Full Diversity Unitary Space-Time Bruhat Constellations

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Abstract — **In this paper, we present a new design of constructing full diversity unitary constellations using differential space-time modulation for any number of transmitter and receiver antennas. These constellations are constructed from a Bruhat decomposition which allows us to choose disjoint non-overlapping cosets of a unitary diagonal subgroup in such a way that full diversity is preserved. Simulations show that our Bruhat constellations perform well in unknown Rayleigh fading channel.**

I. Introduction

Differential unitary space-time modulation [2, 3] was proposed for use with an unknown fading channel. Consider a multiple-antenna system in a Rayleigh fading channel. Let M be the number of transmitter antennas, and $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$ be a *signal constellation*, a set of $M \times M$ unitary matrices. The design problem of differential unitary space-time modulation is to maximize the minimum value of the distance

$$d(l, l') = |\det(V_l - V_{l'})| \geq 0, \tag{1}$$

for all distinct $V_l, V_{l'} \in \mathcal{V}$. A constellation $\mathcal{V}$ for which $d(l, l') > 0$ for all $l \neq l'$ is said to have *full diversity*. The *diversity product* (as defined in [2, 3, 7]) of a constellation $\mathcal{V}$ is calculated by

$$\zeta_{\mathcal{V}} = \frac{1}{2} \min_{l,l'} d(l, l')^{\frac{1}{M}}; \tag{2}$$

this term is used to compare different constellation designs. It has been shown that the finite simple unitary groups having full diversity are the fixed-point free groups [7]. It has also been proven that $U(1)$ and $SU(2)$ are the only two infinite full diversity unitary groups [1]. The application of the Bruhat decomposition to the design of a full diversity unitary subgroup constellations of $U(2)$ and $U(4)$ was first introduced in [5] using a method which was developed for 2 and 4 transmitter antennas respectively, but which is not applicable to an odd number of transmitter antennas.

In this paper, we present a new Bruhat decomposition design for constructing full diversity unitary space-time constellations for any number of transmitter and receiver antennas. A Bruhat constellation $\mathcal{V}$ is constructed from M cosets of a unitary diagonal subgroup $\mathcal{D}$, and our design has a particularly simple structure when M is prime. Although our Bruhat constellation is not a group, we show that it is extremely symmetric in that the minimum value of $d(l, l')$ can be determined by checking only $L-1$ distinct elements in $\mathcal{V}$ which is comparable to a group constellation. Simulations show that proposed Bruhat constellations perform well in unknown fading channel, and also outperform constellations obtained from cyclic groups.

II. Differential Unitary Space-Time Modulation

Consider a multiple-antenna system in a Rayleigh flat fading channel with M transmitter and N receiver antennas. The $M \times N$ received signal matrix X_τ is

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + W_\tau, \quad \tau = 0, 1, \ldots \tag{3}$$

where S_τ is the $M \times M$ transmitted signal matrix, W_τ is the $M \times N$ additive noise matrix and H_τ is the $M \times N$ channel matrix. The fading coefficients of a channel matrix and the additive noise on receiver antenna are independent complex Gaussian variables with mean zero and variance one, $\mathcal{CN}(0, 1)$. Here ρ is the SNR at each receiver antenna. Let V_{z_τ} be an $M \times M$ unitary message matrix at block τ which is chosen from an alphabet $z_\tau \in \{0, 1, \ldots, L-1\}$. The date rate is $R = \log_2 L/M$. In differential unitary space-time modulation, the transmitted signal matrix is

$$S_\tau = V_{z_\tau} S_{\tau-1} \tag{4}$$

with $S_0 = I_M$ where I_M is an $M \times M$ identity matrix. We assume that the channel matrix is constant over two consecutive time periods, that is $H_\tau \approx H_{\tau-1}$. Then the received signal matrix of (3) can be written as

$$X_\tau = V_{z_\tau} X_{\tau-1} + \sqrt{2} W'_\tau \tag{5}$$

where W'_τ is also independent $\mathcal{CN}(0, 1)$. The ML decoder is used for decode a message $\hat{z}_\tau$ to be

$$\hat{z}_\tau = \arg \min_{l=0,1,\ldots,L-1} \|X_\tau - V_l X_{\tau-1}\|. \tag{6}$$

III. Mathematical Preliminaries

Let S_M denote the *symmetric group* of *order* M, that is, the group consisting of all permutations of M elements. We use the notation (123) to denote the cycle sending $1 \to 2 \to 3 \to 1$; every element of S_M may be decomposed uniquely into a product of disjoint cycles. Of particular interest to us is the cycle $\omega = (12 \cdots M)$. For $i > 1$, the permutation ω^i need not to be a single cycle. Set $g = \gcd(i, M)$ and $b = M/g$. Then we may compose

$$\omega^i = \sigma_1 \ \sigma_2 \cdots \sigma_g \tag{7}$$

where σ_r is the cycle $(r, r \oplus i, r \oplus 2i, \cdots, r \oplus (b-1)i)$ where we write $r \oplus k$ for the result of adding k to r mod M.

The group S_M has a *natural representation* on $\mathbb{C}^M$, called the *permutation representation*, in which each element $\tau \in S_M$ is realized as the $M \times M$ matrix of the linear transformation which permutes the standard basis vectors e_1, e_2, ... e_M of $\mathbb{C}^M$ according to τ. For example, in S_3, the permutation $\tau = (123)$

sends e_1 to e_2, e_2 to e_3 and e_3 to e_1. Hence under the natural representation, we identify τ with the (unitary) matrix

$$\tau = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{8}$$

The symmetric group has a special relationship with the *general linear group* $GL(M)$ (that is, the group of all invertible $M \times M$ matrices) defined by the *Bruhat decomposition*. The Bruhat decomposition of $GL(M)$ is the disjoint union of double cosets

$$GL(M) = \bigcup_{\tau \in S_M} B\tau B \tag{9}$$

where B denotes the subgroup of all invertible upper triangular matrices, and τ is written in its natural representation. Whereas the group B is not unitary, it contains the subgroup T of diagonal unitary matrices. Restricting to T in equation (9) defines a unitary subgroup of $GL(M)$. A quick calculation reveals that the double coset $T\tau T$ is equal to the left coset τT for any $\tau \in S_M$. This defines an infinite unitary subgroup of $GL(M)$, $S_M \ltimes T$ (a semi-direct product) as:

$$S_M \ltimes T = \bigcup_{\tau \in S_M} T\tau T = \bigcup_{\tau \in S_M} \tau T. \tag{10}$$

This subgroup does not have full diversity, however, since T itself does not. Thus to obtain a full diversity constellation, we restrict to a full diversity subgroup of T and then construct our (nongroup) constellation from its cosets.

IV. Unitary Bruhat Constellation

In this section, we present the design of a full diversity unitary Bruhat constellation for any number of transmitter antennas using a Bruhat decomposition as given in equation (10).

A. Design Preliminaries

Let M be the number of transmitter antennas. We first construct a cyclic subgroup $\mathcal{D}$ of unitary diagonal matrices, whose cosets will form the constellation. We choose this subgroup, of order $|\mathcal{D}| = L_D$, subject to two constraints, **(C1) and (C2)**, given below. For each $l \in \{0, 1, \ldots, L_D - 1\}$, write the element $D_l \in \mathcal{D}$ as

$$D_l = \mathrm{diag}(d_1, d_2, \ldots, d_M). \tag{11}$$

Then each $d_k = e^{j2\pi u_k l/L_D}$, for some choice of $u_k \in \{1, 2, \ldots, L_D - 1\}$. For each $i \in \{1, 2, \ldots, M-1\}$, we require **(C1)**

$$d_r \cdot d_{r\oplus i} \cdot d_{r\oplus 2i} \cdots d_{r\oplus(b-1)i} = 1 \tag{12}$$

for each $r \in \{1, 2, \ldots, g = \gcd(i, M)\}$, where $b = M/g$ and $\oplus$ denotes addition modulo M. (This condition arises from the decomposition of ω^i into disjoint cycles, as given in (7), and is used in the proof of Theorem 1, below.) Further, choose $\mathcal{D}$ to be a subgroup with full diversity, that is, such that **(C2)**

$$d_{\mathcal{D}}(l, l') = \det(D_l - D_{l'}) > 0 \tag{13}$$

for all $0 \le l, l' < L_D$. In terms of the parameters u_k, the constraints (C1) and (C2) may be rephrased as the requirement that

$$u_r + u_{r\oplus i} + u_{r\oplus 2i} + \cdots + u_{r\oplus(b-1)i} \equiv 0 \mod L_D \tag{14}$$

for each $0 < i < M$ and for each $1 \le r \le g$, and that each u_k is relatively prime to L_D. This latter condition is equivalent to (C2): for suppose instead that $\gcd(u_k, L_D) = t > 1$ for some $1 \le k \le M$. Then $d_k^l = 1$ with $l = L_D/t < L_D$, which implies $\det(D_0 - D_l) = 0$, so that (13) fails. (One can also prove the converse; see [6] for details.) For example, when $M = 3$, we must choose $D_l = \mathrm{diag}(d_1, d_2, d_1^{-1}d_2^{-2}) = \mathrm{diag}(e^{j2\pi u_1 l/L_D}, e^{j2\pi u_2 l/L_D}, e^{-j2\pi(u_1+u_2)l/L_D})$. This will give a full diversity constellation only if u_1, u_2 and $u_1 + u_2$ are relatively prime to L_D. For example, we can choose L_D to be prime. In practice, one chooses the values of u_k, subject to (C1) and (C2), which maximize the minimum value of (13) over all $l \neq l'$, by exhaustive search.

B. Design of a Full Diversity Unitary Bruhat Constellation

We may now define the unitary Bruhat constellation. Set $\lambda = e^{2\pi j/M^2}$, and let ω be an $M \times M$ matrix representing the full permutation $(123 \cdots M)$ in the natural representation. This cycle has the property that ω^i has no fixed points for any $i \in \{1, 2, \ldots, M-1\}$. Consequently, the positions of the nonzero entries in ω^i, or equivalently, in $\omega^i\mathcal{D}$, are entirely disjoint from those for ω^j for any $i \neq j$. Our proposed unitary Bruhat constellation $\mathcal{V}$ is a subset of $S_n \ltimes T$, given by the disjoint union of sets of unitary matrices defined by

$$\mathcal{V} = \bigcup_{i=0}^{M-1} \lambda^i \omega^i \mathcal{D}. \tag{15}$$

Since equation (12), with $i = 1$, implies that $\det(D) = 1$ for all $D \in \mathcal{D}$, it follows that

$$\det(\lambda^k \omega^k D) = \pm e^{2\pi jk/M} \tag{16}$$

for all $k \in \{0, 1, \ldots, M-1\}$.

Theorem 1*: The Bruhat constellation $\mathcal{V}$ has order $L_D M$ and rate $\log_2(L_D M)/2$. It has full diversity whenever $\mathcal{D}$ does. The value of the diversity product (2) $\zeta_{\mathcal{V}}$ will equal*

$$\zeta_{\mathcal{V}} = \min_{1 \le i < M} \left(\zeta_{\mathcal{D}}, \frac{1}{2} |\lambda^{ib} - 1|^{\frac{1}{b}} \right), \tag{17}$$

where $\zeta_{\mathcal{D}}$ denotes the diversity product of $\mathcal{D}$ and for each i, we set $b = M/gcd(i, M)$. The term $\zeta_{\mathcal{D}}$ may be computed explicitly in terms of the parameters u_i as

$$\zeta_{\mathcal{D}} = \frac{1}{2} \min_{l', l''} d_{\mathcal{D}}(l', l'')^{\frac{1}{M}} \tag{18}$$

$$= \min_{l=1,2,\ldots,L_D-1} \left| \prod_{i=1}^{M} \sin \frac{\pi u_i l}{L_D} \right|^{\frac{1}{M}}. \tag{19}$$

Proof: One computes the order and rate directly. Recall that the diversity product depends on the minimum value of the distance (as defined in (1)) between two elements of $\mathcal{V}$. From the definition of $\mathcal{V}$ as a disjoint union of cosets in (15), it follows that the calculation of the distance $d(l, l')$ may be broken into two cases: the *same coset case*, when V_l and $V_{l'}$ lie in the same coset $\lambda^i\omega^i\mathcal{D}$; and the *distinct coset case*, when V_l and $V_{l'}$ lie in two different cosets.

i) Same Coset (sc) Case: Suppose $V_l, V_{l'} \in \mathcal{V}$ lie in the same coset $\lambda^i\omega^i\mathcal{D}$. Then we can write $V_l = \lambda^i\omega^i D$ and $V_{l'} = \lambda^i\omega^i D'$ for some $D, D' \in \mathcal{D}$. We compute $d_{sc}(l, l') = |\det(\lambda^i\omega^i D -$

$\lambda^i\omega^i D')| = |\det(\lambda^i\omega^i)||\det(D-D')| = |\det(D-D')|$. Thus the minimum value of $d_{sc}(l,l')$ equals the minimum distance between two elements of $\mathcal{D}$. Hence, in order to have $d_{sc}(l,l') > 0$, it suffices to require the full diversity of the subgroup $\mathcal{D}$.

ii) Distinct Coset (dc) Case: Suppose now that $V_l, V_{l'} \in \mathcal{V}$ lie in distinct cosets; say $V_l = \lambda^k\omega^k D$ and $V_{l'} = \lambda^j\omega^j D'$, $k < j$ for some $D, D' \in \mathcal{D}$. Then as above, we may restrict ourselves to the case where $V_l \in \mathcal{D}$ and $V_{l'} \in \lambda^i\omega^i\mathcal{D}$, since $|\det(\lambda^k\omega^k D - \lambda^j\omega^j D')| = |\det(\lambda^k\omega^k)||\det(D - \lambda^{j-k}\omega^{j-k}D')| = |\det(D - \lambda^{j-k}\omega^{j-k}D')|$. To compute d_{dc} explicitly, we proceed follows. Recall the decomposition of ω^i into disjoint cycles given in (7). Let $\tilde{S}_i$ be the set of all possible permutations which can be formed from the set of disjoint cycles $\{\sigma_1, \sigma_2, \ldots, \sigma_g\}$ using each cycle at most once. For example, in the symmetric group S_6 with $i = 2$, we have $\omega^2 = (135)(246)$ so $\tilde{S}_{i=2} = \{I, (135), (246), (135)(246)\}$. In general, $|\tilde{S}_i| = 2^g$. For $\sigma \in \tilde{S}_i$, write $\sharp\sigma$ for the number of disjoint cycles occuring in σ. Then, as proven in [6], the distance between V_l and $V_{l'}$ in this case can be computed as

$$d_{dc}(l,l') = |\det(D - \lambda^i\omega^i D')| \quad (20)$$

$$= |\sum_{\sigma\in\tilde{S}_i}(-1)^{\sharp\sigma}\lambda^{\sharp\sigma\cdot i\cdot b}\prod_{j=1}^{M} d_j^{\sigma}|. \quad (21)$$

Here we have defined d_j^{σ} to be equal to d_j if $\sigma(j) = j$ and d'_j if $\sigma(j) \neq j$. Our constraint (C1) thus implies $\prod_{j=1}^{M} d_j^{\sigma} = 1$. For each value of k, $0 \leq k \leq g$, there are $\binom{g}{k}$ elements $\sigma \in \tilde{S}_i$ with $\sharp\sigma = k$. Thus to evaluate this expression,we may sum over k, to yield

$$d_{dc}(l,l') = |\sum_{k=0}^{g}\binom{g}{k}(-1)^k\lambda^{kib}| \quad (22)$$

$$= |\lambda^{ib} - 1|^g. \quad (23)$$

Finally, note that since $i < M$ and $b = M/g < M$, we have that $ib < M^2$. Hence $\lambda^{ib} \neq 1$ for any i, and thus $d_{dc}(l,l') \neq 0$. It now follows, using (2) that

$$\zeta_{\mathcal{V}} = \frac{1}{2}\min_{l,l'}(d_{sc}(l,l'), d_{dc}(l,l'))^{\frac{1}{M}} \quad (24)$$

$$= \min_i\left(\zeta_{\mathcal{D}}, \frac{1}{2}|\lambda^{ib} - 1|^{\frac{g}{M}}\right). \quad (25)$$

Since $b = M/g$, this completes the proof of (16). The proof of the equation (18) is given in [6]. □

Let us illustrate the concept of the proof with an example. Suppose the number of transmitter antennas is $M = 9$ and $\lambda = e^{2\pi j/81}$. Consider $i = 3$: the permutation $\omega^3 = (147)(258)(369) = \sigma_1\sigma_2\sigma_3$ with $g = \gcd(3,9) = 3$ and $b = 9/3 = 3$. We choose a subgroup $\mathcal{D}$ to consist of diagonal matrices $D = \mathrm{diag}(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9)$ which satisfy $d_1d_4d_7 = d_2d_5d_8 = d_3d_6d_9 = 1$ (following from (12)). Clearly $\det(D) = 1$ and from (22) we have the distance between any 2 elements of the distinct cosets $\mathcal{D}$ and $\omega^3\lambda^3\mathcal{D}$ is $d_{dc}(l,l') = |\lambda^9 - 1|^3 = 0.3201$.

Remark: When the number of transmitter antennas M is prime, we have $g = \gcd(i, M) = 1$ for all $i = 1, 2, \ldots, M-1$, so $b = M$. Hence from (22), we have $d_{dc}(l,l') = |\lambda^{iM} - 1|$. This value is equal to the Euclidean distance between two points on a unit circle. It follows that the chosen value of $\lambda = e^{2\pi j/M^2}$ maximizes the minimum value of $d_{dc}(l,l')$ in this case, by ensuring the M points λ^{iM} are equally spaced along the unit circle. Moreover, since the permutation ω^i is a single cycle for each i, the equation (12) simplifies to nothing more than the requirement that $\det D = 1$.

C. Some Examples of a Bruhat Constellation

1. For $M = 2$: From (15), a Bruhat constellation for two transmitter antennas is

$$\mathcal{V}_2 = \mathcal{D} \cup \lambda\omega\mathcal{D} \quad (26)$$

where $\lambda = e^{j2\pi/4}$ and

$$\omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (27)$$

Using (22), we have $d_{dc}(l,l') = |\lambda^2 - 1| = |-1-1| = 2 > 0$. The elements $D_l = \mathrm{diag}(d_1, d_2) \in \mathcal{D}$ must satisfy $d_2 = d_1^{-1}$ by (12) where $d_1 = e^{2\pi j u_1/L_D}$. Then this constellation is

$$\mathcal{V}_2 = \mathcal{D} \cup e^{j\frac{2\pi}{4}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathcal{D} \quad (28)$$

where the value of $u_1 = \{1, 2, \ldots, L_D - 1\}$ is chosen to maximize $d_{sc}(l,l')$ for a given L_D. The order of this constellation is $2L_D$. (We note that our constellation in (27) is equivalent to the optimal dicyclic group constellation [4] for $M = 2$.)

2. For $M = 3$: Our Bruhat constellation for three transmitter antennas of order $3L_D$ is

$$\mathcal{V}_3 = \mathcal{D} \cup \lambda\omega\mathcal{D} \cup \lambda^2\omega^2\mathcal{D} \quad (29)$$

with $\lambda = e^{j2\pi/9}$ and

$$\omega = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (30)$$

Using (22) with $g = 1$ and $b = 3$, the distance $d_{dc}(l,l')$ may be computed as: for $i = 1$, $\omega = (123)$, we have $d_{dc} = |\lambda^3 - 1| = 1.7321 > 0$; for $i = 2$, $\omega^2 = (132)$, we have $d_{dc} = |\lambda^6 - 1| = 1.7321 > 0$. Each $D_l \in \mathcal{D}$ must have $\det(D_l) = 1$; so set

$$D_l = \mathrm{diag}(d_1, d_2, d_1^{-1}d_2^{-1}). \quad (31)$$

To obtain a cyclic subgroup, set each $d_i = e^{j2\pi u_i l/L_D}$, $i = 1, 2$ for some parameters $u_i \in \{1, 2, ..L_D - 1\}$ which are chosen to maximize the value of $d_{sc}(l,l')$ for a given L_D. Table 1 shows some of our Bruhat constellations for $M = 3$ but L_D varying, with best diversity products compared with different 3×3 unitary constellation designs in the literature. We can see that our proposed constellations have diversity products higher than cyclic groups and some constellations obtained from fixed-point free groups.

3. For $M = 4$: For four transmitter antennas, our proposed Bruhat constellation of order $4L_D$ is

$$\mathcal{V}_4 = \mathcal{D} \cup \lambda\omega\mathcal{D} \cup \lambda^2\omega^2\mathcal{D} \cup \lambda^3\omega^3\mathcal{D}. \quad (32)$$

where $\lambda = e^{j2\pi/16}$ and

$$\omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (33)$$

L	R	$\zeta_{\mathcal{V}}$	Designs
9	1.06	0.6004	fpf group $G_{9,1}$ [7]
9	1.06	0.6004	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 1$
57	1.94	0.4845	$\mathcal{S}_{19,3}, u = (1, 7, 11)$ [7]
57	1.94	0.4845	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 7$
63	1.99	0.3301	cyclic $u = (1, 17, 26)$ [3]
63	1.99	0.3851	fpf group $G_{21,4}$ [7]
63	1.99	0.3851	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 4$
64	2.00	0.2765	cyclic $u = (1, 11, 27)$ [3]
69	2.04	0.3548	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 4$
513	3.00	0.1353	fpf group $G_{171,64}$ [7]
513	3.00	0.1436	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 22$
4095	4.00	0.0361	fpf group $G_{1365,16}$ [7]
4095	4.00	0.0471	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 121$

Table 1: Comparison of different 3×3 unitary constellation designs (our proposed constellations are highlighted in grey).

For $i = 1$, $\omega = (1234)$, we have $d_{dc} = |\lambda^4 - 1|$ whenever $\det(D_l) = 1$. For $i = 2$, $\omega^2 = (13)(24)$, $g = \gcd(2, 4) = 2$, $b = 4/2 = 2$; from (22), we can have $d_{dc} = |\lambda^4 - 1|^2$ by setting $d_1 d_3 = d_2 d_4 = 1$. For $i = 3$, $\omega^3 = (1432)$, we have $d_{dc} = |\lambda^{12} - 1|$. Thus the unitary diagonal matrices D_l can be chosen by

$$D_l = \mathrm{diag}(d_1, d_2, d_1^{-1}, d_2^{-1}) \tag{34}$$

where $d_i = e^{j2\pi u_i l/L_D}$, $i = 1, 2$. The values of $u_1, u_2 = \{1, 2, \ldots, L_D - 1\}$ are chosen to maximize $d_{sc}(l, l')$ for each given L_D. For example, set $L_D = 4$, then the order of a proposed constellation is $4 \times 4 = 16$. The choices of $u_1 = 1$ and $u_2 = 1$ give a Bruhat constellation with the diversity product $\zeta_{\mathcal{V}} = 0.5453$, which is equal to that of a constellation obtained from a cyclic group $u = (1, 3, 5, 7)$ [3] at the same data rate $R = 1.00$. For $L_D = 16$, the constellation has $4 \times 16 = 64$ elements. Choose $u_1 = 1$ and $u_2 = 7$; then the diversity product of this Bruhat constellation is $\zeta_{\mathcal{V}} = 0.3827$ which is higher than the constellation $\tilde{G}_4$ obtained by Konishi's method ($\zeta_{\mathcal{V}} = 0.3418$) in [5] with the same order and data rate $R = 1.50$.

4. For $M = 5$: A Bruhat constellation for five transmitter antennas is

$$\mathcal{V}_5 = \mathcal{D} \cup \lambda\omega\mathcal{D} \cup \lambda^2\omega^2\mathcal{D} \cup \lambda^3\omega^3\mathcal{D} \cup \lambda^4\omega^4\mathcal{D} \tag{35}$$

where $\lambda = e^{j2\pi/25}$. For this case, we have $g = \gcd(i, 5) = 1$ for all $i = 1, 2, 3, 4$. The choice of a subgroup of unitary diagonal matrices $\mathcal{D} = \{D_l = \mathrm{diag}(d_1, d_2, d_3, d_4, d_5 | d_1 d_2 d_3 d_4 d_5 = 1\}$ can be made by

$$D_l = \mathrm{diag}(d_1, d_2, d_3, d_4, d_1^{-1} d_2^{-1} d_3^{-1} d_4^{-1}) \tag{36}$$

where $d_i = e^{j2\pi u_i l/L_D}$, $i = 1, 2, 3, 4$. The values of $u_i = \{1, 2, \ldots, L_D - 1\}$ are chosen to maximize $d_{sc}(l, l')$ for a given L_D. The order of this constellation is $5L_D$. Table 2 compares our proposed constellation with different 5×5 unitary constellation designs at the data rate $R \approx 1.00$.

6. For $M = 6$: For six transmitter antennas, our proposed constellation is

$$\mathcal{V}_6 = \mathcal{D} \cup \lambda\omega\mathcal{D} \cup \lambda^2\omega^2\mathcal{D} \cup \lambda^3\omega^3\mathcal{D} \cup \lambda^4\omega^4\mathcal{D} \cup \lambda^5\omega^5\mathcal{D} \tag{37}$$

L	R	$\zeta_{\mathcal{V}}$	Designs
32	1.00	0.4095	cyclic $u = (1, 5, 7, 9, 11)$ [3]
33	1.01	0.5580	$\mathcal{S}_{11,3}$ $u = (1, 3, 4, 5, 9)$ [7]
35	1.03	0.5164	$\mathcal{V}$ $u_1 = 1, u_2 = 1, u_3 = 4, u_4 = 5$

Table 2: Comparison of different 5×5 unitary constellation designs (proposed constellation is highlighted in grey).

where $\lambda = e^{j2\pi/36}$. For $i = 1$ and 5, $g = \gcd(i, 6) = 1$, the condition of (12) on the unitary diagonal matrix D_l is simply $d_1 d_2 d_3 d_4 d_5 d_6 = 1$. For $i = 2, 3, 4$, the conditions on D_l are listed below:

i	d_{dc}	Constraint on D_l
2	$\lvert\lambda^6 - 1\rvert^2$	$d_1 d_3 d_5 = d_2 d_4 d_6 = 1$
3	$\lvert\lambda^6 - 1\rvert^3$	$d_1 d_4 = d_2 d_5 = d_3 d_6 = 1$
4	$\lvert\lambda^{12} - 1\rvert^2$	$d_1 d_3 d_5 = d_2 d_4 d_6 = 1$

Consequently we may choose a subgroup $\mathcal{D} = \{D_l = \mathrm{diag}(d_1, d_2, d_3, d_4, d_5, d_6)\}$ by

$$D_l = \mathrm{diag}(d_1, d_2, d_1^{-1} d_2, d_1^{-1}, d_2^{-1}, d_1 d_2^{-1}) \tag{38}$$

where $d_1 = e^{j2\pi u_1 l/L_D}$ and $d_2 = e^{j2\pi u_2 l/L_D}$. The values of u_1 and u_2 are chosen to maximize $d_{sc}(l, l')$ for a given L_D. Here $6L_D$ is the order of this constellation. Table 3 gives some of our proposed constellations with their best diversity products compared with different 6×6 unitary constellation designs at the data rate $R \approx 1.00$.

L	R	$\zeta_{\mathcal{V}}$	Designs
64	1.00	0.3792	cyclic $u = (1, 7, 15, 23, 25, 31)$ [3]
66	1.01	0.4865	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 3$
72	1.03	0.5000	$\mathcal{S}_{12,6}$ $u = (1, 1, 7, 7, 7, 1)$ [7]
78	1.05	0.5000	Bruhat $\mathcal{V}$ $u_1 = 1, u_2 = 10$

Table 3: Comparison of different 6×6 unitary constellation designs (our proposed constellations are highlighted in grey).

V. Performance

The performance is considered by plotting the block error rate, *bler*, against signal-to-noise ratio, SNR, in unknown Rayleigh flat fading channels. We use a differential modulation as explained in Section II to transmit signals. The fading coefficients and additive noise are independent complex Gaussian variables with mean zero and variance one. The channel matrix is assumed to be constant within two consecutive time periods. The maximum-likelihood decoder (6) is used for decoding.

Figure 1 displays the block error rate performance of our 3×3 Bruhat constellation with $u_1 = 1, u_2 = 4$ at $R = 2.04$ compared with the cyclic group $u = (1, 11, 27)$ at $R = 2.00$ as given in Table 1 with the number of receiver antenna, $N = 1$. We can see that the proposed constellation outperforms the cyclic group. The gain improvement is ≈ 2 dB.

Figure 2 shows the block error rate performance of our 5×5 Bruhat constellation with $u_1 = 1, u_2 = 1, u_3 = 2$ at $R = 1.03$

compared with the cyclic group $u = (1, 5, 7, 9, 11)$ at $R = 1.00$ and the nongroup $\mathcal{S}_{11,3}$ at $R = 1.01$ as given in Table 2 with $N = 1$. We can see that our Bruhat constellation outperforms the cyclic group and compares well with the nongroup constellation.

Figure 3 displays the block error rate performance of our 6×6 Bruhat constellation with $u_1 = 1, u_2 = 3$ at $R = 1.01$ with the cyclic group $u = (1, 7, 15, 23, 25, 31)$ at $R = 1.00$ as given in Table 3 with $N = 1$. We can see that our proposed constellation $\mathcal{V}$ again outperforms the cyclic group with ≈ 1 dB gain improvement.

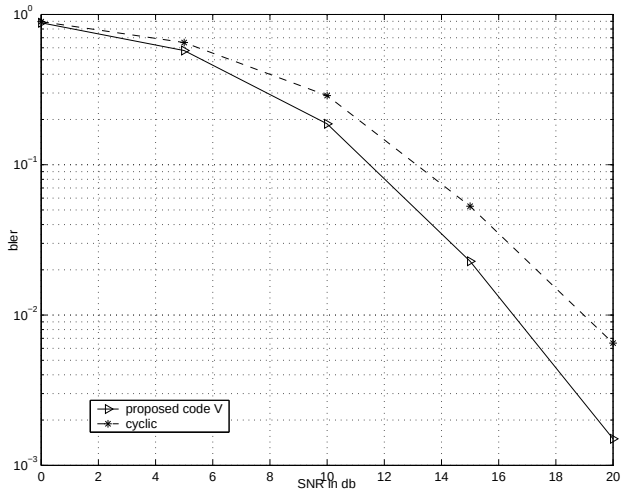

Figure 1: Block error rate performance for $M = 3, N = 1$ at $R \approx 2.00$ of our proposed constellation and cyclic group

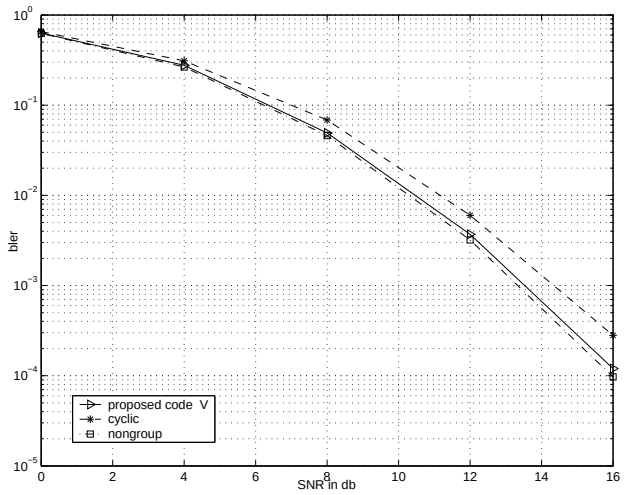

Figure 2: Block error rate performance for $M = 5, N = 1$ at $R \approx 1.00$ of our proposed constellation, cyclic group and nongroup

VI. Conclusion

We have proposed the design of a full diversity unitary space-time Bruhat constellation $\mathcal{V}$ for any number of antennas using the Bruhat decomposition. Our proposed design has a simple structure for a case where the number of transmitter antennas is prime. We have shown that the optimization of the diversity product is not computationally intensive when the number of transmitters is a composite number, for example, when the number of transmitter antennas is 6, there are only two parameter values, u_1 and u_2, for searching (see equation (37)). Simulations show that our proposed Bruhat constellations performs well without the knowledge of the information of the channel, and also outperform cyclic group constellation designs.

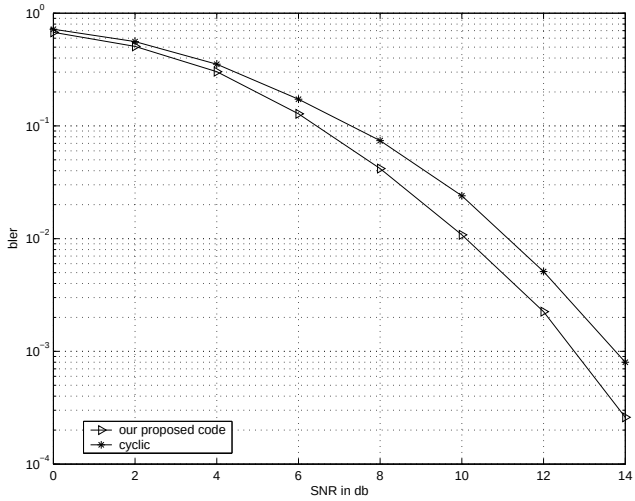

Figure 3: Block error rate performance for $M = 6, N = 1$ at $R \approx 1.00$ of our proposed constellation and cyclic group

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