicting our choice of x. Hence, we now know that for all non-zero x_α the connected component C_α of the root α is an odd quadratic form grading, α is the unique short root in $C_\alpha\cap R_1$ and for every other non-zero $x_{\alpha'}$ the root α' lies in a different connected component. Since then $E_2^\sigma(\alpha)x_{\alpha'}=0$ it follows again that $x_\alpha\in I^\sigma$, leading to the final contradiction and hence to the proof of (1). The proof of (a) is then immediate from the definitions, using that every 3-graded root system is a direct sum of irreducible 3-graded root systems, see [**23**] or [**11**]. (b) follows from the definition of the product in A_α, [**21**, (3.2.1)].

In view of this result we will assume for the remainder of this section that V is a Jordan superpair covered by a connected grid $\mathcal{G}$, equivalently, the associated 3-graded root system of $\mathcal{G}$ is irreducible. Hence, V is one of the standard examples of §1. We start with the case $|\mathcal{G}|=1$, i.e., $R=\mathrm{A}_1$.

3.4. Example: $V=\mathbb{J}=(J,J)$ **for a unital Jordan superalgebra** J**.** The ideals of $\mathbb{J}$ are easily described:

$$(I^+,I^-) \textit{ is an ideal of } (J,J) \textit{ if and only if } I^+=I^- \textit{ is an ideal of } J.$$

Indeed, let (I^+,I^-) be an ideal of $V=(J,J)$. We then have $I^\sigma=U_{\bar 0}(1)I^\sigma\in Q_{\bar 0}(V_{\bar 0}^{-\sigma})I^\sigma\subset I^{-\sigma}$, i.e., $I^+=I^-$. Moreover, $I=I^+=I^-$ is an ideal of J since $U_{\bar 0}(I_{\bar 0})J=Q_{\bar 0}(I_{\bar 0}^+)V^-\subset I^+=I$, $U_{\bar 0}(J_{\bar 0})I=Q_{\bar 0}(V_{\bar 0}^-)I^+\subset I^-=I$, $\{I,J,J\}=\{I^+,V^-,V^+\}\subset I^+=I$ and $\{J,I,J\}=\{V^+,I^-,V^+\}\subset I^-=I$.

Semiprimeness and primeness of a unital Jordan superalgebra J are defined in the same way as for Jordan superpairs. It therefore follows from the above that

$$\begin{aligned}&\mathbb{J} \textit{ is semiprime, prime or simple}\\ &\iff J \textit{ is respectively semiprime, prime or simple.}\end{aligned} \tag{1}$$

We next consider the case $R=\mathrm{B}_2$.

3.5. Jordan superpairs covered by a triangle. By 1.7 these are the Jordan superpairs $V\cong\mathbb{J}=(J,J)$ for a unital Jordan superalgebra J over S which contains two strongly connected supplementary orthogonal idempotents c_1 and c_2 and hence has Peirce decomposition $\mathfrak{P}$, which for convenient notation we write in the form $J = J_2 \oplus J_1 \oplus J_0$ where $J_i = J_i(c_1)$ denotes the Peirce spaces of c_1.

We have seen in 3.4 that the ideals of superpairs $\mathbb{J}$ are of the form $\mathbb{I} = (I, I)$, where I is an ideal of J. By 3.3.1, I is a direct sum of its Peirce components

$$I = I_2 \oplus I_1 \oplus I_0 \quad \text{where } I_i = I \cap J_i. \tag{1}$$

The criterion 3.4.1 also holds in this case.

Since $I \subset J$ is an ideal if and only if the Grassmann envelope $G(I)$ is an ideal of $G(J)$ the Ideal Criterion in [**16**, 1.5] provides a description of the ideals of J.

We will next consider a class of Jordan superpairs which are superextensions of Jordan pairs in the sense of 1.11. This class covers 4 types of standard examples, see 3.7 below.

3.6. Theorem. *Suppose A is a superextension of S and $V = A \otimes_k U$ is the A-extension of a Jordan pair U over k which is split of type $\mathcal{G}$, i.e., $U^\sigma = \bigoplus_{\alpha \in R_1} k \cdot g_\alpha^\sigma$ for $\sigma = \pm$, where $\mathcal{G} = \{g_\alpha : \alpha \in R_1\}$ is a connected ortho-collinear standard grid and A is the McCrimmon-Meyberg superalgebra algebra of some collinear pair $g_\alpha, g_\beta \in \mathcal{G}$.*

(a) *The ideals of V are precisely the spaces $B \otimes_k U$ where B is an ideal of the superextension A of S.*

(b) *V is semiprime, prime or simple if and only if A is respectively semiprime, prime and simple. In particular:*

- (i) *If $\frac{1}{2} \in S$ then V is semiprime or prime if and only if $V_{\bar{1}} = 0$ and $V = V_{\bar{0}} = A_{\bar{0}} \otimes_k U$ where $A_{\bar{0}}$ is, respectively, a semiprime or prime extension.*
- (ii) *V is simple if and only if either $A = A_{\bar{0}}$ is a field (and hence $V = V_{\bar{0}}$ is a simple Jordan pair) or $A_{\bar{0}}$ is a field of characteristic 2 and $V = \mathbb{D}(A_{\bar{0}} \otimes_k U)$.*

Proof. (a) Let $\otimes = \otimes_k$ throughout. It is easily checked that $B \otimes U$ is an ideal of V for any ideal B of A. Conversely, by 3.3, any ideal I of V has the form $I^\sigma = \bigoplus_{\alpha \in R_1} I_\alpha^\sigma$ where each I_α^σ is an S-submodule of $V_\alpha^\sigma = A \otimes k g_\alpha^\sigma$ and can therefore be written in the form $I_\alpha^\sigma = B_\alpha^\sigma \otimes k g_\alpha^\sigma$ where $(B_\alpha^\sigma : \sigma = \pm, \alpha \in R_1)$ is a family of S-submodules of A. Since $Q_{\bar{0}}(1 \otimes g_\alpha^\sigma) Q_{\bar{0}}(1 \otimes g_\alpha^{-\sigma}) | V_2^\sigma(g_\alpha) = \mathrm{Id}$ we have $Q_{\bar{0}}(1 \otimes g_\alpha^\sigma) I_\alpha^{-\sigma} = I_\alpha^\sigma$ and hence $B^{-\sigma} \otimes k g_\alpha^\sigma = Q_{\bar{0}}(1 \otimes g_\alpha^\sigma)(B_\alpha^{-\sigma} \otimes k g_\alpha^{-\sigma}) = Q_{\bar{0}}(1 \otimes g_\alpha^\sigma) I_\alpha^{-\sigma} = I^\sigma \cap V_\alpha^\sigma = B_\alpha^\sigma \otimes k g_\alpha^\sigma$. We can therefore assume $B_\alpha^+ = B_\alpha^- =: B_\alpha$ for all $\alpha \in R_1$.

Suppose $\alpha \top \beta$. Since the exchange automorphism $t_{\alpha\beta} = t_{g_\alpha, g_\beta}$ acts on V_α^+ as the endomorphism $D(g_\beta^+, g_\alpha^-) | V_\alpha^+$ it maps I_α^+ onto I_β^+ which implies $B_\alpha \otimes k g_\beta^+ = t_{\alpha\beta}(B_\alpha \otimes k g_\alpha^+) = B_\beta \otimes k g_\beta^+$. Because $\mathcal{G}$ is connected, i.e., (R, R_1) is irreducible, any two distinct roots $\alpha, \beta \in R_1$ are connected by a chain of collinear roots, in fact either $\alpha \top \beta$ or $\alpha \perp \beta$ in which case there exists $\gamma \in R_1$ such that $\alpha \top \gamma \top \beta$. Therefore $B_\beta \otimes k g_\beta^+ = B_\alpha \otimes k g_\beta^+$ and we can assume that $B_\alpha =: B$ is independent of α and so $I = B \otimes U$. Since for some collinear pair α, β, A is the McCrimmon-Meyberg superalgebra algebra it follows from 3.3(b) that B is an ideal of A.

(b) The simplicity criterion is immediate from (a). The proof for semiprimeness and primeness can be done at the same time. Let B and C be ideals of A with $B = C$ in the case of semiprimeness. First suppose V is semiprime or prime and that $BC = 0$. It then follows from the definition of the product in V that the ideals $B \otimes U$ and $C \otimes U$ of V satisfy $(B \otimes U) \diamond (C \otimes U) = 0$. By (semi)primeness, $B \otimes U = 0$ or $C \otimes U = 0$ and consequently B or C is zero. Conversely, suppose that A is prime and that $(B \otimes U) \diamond (C \otimes U) = 0$. Let g_α and g_β in $\mathcal{G}$ be two collinear idempotents defining A. For all $b, \tilde{b} \in B$ and all $c \in C$, we have that $b \otimes g_\alpha^+, \tilde{b} \otimes g_\beta^+ \in B \otimes U^+$ and $c \otimes g_\alpha^- \in C \otimes U^-$, so $0 = \{b \otimes g_\alpha^+ \,, c \otimes g_\alpha^- \,, \tilde{b} \otimes g_\beta^+\} = (bc\tilde{b}) \otimes g_\beta^+$, and $bc\tilde{b} = 0$ follows. Thus $BCB = 0$ and since A is prime, B or C are zero. The statements (i) and (ii) follow from 2.6 and 3.2.2.

3.7. Example: Jordan superpairs covered by an even quadratic form grid, an alternating grid, a Bi-Cayley grid or an Albert grid. In all these cases the assumptions of Theorem 3.6 are satisfied. Hence we know:

(1) The ideals of an alternating matrix superpair $\mathbb{A}_I(A)$, $|I| \geq 4$, are exactly the subpairs $\mathbb{A}_I(B)$ where B is an ideal of A.

(2) The ideals of an even quadratic form superpair $\mathbb{EQ}_I(A)$, $|I| \geq 3$, are exactly the subpairs of the form $\mathbb{EQ}_I(B)$ where B is an ideal of A.

(3) The ideals of a Bi-Cayley superpair $\mathbb{B}(A)$ are exactly the subpairs of the form $\mathbb{B}(B)$, where B is an ideal of A.

(4) The ideals of an Albert superpair $\mathbb{AB}(A)$ are exactly the subpairs $\mathbb{AB}(B)$ where B is an ideal of A.

Moreover, in these cases semiprimeness, primeness and simplicity is determined by 3.6(b) and 2.6. In particular, *if* $\frac{1}{2} \in A$ *a semiprime Jordan superpair* V *covered by an even quadratic form grid, an alternating grid, a Bi-Cayley grid or an Albert grid necessarily has* $V_{\bar{1}} = 0$ *and hence is a Jordan pair.* For prime Jordan superpairs this can be considered a (weak) analogue of 2.2.

In the following we can restrict our attention to the case of rectangular, hermitian and odd quadratic form grids.

3.8. Proposition. *The ideals of a rectangular matrix superpair* $\mathbb{M}_{JK}(A)$, $|J| + |K| \geq 3$, *are exactly the spaces* $\mathbb{M}_{JK}(B)$, *where* B *is an ideal of* A. *Moreover,* $\mathbb{M}_{JK}(A)$ *is semiprime, prime or simple if and only if* A *is respectively semiprime, prime and simple. In particular:*

(i) *A prime rectangular matrix superpair over a field of characteristic* $\neq 2, 3$ *either has a prime associative coordinate superalgebra or is of type* $\mathbb{M}_{12}(A)$ *for* $A = A_{\bar{0}}$ *a prime Cayley-Dickson algebra and hence has* $\mathbb{M}_{12}(A)_{\bar{1}} = 0$.

(ii) $\mathbb{M}_{JK}(A)$ *is simple if and only if either* A *is simple as algebra or* $\mathbb{M}_{JK}(A) = \mathbb{D}(\mathbb{M}_{JK}(A_{\bar{0}}))$ *and* $A_{\bar{0}}$ *is simple.*

Proof. This proof is parallel to the proof of 3.6. Nevertheless, for the convenience of a reader uninitiated to the techniques used there, we include the details. Let $I = (I^+, I^-)$ be an ideal of $V = \mathbb{M}_{IJ}(A)$. By 3.3 we then have $I^\sigma = \oplus_{jk} I^\sigma_{jk}$ where $I^\sigma_{jk} = I^\sigma \cap V_2^\sigma(e_{jk}) =$

$B^\sigma_{jk} e^\sigma_{jk}$ for graded subspaces B^σ_{ij} of A and $\mathcal{R} = \{e_{jk} : j \in J, k \in K\}$ the usual rectangular grid. If $be^\sigma_{jk} \in I^\sigma$, then:

$$\begin{aligned} \{e^\sigma_{lk}\,, e^{-\sigma}_{jk}\,, be^\sigma_{jk}\} &= be^\sigma_{lk} \in I^\sigma, \text{ for } l \neq j, \\ \{e^\sigma_{jm}\,, e^{-\sigma}_{jk}\,, be^\sigma_{jk}\} &= be^\sigma_{jm} \in I^+, \text{ for } m \neq k, \text{ and} \\ Q_{\bar{0}}(e^\sigma_{jk})(be^{-\sigma}_{jk}) &= be^\sigma_{jk} \in I^{-\sigma}. \end{aligned}$$

These formulas imply that $B^+_{jk} = B^-_{jk}$ is independent of i, j and that

$$I = \mathbb{M}_{JK}(B) = (\mathrm{Mat}(J, K; B), \mathrm{Mat}(K, J; B))$$

for some submodule B of A. By 3.3(b), B is an ideal of A. Conversely, if B is an ideal of A, it is immediate to see that $\mathbb{M}_{JK}(B)$ is an ideal of $\mathbb{M}_{JK}(A)$. The remaining part is proven in the same way as 3.6(b). In the last part, one obtains $B(CB) = 0$ and one can then use [**33**, 5.5 Prop.1] to conclude that $B = 0$ or $C = 0$. The statement (i) follows from 2.2, (ii) is a consequence of 2.5 and 3.2.3.

3.9. More examples of Jordan superpairs. The description of the rectangular matrix superpairs $\mathbb{M}_{JK}(A)$ for a simple Artinian A is best done in the following more general set-up.

Let E be an associative superalgebra over S. A *3-grading of E* is a $\mathbb{Z}$-grading of the form $E = E_1 \oplus E_0 \oplus E_{-1}$. Thus, $E_i E_j \subset E_{i+j}$ with the understanding that $E_{i+j} = 0$ if $i + j \notin \{\pm 1, 0\}$. Note that the E_i are assumed to be S-submodules, hence respecting the $\mathbb{Z}_2$-grading of E. If $E = E_1 \oplus E_0 \oplus E_{-1}$ is a 3-grading one easily sees that (E_1, E_{-1}) is a subpair of the special Jordan superpair (E, E) and hence in particular a Jordan superpair.

A natural example of a 3-graded associative superalgebra $E = \mathrm{End}_A(\mathcal{M} \oplus \mathcal{P})$ where A is an associative superalgebra and $\mathcal{M}$ and $\mathcal{P}$ are two $\mathbb{Z}_2$-graded A-bimodules. Here E has a 3-grading given by $E_1 = \mathrm{Hom}_A(\mathcal{P}, \mathcal{M})$, $E_0 = \mathrm{End}_A(\mathcal{M}, \mathcal{M}) \oplus \mathrm{End}_A(\mathcal{P}, \mathcal{P})$ and $E_{-1} = \mathrm{Hom}_A(\mathcal{M}, \mathcal{P})$. In particular, if $\mathcal{M}$ and $\mathcal{P}$ are free the E_i can be identified with matrices over A, 2.3.2, and we obtain the Jordan superpair

$$\mathbb{M}_{M|N,P|Q}(A) = (\mathrm{Mat}(M|N, P|Q; A), \mathrm{Mat}(P|Q, M|N; A)) \tag{1}$$

for which the homogenous parts of $\mathbb{M}_{M|N,P|Q}(A)^+$ are given by 2.3.1. For $A = A_{\bar{0}} = k$ a field of characteristic 0 this type of Jordan superpairs is one of the examples in [**9**]. It is straightforward, using the techniques of the proof of 3.8, to show that the ideals of $\mathbb{M}_{M|N,P|Q}(A)$ are given by $\mathbb{M}_{M|N,P|Q}(B)$ where B is an ideal of A. Hence, $\mathbb{M}_{M|N,P|Q}(A)$ *is semiprime, prime or simple if and only if A is so.*

Note that the rectangular matrix superpairs with associative coordinate algebras A are special cases of this construction: $\mathbb{M}_{JK}(A) = \mathbb{M}_{J|\emptyset,K|\emptyset}(A)$. But these are not the only examples of $\mathbb{M}_{M|N,P|Q}(A)$ covered by a rectangular grid. Indeed, for the associative superalgebra $\mathrm{Mat}_{P|Q}(A)$ of 2.3.3 as coordinate algebra we obtain

$$\mathbb{M}_{JK}(\mathrm{Mat}_{P|Q}(A)) \cong \mathbb{M}_{J\times P|J\times Q, K\times P|K\times Q}(A). \tag{2}$$

Note that by 3.8 and 2.7 this in particular describes the structure of the simple rectangular matrix superpairs $\mathbb{M}_{JK}(A)$ with a simple Artinian associative A. For example it follows from 2.5 and 3.2.3 that the finite-dimensional simple rectangular matrix superpairs over an algebraically closed field k are isomorphic to

(i) $\mathbb{M}_{JK}(\mathbb{D}(k)) = \mathbb{D}(\mathbb{M}_{JK}(k))$ as in 3.2.3 for finite sets J, K, or to
(ii) $\mathbb{M}_{JK}(\mathrm{Mat}_{p|q}(k))$ as in (2).

3.10. Proposition. *Let $V = \mathbb{H}_I(A, A^0, \pi)$ be a hermitian matrix superpair with $|I| \geq 3$.*

(a) *The ideals of V are exactly the $\mathbb{H}_I(B, B^0, \pi) = (\mathrm{H}_I(B, B^0, \pi), \mathrm{H}_I(B, B^0, \pi))$, defined as in* 1.8.3, *where B is a π-invariant ideal of A and $B^0 \subset A^0 \cap B$ satisfies the conditions*

(1) $b_{\bar{0}} a^0 b_{\bar{0}}^{\pi} \in B^0$ *if* $b_{\bar{0}} \in B_{\bar{0}}$ *and* $a^0 \in A^0$,
(2) $a_{\bar{0}} b^0 a_{\bar{0}}^{\pi} \in B^0$ *if* $b^0 \in B^0$ *and* $a_{\bar{0}} \in A_{\bar{0}}$, *and*
(3) $b + b^{\pi} \in B^0$ *for* $b \in B$.

In particular, if $\frac{1}{2} \in S$ then $B^0 = A^0 \cap B$.

(b) *V is semiprime or prime if and only if A is respectively π-semiprime or π-prime.*

(c) *If V is simple, then A is a π-simple superalgebra. Conversely, if A is π-simple and A^0 is the span of all traces and norms, i.e., $A^0 = A^0_{\min}$ as defined in* [**21**, 4.10], *then V is simple.*

In particular, if $\frac{1}{2} \in A$ then V is simple if and only if A is π-simple, and if $S = k$ is a field of characteristic $\neq 2, 3$ then there are exactly the following possibilities for a simple V:

(i) *$V \cong \mathbb{M}_{II}(B)$ for a simple associative superalgebra B,*
(ii) *A is a simple associative superalgebra, or*
(iii) *$A = A_{\bar{0}}$ is a simple Cayley-Dickson algebra and hence $V = V_{\bar{0}}$.*

Proof. (a) can be proven by taking Grassmann envelopes and applying [**22**, IV, Lemma 1.6] to the polarized Jordan triples system associated to $G(\mathbb{H}_I(A, A^0, \pi))$. For the convenience of the reader we include a direct proof.

Let $L = (L^+, L^-)$ an ideal of $V = \mathbb{H}_I(A, A^0, \pi)$. By 3.3 we have $L^\sigma = \oplus_{ij} L^\sigma_{ij}$ where $L^\sigma_{ij} = L^\sigma \cap A[ij]$ for $i \neq j$, $L^\sigma_{ii} = L^\sigma \cap A^0[ii]$ and, as usual, $\sigma = \pm$. We will first show that the off-diagonal L^σ_{ij} are independent of i, j. This follows from the following two formulas where $bh^\sigma_{ij} \in L^\sigma$, $i \neq j$:

$$\{bh^\sigma_{ij}\,,\, h^{-\sigma}_{ji}\,,\, h^\sigma_{ik}\} = bh^\sigma_{ik} \in L^\sigma \text{ for all } k \neq i, \tag{1}$$
$$\{bh^\sigma_{ij}\,,\, h^{-\sigma}_{ij}\,,\, h^\sigma_{jk}\} = b^\pi h^\sigma_{jk} = bh^\sigma_{kj} \in L^\sigma \text{ for all } k \neq j. \tag{2}$$

Next, we claim $L^+_{ij} = L^-_{ij}$. Indeed, if $bh^\sigma_{ij} \in L^\sigma$ then $Q_{\bar{0}}(h^{-\sigma}_{ij}) bh^\sigma_{ij} = b^\pi h^{-\sigma}_{ij} \in L^{-\sigma}$. For $k \neq i, j$ we then obtain $bh^{-\sigma}_{ki} = b^\pi h^{-\sigma}_{ik} \in L^{-\sigma}$ by (1). We then apply again (1) to get that $bh^{-\sigma}_{kj} \in L^{-\sigma}$ and finally $bh^{-\sigma}_{ij} \in L^{-\sigma}$ by (2).

Similarly, we have $L^\sigma_{ii} = L^\tau_{jj}$ for all $i, j \in I$ and $\sigma, \tau = \pm$. Indeed, if $b_0 h^\sigma_{ii} \in L^\sigma$, then $Q_{\bar{0}}(h^{-\sigma}_{ji}) b_0 h^\sigma_{ii} = b_0[jj] = b_0 h^{-\sigma}_{jj} \in L^{-\sigma}$ for all $j \in I$, and applying $Q_{\bar{0}}(h^\sigma_{jj})$ again, we also have that $b_0 h^\sigma_{jj} \in L^\sigma$ for all $j \in I$.

We have now shown that $L = \mathbb{H}_I(B, B^0, \pi)$ for certain subspaces $B \subset A$ and $B^0 \subset A^0$. In fact, since $\{b_0[ii], [ij], [jj]\} = b_0[ij]$ for $i \neq j$ we also have $B^0 \subset B$. Since A is the McCrimmon-Meyberg superalgebra algebra for some collinear pair (h_{ij}, h_{ik}) it follows from 3.3(b) that B is a π-invariant ideal of A. The conditions (1) and (2) then follow from $Q_{\bar{0}}(a_{\bar{0}}[ij])b^0[jj] = a_{\bar{0}}b^0a_{\bar{0}}^\pi[ii]$ while (3) is a consequence of $\{a[ij], b[jk], c[ki]\} = (a(bc) + (-1)^{|a||b|+|a||c|+|b||c|}(c^\pi b^\pi)a^\pi)[ii]$.

Conversely, it is easy to check that $\mathbb{H}_I(B, B^0, \pi)$ is an ideal of $\mathbb{H}_I(A, A^0, \pi)$ whenever B and B^0 satisfy the conditions above. If $\frac{1}{2} \in A$ then $A^0 = \mathrm{H}(A, \pi) \cap \mathrm{N}(A)$ and $\mathrm{H}(B, \pi) \subset B^0$ by (3), proving $B^0 = A^0 \cap B$.

(b) For any π-invariant ideal B of A, $B^0 = B \cap A^0$ satisfies the conditions (1)–(3) of (a), hence gives rise to an ideal of V. We can then argue as in the proof of 3.6(b) above to establish (b).

(c) It is immediate that V simple implies A π-simple. For the converse, suppose we have a non-zero ideal L of V given by B and B^0 as above. Then $0 \neq B$ and hence $B = A$ by π-simplicity. Moreover, B^0 must contain at least all traces and norms of A. Thus $A^0 = B^0$ and $L = V$ follows. If $\frac{1}{2} \in A$ then A^0 is indeed the span of all traces and norms, [**21**, 4.10]. The last statement follows from 2.2 and 2.9. Indeed, in case (i) of 2.9 we have case (i) above.

Remark. Our simplicity result provides a partial generalization of results in [**3**, §6]. It easily follows from the results proven there that $\mathrm{H}_I(A, A^0, \pi)$ is a simple Jordan algebra and hence $\mathbb{H}_I(A, A^0, \pi)$ is a simple Jordan pair whenever $|I| \geq 2$ and A is π-simple and associative over a ring k containing $\frac{1}{2}$.

3.11. Examples of simple hermitian matrix superpairs. We will describe simple superpairs $\mathbb{H}_I(A, A^0, \pi)$ for a simple associative superalgebra A. Since $\mathbb{H}_I(A, A^0, \pi) = (\mathrm{H}_I(A, A^0, \pi), \mathrm{H}_I(A, A^0, \pi))$ is the Jordan superpair associated to the simple (by 3.4.1) Jordan superalgebra $\mathrm{H}_I(A, A^0, \pi)$ it suffices to describe the latter. Recall from 2.10 that there are three possibilities for a simple A with involution, namely cases (ii), (iii) and (iv). We will consider (ii) and (iii) only.

(a) Suppose $A = \mathbb{D}(B)$ for a simple associative algebra B. Since $Z(B)$ is a field, there are two cases here.

(a.1) $Z(B)$ has characteristic $\neq 2$: Then $A^0 = \mathrm{H}(A, \pi) = B^0 \oplus (1+z)uB^0$ for $B^0 = \mathrm{H}(B, \pi)$ by 2.4.6. Using 2.4.7 we obtain

$$\mathrm{H}_I(\mathbb{D}(B), \mathbb{D}(B)^0, \pi) = \mathrm{H}_I(B, B^0, \pi) \oplus u(1+z)\mathrm{H}_I(B, B^0, \pi).$$

Thus, the corresponding matrix pair $\mathbb{H}_I(A, A^0, \pi)$ is a (slightly generalized) double of the hermitian matrix pair $\mathbb{H}_I(B, B^0, \pi)$. Note that this case does not occur if $S = k$ is an algebraically closed field.

(a.2) $Z(B)$ has characteristic 2. Then $(b_1 \oplus b_2u)^\pi = b_1^\pi + ub_2^\pi$ for $b_i \in B$ by 2.4, hence $\mathrm{H}(A, \pi) = \mathrm{H}(B, \pi) \oplus u\mathrm{H}(B, \pi)$. Also the ample subspace $A^0 = B^0 \oplus uC^0$ where B^0 is an ample subspace of B, C^0 contains all traces $b + b^\pi$ for $b \in B$ and $ac^0b^\pi + bc^0a^\pi \in B^0$ for $a, b \in B$ and $c^0 \in C^0$. We have

$$\mathrm{H}_I(\mathbb{D}(B), \mathbb{D}(B)_0, \pi) = \mathrm{H}_I(B, B^0, \pi) \oplus uH_I(B, C^0, \pi),$$

and so again $\mathbb{H}_I(A, A^0, \pi)$ is a (slightly generalized) double.

(b) In case (iii) of 2.10 we can write $A = (B_1 \oplus B_2) \oplus (C_1 \oplus C_2)$ as a Morita context

$$A = \begin{pmatrix} B_1 & C_1 \\ C_2 & B_2 \end{pmatrix} \quad \subset \quad \mathrm{Mat}_{1|1}(A).$$

We identify $\mathrm{Mat}(I, I; \mathrm{Mat}_{1|1}(A)) = \mathrm{Mat}_{1|1}(\mathrm{Mat}(I, I; A))$. Also, we extend the involution of A to $\mathrm{Mat}(I, I; A)$ by $(a_{ij})^\pi = (a_{ji}^\pi)$. There are again two cases here, depending on whether π exchanges the B_i or the C_i. For simplicity, we will abbreviate $\mathrm{H}_I(A, \pi) = \mathrm{H}_I(A, \mathrm{H}(A, \pi), \pi)$.

(b.1) Suppose $B_1^\pi = B_2$ and $C_i^\pi = C_i$ for $i = 1, 2$. We let $C_{i,\min} = \{c_i + c_i^\pi : c_i \in C_i\}$ denote the set of all traces. Recall ([**21**, 4.10]) that the ample subspace A^0 of (A, π) contains $A^0_{\min}$, the span of all traces $a + a^\pi$ and norms aa^π for $a \in A$, and is itself contained in $A^0_{\max} = \mathrm{H}(A, \pi)$. Hence in our situation

$$\begin{aligned} A^0_{\min} &= \left\{ \begin{pmatrix} b_1 & c_1 \\ c_2 & b_1^\pi \end{pmatrix} : b_1 \in B_1, c_i \in C_{i,\min} \right\} \subset A^0 \\ A^0 \subset A^0_{\max} &= \left\{ \begin{pmatrix} b_1 & c_1 \\ c_2 & b_1^\pi \end{pmatrix} : b_1 \in B_1, c_i \in \mathrm{H}(C_i, \pi|C_i) \right\}. \end{aligned}$$

Therefore A^0 can be written as

$$A^0 = \left\{ \begin{pmatrix} b_1 & c_1 \\ c_2 & b_1^\pi \end{pmatrix} : b_1 \in B_1, c_i \in C_i^0, i = 1, 2 \right\}$$

for submodules C_i^0 with $C_{i,\min} \subset C_i^0 \subset H(C_i, \pi|C_i)$. It follows that

$$\begin{pmatrix} b_1 & c_1 \\ c_2 & b_2 \end{pmatrix} \in \mathrm{H}_I(A, A^0, \pi) \iff \begin{cases} b_i = b_j^\pi \in \mathrm{Mat}(I, I; B_i) & \text{for } \{i, j\} = \{1, 2\} \\ c_i \in \mathrm{H}_I(C_i, C_i^0, \pi|C_i) & \text{for } i = 1, 2. \end{cases}$$

In particular, for $A = \mathrm{Mat}_{Q|Q}(B)$ and B an associative algebra, we have the *supertranspose involution*

$$A \ni \begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \stackrel{\tau}{\longmapsto} \quad \begin{pmatrix} z^{\mathrm{T}} & -x^{\mathrm{T}} \\ y^{\mathrm{T}} & w^{\mathrm{T}} \end{pmatrix} \in A, \tag{1}$$

where z^{T} is the usual transpose of z ([**26**, Prop.13]). As fixed point set of an involution,

$$\begin{aligned} \mathrm{H}(\mathrm{Mat}_{Q|Q}(B), \tau) &= \left\{ \begin{pmatrix} a & b \\ c & a^{\mathrm{T}} \end{pmatrix} \in \mathrm{Mat}_{Q|Q}(B) : b \text{ skew}, c \text{ symmetric} \right\} \\ &=: \mathrm{P}_Q(B) \end{aligned} \tag{2}$$

is a Jordan superalgebra. We have a canonical isomorphism

$$\mathrm{H}_I(\mathrm{Mat}_{Q|Q}(B), \tau) \cong \mathrm{P}_{I \times Q}(B). \tag{3}$$

By 3.10(c) above and 3.4.1 we know that $\mathrm{H}_I(\mathrm{Mat}_{Q|Q}(B)$ is a simple Jordan superalgebra if $|I| \geq 3$ and B is simple and defined over k containing $\frac{1}{2}$. Because of (3), this can also be deduced from [**3**, §6]. For finite Q and $B = k$ an algebraically closed field of characteristic $\neq 2$ the simple Jordan superalgebra $\mathrm{P}_Q(k)$ appears (of course) also in [**27**]. We note that if one replaces the involution $z \mapsto z^{\mathrm{T}}$ above by a symplectic involution, assuming that Q has finite even order, one gets a Jordan superalgebra isomorphic to $P_n(k)$, [**27**]. Indeed, if s is the block diagonal matrix inducing the symplectic involution $d \mapsto sd^{\mathrm{T}}s^{\mathrm{T}}$, i.e.,

$$s = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots \right), \tag{4}$$

and we change τ to the conjugate involution

$$A \ni \begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \xmapsto{\hat{\tau}} \quad \begin{pmatrix} sz^{\mathrm{T}}s^{\mathrm{T}} & sx^{\mathrm{T}}s^{\mathrm{T}} \\ -sy^{\mathrm{T}}s^{\mathrm{T}} & sw^{\mathrm{T}}s^{\mathrm{T}} \end{pmatrix} \in A,$$

the fixed point set with respect to $\hat{\tau}$ is

$$\mathrm{H}(\mathrm{Mat}_{Q|Q}(k), \hat{\tau})) = \left\{ \begin{pmatrix} a & b \\ c & sa^{\mathrm{T}}s^{\mathrm{T}} \end{pmatrix} \in \mathrm{Mat}_{Q|Q}(k) : b = sb^{\mathrm{T}}s^{\mathrm{T}}, c = -sc^{\mathrm{T}}s^{\mathrm{T}} \right\},$$

and an isomorphism between $\mathrm{H}(\mathrm{Mat}_{Q|Q}(k), \hat{\tau})$ and $P_{2Q}(k)$ is given by

$$\begin{array}{ccc} \mathrm{H}(\mathrm{Mat}_{Q|Q}(k), \hat{\tau}) & \to & P_{2Q}(k) \\ \begin{pmatrix} a & b \\ c & sa^{\mathrm{T}}s^{\mathrm{T}} \end{pmatrix} & \mapsto & \begin{pmatrix} a & bs \\ s^{\mathrm{T}}c & a^{\mathrm{T}} \end{pmatrix}. \end{array}$$

(b.2) Suppose $B_i^\pi = B_i$ and $C_1^\pi = C_2$. Then

$$A^0 = \left\{ \begin{pmatrix} b_1 & c_1 \\ c_1^\pi & b_2 \end{pmatrix} : b_i \in B_i^0, c_1 \in C_1 \right\}$$

where B_i^0 are ample subspaces of $(B_i, \pi|A_i)$ and

$$\begin{pmatrix} b_1 & c_1 \\ c_2 & b_2 \end{pmatrix} \in \mathrm{H}_I(A, A^0, \pi) \iff \begin{cases} b_i \in \mathrm{H}_I(B_i, B_i^0, \pi|B_i) & \text{for } i = 1, 2, \\ c_i = c_j^\pi \in \mathrm{Mat}(I, I; C_i) & \text{for } \{i, j\} = \{1, 2\}\ . \end{cases}$$

In particular, for $A = \mathrm{Mat}_{P|Q}(B)$ and B an associative algebra, an involution of this type is the so-called *orthosymplectic involution*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\sigma} \begin{pmatrix} a^{\mathrm{T}} & c^{\mathrm{T}}s^{\mathrm{T}} \\ sb^{\mathrm{T}} & sd^{\mathrm{T}}s^{\mathrm{T}} \end{pmatrix}, \tag{5}$$

where s is the matrix (4) and Q is either of finite even order or infinite. The fixed point superalgebra

$$\begin{aligned} \mathrm{H}(\mathrm{Mat}_{P|Q}(B), \sigma) &= \left\{ \begin{pmatrix} a & b \\ sb^{\mathrm{T}} & d \end{pmatrix} \in \mathrm{Mat}_{P|Q}(B) : a = a^{\mathrm{T}}, d = sd^{\mathrm{T}}s^{\mathrm{T}} \right\} \\ &=: \mathrm{OSP}_{P|Q}(B) \end{aligned} \tag{6}$$

is called the *orthosymplectic (Jordan) superalgebra* and denoted BC in [**5**]. We have

$$\mathrm{H}_I(\mathrm{Mat}_{P|Q}(B)) \cong \mathrm{OSP}_{I\times P|I\times Q}(B) \tag{7}$$

which is a simple Jordan superalgebra if $|I| \geq 3$ and B is simple over a ring k containing $\frac{1}{2}$.

For finite non-empty P, Q and B a division algebra, the involutions of $A = \mathrm{Mat}_{P|Q}(B)$ leaving the two ideals of $A_{\bar{0}}$ invariant are described in [**26**, Prop. 14]. In particular, if $B = k$ is an algebraically closed field then all such involutions are isomorphic to the orthosymplectic involution σ.

It remains to consider the standard example of an odd quadratic form superpair. To do so, we will need the following definition.

3.12. Quadratic form superpairs. Let A be a superextension of S. An A-quadratic form $q = (q_{\bar{0}}, b) : X \to A$, as defined in [**21**, 1.9], is called *nondegenerate* if for $x_\mu \in X_\mu$, $\mu = \bar{0}, \bar{1}$ we have

$$q_{\bar{0}}(A_\mu x_\mu) = 0 = b(x_\mu, X) \quad \Longrightarrow \quad x_\mu = 0.$$

Of course, $q_{\bar{0}}(A_{\bar{0}} x_{\bar{0}}) = 0 \iff q_{\bar{0}}(x_{\bar{0}}) = 0$. We note that the orthogonal sum of nondegenerate forms is again nondegenerate and that any hyperbolic form ([**21**, 4.13]) is nondegenerate.

Let $V = (X, X)$ be the corresponding quadratic form superpair, [**21**, 2.9]. Then

$$V \textit{ semiprime} \quad \Longrightarrow \quad q \textit{ nondegenerate.} \tag{1}$$

Indeed, if $x_\mu \in X_\mu$ for $\mu = \bar{0}, \bar{1}$ is such that $q_{\bar{0}}(A_\mu x_\mu) = 0 = b(x_\mu, X)$ then (Ax_μ, Ax_μ) is an ideal of V whose Jordan cube is zero whence $x_\mu = 0$.

3.13. Proposition. *Let $V = \mathbb{OQ}_I(A, q_X)$ be an odd quadratic form superpair with $|I| \geq 2$.*

(a) *The ideals of V are exactly the subpairs of the form $Y \oplus \mathbb{EQ}_I(B)$, where B is an ideal of A and Y is an A-submodule of X satisfying*

(i) $q_X(Y_{\bar{0}}) \subset B_{\bar{0}}$,
(ii) $b_X(Y, X) \subset B$, *where $b(\ ,\ )$ is the polar of q_X, and*
(iii) $BX \subset Y$.

(b) *If V is semiprime then q_X is nondegenerate, equivalently, $q_I \oplus q_X$ is nondegenerate.*

(c) *Suppose q_X is nondegenerate. If A is semiprime, prime or simple then V is respectively semiprime, prime or simple.*

(d) *Suppose $\frac{1}{2} \in A$ or $A = A_{\bar{0}}$. Then V is semiprime, prime or simple if and only if q_X is nondegenerate and A is respectively semiprime, prime or a field.*

Proof. This follows by considering the Grassmann envelope and applying [**22**, IV, Lemma 1.7]. A direct proof goes as follows. Let $L = (L^+, L^-)$ be an ideal of $\mathbb{OQ}_I(A, q_X)$. By 3.3 we have that $L^\sigma = (X \cap L^\sigma) \oplus (\oplus_{\pm i} L^\sigma_{\pm i})$ where $L^\sigma_{\pm i} = L^\sigma \cap V_2^\sigma(g_{\pm i})$ for $\sigma = \pm$.

In particular, $\bigoplus_{\pm i} L_{\pm i}$ is an ideal of the subpair $\bigoplus_{\pm i} V_2(g_{\pm i}) = \mathbb{EQ}_I(A)$ which, by 3.7, is therefore of the form $\mathbb{EQ}_I(B)$ for an ideal B of A. Moreover, if $x \in X \cap L^\sigma$ then $\{g_{+i}^{-\sigma}, x, g_{-i}^{-\sigma}\} = x \in L^{-\sigma}$, i.e., $X \cap L^+ = X \cap L^-$. Therefore $L = (Y, Y) \oplus \mathbb{EQ}_I(B)$ for a certain S-subspace $Y = X \cap L^\sigma$. Now, we just need to check the properties satisfied by B and Y:

- Y is an A-submodule of X: For $x \in Y$ we know that $x \in L^+$, so for $a \in A$ we get $\{ag_{+i}^+, g_{+i}^-, x\} = ax \in L^+$, i.e., $ax \in Y$.
- $q_X(Y_{\bar{0}}) \subseteq B$: If $x \in Y_{\bar{0}}$, then $x \in L^+$, so $Q_{\bar{0}}(x) g_{+i}^- = q_X(x) g_{-i}^+ \in L^+$, i.e., $q_X(x) \in B_{\bar{0}}$.
- $b_X(Y, X) \subseteq B$: Take $x \in Y$ and $y \in X$. Then $x \in L^+$ and $\{x, y, g_{+i}^+\} = b_X(x, y) h_{+i} \in L^+$, so $b_X(x, y) \in B$.
- $BX \subseteq Y$: Take $b \in B$ (so $bg_{+i}^+ \in L^+$) and $x \in X$. Then $\{bg_{+i}^+ \, g_{+i}^- \, x\} = bx \in L^+$, so $bx \in Y$.

Conversely, it is straightforward to check that any pair B, Y satisfying the conditions above gives rise to an ideal of $\mathbb{OQ}_I(A, q_X)$.

(b) is immediate from 3.12. For the proof of (c) let us first observe that any non-zero ideal of V, given by Y, B as in (a) above has $B \neq 0$. Indeed, if $B = 0$ nondegeneracy of q_X implies $Y = 0$. Now suppose we have two ideals of V, given by Y, B and Z, C respectively, such that their product vanishes. Then $BCB = 0$ follows. So one of B or C must be zero and the corresponding ideal of V is zero. This proves the statements concerning semiprimeness and primeness. For simplicity, consider a non-zero ideal U of V, given by Y, B. As we just observed, B is then a non-zero ideal of A whence $B = A$ and consequently $X = AX \subset Y$, proving $U = V$.

(d) Because of (b) and (c) we only have to prove that if V is semiprime, prime or simple then so is A. In general, for B an ideal of A, the ideal of V generated by $\mathbb{EQ}_I(B) \subset V$ is given by the data

$$\begin{aligned} B' &= B + q(B_{\bar{1}} x_{\bar{1}}) + A_{\bar{1}} q_X(B_{\bar{1}} X_{\bar{1}}) \subset A \quad \text{and} \\ Y &= BX + q_X(B_{\bar{1}} X_{\bar{1}}) X. \end{aligned}$$

Under our assumptions $B' = B$ and $Y = BX$. It is then straightforward to see that two ideals B, C of A generate ideals of V whose Jordan product is zero, whence B or C vanishes, and A is semiprime or prime if V is so. Also, if V is simple then so is A, and it follows from 2.6(c) that A is a field.

Examples. (a) If $A = A_{\bar{0}}$ we do not necessarily have $V = V_{\bar{0}}$. Rather, V is a quadratic form superpair of a quadratic form q defined on a superspace $M = M_{\bar{0}} \oplus M_{\bar{1}}$ which is an orthogonal sum of a quadratic form on $M_{\bar{0}} = \mathrm{H}_I(A) \oplus X_{\bar{0}}$ and an alternating form on $M_{\bar{1}} = X_{\bar{1}}$. In particular, by 2.6(a) we have $A = A_{\bar{0}}$ as soon as $\frac{1}{2} \in A$ and V is semiprime.

(b) Suppose A is simple. We then either have $A = A_{\bar{0}}$ or $A = \mathbb{D}(A_{\bar{0}})$ where $A_{\bar{0}}$ is a field of characteristic 2. In the latter case $\mathbb{OQ}(A, q_X) = \mathbb{D}(\mathbb{OQ}_I(A_{\bar{0}}, q_{\bar{0}}))$ is the double of an odd quadratic form pair by 3.2.4.

3.14. Concluding remarks. Let V be a standard example $V = \mathbb{V}(\mathcal{G}, \mathcal{C})$ with supercoordinate system $\mathcal{C}$, as introduced in section §1. The results obtained in this section show that within a fixed type, ideals are determined in a uniform way by data depending on $\mathcal{C}$ and not on $\mathcal{G}$.

Suppose $S = k$ is a field of characteristic $\neq 2, 3$ and V is simple. It is remarkable that then either $V = V_{\bar{0}}$ or $V = \mathbb{J} = (J, J)$ for a simple Jordan superalgebra or V is a subpair of a rectangular matrix superpair, and hence obviously special, or V is a quadratic form superpair and hence also special by McCrimmon's recent result [**15**]: a Jordan superpair is special if and only if its Grassmann envelope is a special Jordan pair over $G_{\bar{0}}$.

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