Pairwise Error Probability of Space-Time Codes for a Keyhole Channel

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Abstract

We present a closed form upper bound for the average pairwise error probabilities (PEP) of space-time codes for a keyhole channel. It is derived from the exact conditional PEP for given fading channel coefficients, using a moment generating function-based approach. We include simulation results for varying numbers of antennas which affirm that the proposed PEP serves as a tight bound for codes in a keyhole channel.

Index Terms– space-time codes, keyhole channel, pairwise error probability, moment generating function.

1 Introduction

Recently, the papers [1, 4, 6] have shown that zero or low correlation of fading channel coefficients of a multiple-input multiple-output channel does not guarantee high channel capacity. *Keyholes*, whose

fading coefficients are uncorrelated, cause a rank deficiency of the channel matrix. This degrades the channel capacity [1, 6, 11] and the performance of space-time codes over the channel [10]. One example of keyholes, in practical situations, is the roof edge diffraction phenomenon, as explained in [1].

The symbol error rate (SER) of space-time block codes in a keyhole channel was studied in [10], using a Moment Generating Function (MGF)-based approach to the conditional SER for a given instantaneous SNR. The proposed SERs in [10] are exact, but apply only to block codes.

More generally, the calculation of average PEP and a design criterion of space-time codes in a keyhole channel have been presented in [9] (also in [7] for space-time trellis codes for two transmitter and one receiver antennas). When the (square moduli of the) eigenvalues of a codeword pair difference are not distinct (such as occurs, for example, in the 2×2 orthogonal space-time block code [14]; see Appendix B), the PEPs proposed in [7, 9] do not apply.

In this paper, we derive a tight upper bound on the average PEPs of full rank space-time codes for a keyhole channel, using a moment generating function-based approach. To obtain this upper bound, we use two closed-form upper bounds, denoted Q_1 and Q_2 in (7), of the Q-function, which is used in the exact conditional PEP for given fading channel coefficients as given in [12]. The two upper bounds are applicable everywhere but are especially tight in low and high SNR respectively.

The advantage of our proposed bound on PEP is its closed form solution using hypergeometric functions, and its dependence only on the minimum eigenvalue of a codeword pair difference. This, in turn, implies a simple max-min eigenvalue computation which can be done by a power method [3]. Our proposed PEP applies more generally than does the SER of [10]; for example, to space-time trellis codes [13]. Furthermore, our proposed PEP offers an improvement over the one suggested in [9], as it applies to all full rank space-time codes for any number of antennas.

Notation: We use bold lower case letters $\boldsymbol{c}$ to denote vectors and bold capital letters $\boldsymbol{H}$ to

denote matrices. The Frobenius norm is denoted $|| \cdot ||$, while the transpose and conjugate transpose are denoted ^{T} and † respectively. An expectation is indicated by $E[\cdot]$, and $\mathcal{CN}(0,1)$ denotes a complex Gaussian random variable with zero mean and unit variance and is also circular symmetric. Finally, $\mathbb{C}$ denotes the set of complex numbers.

2   Keyhole Channel

Consider a screen with a small hole (called a keyhole) between M transmitter and N receiver antennas in a quasi-static frequency-flat fading channel. See Figure 1. The transmitted signals can propagate to the receiver antennas only by passing through the keyhole. This situation is modelled by assuming that the fading coefficients of the $N \times M$ channel matrix $\boldsymbol{H}$ are a product of two uncorrelated Gaussian random variables.

Specifically, set $\boldsymbol{\beta} \in \mathbb{C}^{N \times 1} = [\beta_1, \beta_2, \ldots, \beta_N]^T$ and $\boldsymbol{\alpha} \in \mathbb{C}^{M \times 1} = [\alpha_1, \alpha_2, \ldots, \alpha_M]^T$, where β_i and α_j are assumed to be independent $\mathcal{CN}(0,1)$ random variables, representing scatterings at the transmitter and receiver antennas respectively. Then, by [1], the channel matrix is $\boldsymbol{H} = \boldsymbol{\beta}\boldsymbol{\alpha}^T$. The matrix has rank$(\boldsymbol{H}) = 1$; in other words, a keyhole causes a rank deficiency of the channel matrix $\boldsymbol{H}$.

Let $\boldsymbol{X}$ be an $M \times M$ transmitted matrix. Then the corresponding $N \times M$ received matrix $\boldsymbol{Y}$ is modelled by $\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{X} + \boldsymbol{W}$ where $\boldsymbol{W}$ is an $N \times M$ additive noise matrix whose elements are independent $\mathcal{CN}(0,1)$. The receiver uses maximum-likelihood criterion to decode the message.

3   Moment Generating Function (MGF)-Based Approach

Let $P(E|Y = y)$ be the conditional probability of error for a given nonnegative random variable Y, and let $p_Y(y)$ be the probability density function (pdf) of Y. The average probability of error is

computed by $P(E) = \int_0^\infty P(E|Y = y)p_Y(y)dy$, whereas the MGF of Y is defined as

$$M_Y(s) = E[e^{sY}] = \int_0^\infty e^{sy}p_Y(y)dy. \tag{1}$$

Suppose the conditional probability of error for a given Y is of the form $P(E|Y = y) = ke^{-sy}$ where k and s are positive constants. Then the average probability of error is computed as

$$P(E) = \int_0^\infty ke^{-sy}p_Y(y)dy = kM_Y(-s). \tag{2}$$

Now consider a keyhole channel whose channel matrix is $\boldsymbol{H} = \boldsymbol{\beta}\boldsymbol{\alpha}^T$ and set $Y = ||\boldsymbol{H}||^2$. Then $Y = ||\boldsymbol{H}||^2 = ||\boldsymbol{\alpha}||^2||\boldsymbol{\beta}||^2 = UV$, where now U and V are Chi-square random variables with $2M$ and $2N$ degrees of freedom respectively (we note that setting Y which is a product of two random variables will be used later to analyze an average PEP from a conditional PEP for a given product of two random variables of (11) in Section 4). For $y \geq 0$, the pdf of $Y = UV$ is computed by [10]

$$p_Y(y) = \int_{-\infty}^\infty \frac{1}{|u|}p_U(u)p_V\left(\frac{y}{u}\right)du = \int_0^\infty \frac{t^{M-N-1}e^{-t-\frac{y}{t}}y^{N-1}}{\Gamma(M)\Gamma(N)}dt \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. To express its MGF, we recall the *degenerate hypergeometric function* $\Psi(a, b; z)$ and the *generalized hypergeometric function* $_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$; see Appendix A. It is shown in [10] that the MGF of $Y = UV$ is given by

$$M_Y(s) = s^{-M}\Psi(M, M - N + 1; s^{-1}) \tag{4}$$

$$= {}_2F_0(M, N; ; -s). \tag{5}$$

4 Pairwise Error Probability (PEP)

Consider a multiple-input multiple output system with M transmitter and N receiver antennas in a Rayleigh flat fading channel with fading coefficients $\{\beta_i\alpha_j\}_{i=1,j=1}^{N,M}$. Let $\boldsymbol{H} = \boldsymbol{\beta}\boldsymbol{\alpha}^T$ denote the corresponding channel matrix, let ρ denote the SNR at each receiver antenna, and assume that the

expected value of the sum of all transmitted signal powers equals to one. Further, recall that the Q-function is defined by $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, and let us use $\dagger$ to denote the conjugate transpose of a matrix and tr to denote its trace.

Then the exact conditional PEP that the receiver antennas erroneously decodes $\boldsymbol{e}$ from a codeword $\boldsymbol{c}$ is given by [12]

$$P_{\text{PEP}}(\boldsymbol{c} \rightarrow \boldsymbol{e}|\beta_i \alpha_j) = Q\left(\sqrt{\frac{\rho}{2} \text{tr}(\boldsymbol{H}\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger \boldsymbol{H}^\dagger)} \right) \tag{6}$$

where $\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e}) = \boldsymbol{e} - \boldsymbol{c}$ is defined as the difference matrix of the codewords $\boldsymbol{c}$ and $\boldsymbol{e}$.

The codewords are in general $M \times l$ matrices, where l is the length of the codeword sequence. If the space-time code is to be *full rank*[1] then we must have $l \geq M$.

To bound the value of P_{PEP}, we first note that

$$Q(x) \leq \begin{cases} Q_1(x) = \frac{1}{2} e^{-\frac{x^2}{2}} & \text{if } 0 \leq x \leq \sqrt{\frac{2}{\pi}}; \\ Q_2(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} & \text{if } x \geq \sqrt{\frac{2}{\pi}}. \end{cases} \tag{7}$$

See Figure 2, which plots $Q(x)$ compared with $Q_1(x)$ and $Q_2(x)$. We can see that when $x \leq \sqrt{\frac{2}{\pi}}$, $Q_1(x)$ gives the better upper bound of $Q(x)$, whereas when $x \geq \sqrt{\frac{2}{\pi}}$, $Q_2(x)$ gives the better upper bound. Hence (7) is equivalent to saying that we approximate $Q(x)$ by $\min(Q_1(x), Q_2(x))$.

4.1 Using the Upper Bound Q_1

Using the upper bound Q_1 as given in (7), the exact PEP in (6) is bounded by

$$P_{\text{PEP 1}}(\boldsymbol{c} \rightarrow \boldsymbol{e}|\beta_i \alpha_j) \leq \frac{1}{2} \exp\left(-\frac{\rho}{4} \text{tr}(\boldsymbol{H}\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger \boldsymbol{H}^\dagger) \right). \tag{8}$$

Recall that the $M \times M$ matrix $\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger$ is hermitian and positive semi-definite. Hence it admits a singular value decomposition $\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger$, where $\boldsymbol{U}$ is an $M \times M$ unitary matrix whose

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[1]If $\boldsymbol{A}$ is an $M \times l$ matrix, then $\text{rank}(\boldsymbol{A}) \leq M$ and $\text{rank}(\boldsymbol{A}) \leq l$ [8]. In this paper, we say a matrix has full rank when $\text{rank}(\boldsymbol{A}) = M$, the number of columns of $\boldsymbol{A}$.

columns are eigenvectors of $\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger$, and $\boldsymbol{D} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M)$ is an $M \times M$ diagonal matrix whose diagonal entries, which are listed in decreasing order, are the corresponding eigenvalues of $\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger$. In what follows, we will assume that $\text{rank}(\boldsymbol{B}) = M$; that is, we assume that $\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger$ is positive definite (not just positive semi-definite) and hence that $\lambda_i > 0$ for all i [13].

Consider now the specific case of a keyhole channel. Substituting $\boldsymbol{H} = \boldsymbol{\beta}\boldsymbol{\alpha}^T$ and $\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger$ in (8) gives

$$
\begin{aligned}
P_{\text{PEP 1}}(\boldsymbol{c} \to \boldsymbol{e}|\beta_i\alpha_j) &\le \frac{1}{2}\exp\left(-\frac{\rho}{4}\text{tr}(\boldsymbol{\beta}\boldsymbol{\alpha}^T\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger(\boldsymbol{\beta}\boldsymbol{\alpha}^T)^\dagger)\right). \\
&= \frac{1}{2}\exp\left(-\frac{\rho}{4}\text{tr}(\boldsymbol{\beta}(\boldsymbol{\alpha}^T\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger(\boldsymbol{\alpha}^T)^\dagger)\boldsymbol{\beta}^\dagger)\right).
\end{aligned}
\tag{9}
$$

Set $\boldsymbol{\gamma} = \boldsymbol{\alpha}^T\boldsymbol{U} = [\gamma_1, \gamma_2, \ldots, \gamma_M]$; since $\boldsymbol{U}$ is unitary, it follows that $\|\boldsymbol{\gamma}\| = \|\boldsymbol{\alpha}\|$. Then we may write

$$
\boldsymbol{\alpha}^T\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger(\boldsymbol{\alpha}^T)^\dagger = \boldsymbol{\gamma}^T\boldsymbol{D}(\boldsymbol{\gamma}^T)^\dagger = \lambda_1|\gamma_1|^2 + \lambda_2|\gamma_2|^2 + \cdots + \lambda_M|\gamma_M|^2.
$$

Recalling that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_M > 0$, we may further simplify the above expression to deduce that

$$
\begin{aligned}
\boldsymbol{\alpha}^T\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger(\boldsymbol{\alpha}^T)^\dagger &= \lambda_M\left(\frac{\lambda_1}{\lambda_M}|\gamma_1|^2 + \frac{\lambda_2}{\lambda_M}|\gamma_2|^2 + \cdots + |\gamma_M|^2\right) \\
&\ge \lambda_M(|\gamma_1|^2 + |\gamma_2|^2 + \cdots + |\gamma_M|^2) \\
&= \lambda_M\|\boldsymbol{\gamma}\|^2 \\
&= \lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}}\|\boldsymbol{\alpha}\|^2
\end{aligned}
\tag{10}
$$

where we have defined $\lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}} = \lambda_M$, the minimum eigenvalue of the matrix $\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger$.

It now follows that the conditional PEP in the equation (9) is

$$
\begin{aligned}
P_{\text{PEP 1}}(\boldsymbol{c} \to \boldsymbol{e}|\beta_i\alpha_j) &\le \frac{1}{2}\exp\left(-\frac{\rho}{4}(\text{tr}(\boldsymbol{\beta}\lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}}\|\boldsymbol{\alpha}\|^2\boldsymbol{\beta}^\dagger)\right) \\
&= \frac{1}{2}\exp\left(-\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}}\|\boldsymbol{\alpha}\|^2(\text{tr}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger))\right) \\
&= \frac{1}{2}\exp\left(-\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}}\|\boldsymbol{\alpha}\|^2\|\boldsymbol{\beta}\|^2\right) \\
&= \frac{1}{2}\exp\left(-\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}}Y\right),
\end{aligned}
\tag{11}
$$

6

where we recall that $Y = UV = ||\boldsymbol{\alpha}||^2||\boldsymbol{\beta}||^2$, from Section 3.

We note that the bound on conditional PEP determined in (11) has the form ke^{-sY}, with $k = 1/2$ and $s = \frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}$, and $Y = UV$ with U, V Chi-square random variables with $2M$ and $2N$ degrees of freedom, respectively. It follows that we may apply an MGF-based approach to bounded the average probability of error. That is, we deduce from (2) through (5) that the average probability of error for space-time codes on a keyhole channel, $P(E)$, is bounded by

$$
\begin{aligned}
P_{\text{PEP 1}} \;\; &\leq \;\; kM_Y(-s) \\
&= \;\; \frac{1}{2} \;_2F_0\left(M, N; ; -\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}\right) \quad\quad\quad\quad\quad (12) \\
&= \;\; \frac{1}{2}(\lambda_{\min}^{\boldsymbol{c,e}})^{-M}\left(\frac{\rho}{4}\right)^{-M}\Psi\left(M, M-N+1; \left(\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}\right)^{-1}\right). \quad\quad (13)
\end{aligned}
$$

4.2 Using the Upper Bound Q_2

Using the upper bound Q_2 as given in (7) to bound the exact conditional PEP in (6) gives the bound

$$
P_{\text{PEP}}(\boldsymbol{c} \rightarrow \boldsymbol{e}|\beta_i\alpha_j) \leq \frac{\exp\left(-\frac{\rho}{4}\text{tr}(\boldsymbol{H}\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger\boldsymbol{H}^\dagger)\right)}{\sqrt{2\pi \cdot \frac{\rho}{2}\text{tr}(\boldsymbol{H}\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger\boldsymbol{H}^\dagger)}}. \quad\quad (14)
$$

Repeating the argument of the preceding section, with $\boldsymbol{H} = \boldsymbol{\beta}\boldsymbol{\alpha}^T$, we deduce that

$$
\text{tr}(\boldsymbol{H}\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})\boldsymbol{B}(\boldsymbol{c},\boldsymbol{e})^\dagger\boldsymbol{H}^\dagger) = \boldsymbol{\alpha}^T\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger(\boldsymbol{\alpha}^T)^\dagger\text{tr}(\boldsymbol{\beta}\boldsymbol{\beta}^\dagger) \geq \lambda_{\min}^{\boldsymbol{c,e}}||\boldsymbol{\alpha}||^2||\boldsymbol{\beta}||^2.
$$

Hence we may bound the numerator of (14) from above and the denominator of (14) from below to obtain

$$
P_{\text{PEP}}(\boldsymbol{c} \rightarrow \boldsymbol{e}|\alpha_i\beta_j) \leq \frac{\exp(-\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}Y)}{\sqrt{2\pi \cdot \frac{\rho}{2}Y\lambda_{\min}^{\boldsymbol{c,e}}}} = \frac{\exp(-\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}Y)}{\sqrt{\pi\rho\lambda_{\min}^{\boldsymbol{c,e}}} \cdot \sqrt{Y}}. \quad\quad (15)
$$

Recall that Y is defined as $Y = UV = ||\boldsymbol{\alpha}||^2||\boldsymbol{\beta}||^2$ with U and V Chi-square random variables with $2M$ and $2N$ degrees of freedom respectively. Hence to compute the average probability of error $P(E)$ from the bound on condition PEP given in (15), we define a new term called the *modified MGF* of Y as

$$
M_Y'(s) \triangleq E\left[\frac{e^{-sY}}{\sqrt{Y}}\right] = \int_0^\infty \frac{e^{-sy}}{\sqrt{y}} \; p_Y(y)dy. \quad\quad (16)
$$

7

Using the pdf of Y given in (3), the modified MGF of Y can be computed by

$$
\begin{aligned}
M_Y'(s) &= \int_0^\infty \int_0^\infty \frac{t^{M-N-1}e^{-t-\left(s+\frac{1}{t}\right)y}y^{N-\frac{3}{2}}}{\Gamma(M)\Gamma(N)}dtdy \\
&= \frac{1}{\Gamma(M)\Gamma(N)} \int_0^\infty \int_0^\infty t^{M-\frac{3}{2}-\left(N-\frac{1}{2}\right)}e^{-t}e^{-\left(s+\frac{1}{t}\right)y}y^{N-\frac{3}{2}}dtdy \\
&= \frac{1}{\Gamma(M)\Gamma(N)} \int_0^\infty \frac{t^{M-\frac{3}{2}}e^{-t}}{t^{N-\frac{1}{2}}} \frac{\Gamma\left(N-\frac{1}{2}\right)}{\left(s+\frac{1}{t}\right)^{N-\frac{1}{2}}}dt \quad \text{from } \int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}, \ n > -1 \text{ and } a > 0 \\
&= \frac{\Gamma\left(N-\frac{1}{2}\right)}{\Gamma(N)\Gamma(M)} \int_0^\infty \frac{e^{-t}t^{M-\frac{3}{2}}}{(1+st)^{N-\frac{1}{2}}}dt \\
&= \frac{\Gamma\left(N-\frac{1}{2}\right)}{\Gamma(N)\Gamma(M)} \int_0^\infty \frac{e^{-\frac{u}{s}}\left(\frac{u}{s}\right)^{M-\frac{3}{2}}}{(1+u)^{N-\frac{1}{2}}}\frac{du}{s} \quad \text{from setting } u = st \to du = sdt \\
&= \frac{\Gamma\left(N-\frac{1}{2}\right)}{\Gamma(N)\Gamma(M)}s^{-\left(M-\frac{1}{2}\right)}\frac{\Gamma\left(M-\frac{1}{2}\right)}{\Gamma\left(M-\frac{1}{2}\right)} \int_0^\infty e^{-\frac{u}{s}}u^{\left(M-\frac{1}{2}\right)-1}(1+u)^{(M-N+1)-\left(M-\frac{1}{2}\right)-1}du \\
&= \frac{\Gamma\left(N-\frac{1}{2}\right)\Gamma\left(M-\frac{1}{2}\right)}{\Gamma(N)\Gamma(M)}s^{-\left(M-\frac{1}{2}\right)}\Psi\left(M-\frac{1}{2},M-N+1;s^{-1}\right) \\
&= \frac{\Gamma\left(N-\frac{1}{2}\right)\Gamma\left(M-\frac{1}{2}\right)}{\Gamma(N)\Gamma(M)}\ {}_2F_0\left(M-\frac{1}{2},N-\frac{1}{2};;-s\right).
\end{aligned}
\tag{17}
$$

Hence, using the modified MGF computed in (17) and the conditional PEP in (15), that is, setting $k = \sqrt{\pi\rho\lambda_{\min}^{\boldsymbol{c,e}}}^{-1}$ and $s = \frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}$, we deduce that the average PEP of space-time codes using the upper bound Q_2 is

$$
\begin{aligned}
P_{\text{PEP 2}} &\leq \int_0^\infty \frac{\exp(-\frac{\rho}{4}Y\lambda_{\min}^{\boldsymbol{c,e}})}{\sqrt{\pi\rho\lambda_{\min}^{\boldsymbol{c,e}}}\cdot\sqrt{Y}}\cdot p_Y(y)dy = \frac{1}{\sqrt{\pi\rho\lambda_{\min}^{\boldsymbol{c,e}}}}M_Y'(\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}) \\
&= \frac{1}{\sqrt{\pi\rho\lambda_{\min}^{\boldsymbol{c,e}}}}\frac{\Gamma(N-\frac{1}{2})}{\Gamma(N)}\frac{\Gamma(M-\frac{1}{2})}{\Gamma(M)}\ {}_2F_0(M-\frac{1}{2},N-\frac{1}{2};;-\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}).
\end{aligned}
\tag{18}
$$

It is also written in the terms of the degenerate hypergeometric function as

$$
P_{\text{PEP 2}} \leq \frac{1}{\sqrt{\pi}}\frac{\Gamma(N-\frac{1}{2})}{\Gamma(N)}\frac{\Gamma(M-\frac{1}{2})}{\Gamma(M)}(\lambda_{\min}^{\boldsymbol{c,e}})^{-M}\left(\frac{\rho}{4}\right)^{-M}\Psi\left(M-\frac{1}{2},M-N+1;\left(\frac{\rho}{4}\lambda_{\min}^{\boldsymbol{c,e}}\right)^{-1}\right).
\tag{19}
$$

Consequently, we can summarize our proposed bound on the average PEP of erroneously decoding $\boldsymbol{e}$ for $\boldsymbol{c}$ over a keyhole channel as

$$
P_{\text{PEP}} = \min(P_{\text{PEP 1}}, P_{\text{PEP 2}})
\tag{20}
$$

where $P_{\text{PEP 1}}$ and $P_{\text{PEP 2}}$ are as defined in (12) and (18) respectively.

5   Block Error Rate and Union Bound Performance

Let $\mathcal{V}$ be a signal constellation, $\mathcal{V} = \{\boldsymbol{V}_l\}_{l=0}^{L-1}$, where the number of columns of V_l is equal to M, the number of transmitter antennas. The average PEP for a given transmitted V_l is

$$P_{\text{PEP}|V_l} \leq \sum_{l'=0, l' \neq l}^{L-1} P_{\text{PEP}}(l, l') \tag{21}$$

where $P_{\text{PEP}}(l, l')$ is defined in (20). We may assume that all codewords occur with the same probability, hence, the block error rate (BLER) of space-time codes of order L for a keyhole channel is computed by

$$P_{\text{BLER}} = \frac{1}{L} \sum_{l=0}^{L-1} P_{\text{PEP}|V_l} \leq \frac{1}{L} \sum_{l=0}^{L-1} \sum_{l'=0, l' \neq l}^{L-1} P_{\text{PEP}}(l, l'). \tag{22}$$

In the case that $\mathcal{V}$ is suitably symmetric, e.g. $\mathcal{V}$ is a unitary group constellation, the block error rate given in (22) is reduced to

$$P_{\text{BLER}} \leq \sum_{l=1}^{L-1} P_{\text{PEP}}(l, 0). \tag{23}$$

We note that a union bound on the block error rate may be determined by defining $\lambda_{\min} = \min_{l,l' = \{0,1,\dots,L-1\}, \; l \neq l'}(\lambda_{\min}^{l,l'})$; then for a space-time code of order L is approximated by $PU_{\text{BLER}} \leq (L-1)P_{\text{PEP}}(\lambda_{\min}^{\boldsymbol{c},\boldsymbol{e}} = \lambda_{\min})$.

Performance of 2×2 Orthogonal Space-Time Block Codes

We compare simulation results with the analytical block error rate from (22) using the proposed bound on PEPs in (20), for a 2×2 orthogonal space-time block code [14]. Since this code is a group (in fact a subgroup of the special unitary group of rank 2 $SU(2)$), we can compute its block error rate using (23).)

The performance is considered by plotting block error rate against SNR in dB, using a Monte-Carlo simulation over a keyhole channel. We consider 2×2 orthogonal space-time block codes for $M = 2$

transmitter antennas. The transmitted signal matrix is

$$\boldsymbol{X} = \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix},$$ (24)

where x and y are chosen from BPSK signals. Hence $L = 4$ and this code has data rate $R = \log_2 4/2 = 1$ bits/s/Hz. Figure 3 shows the block error rate performance for $N = 1, 2$ and 3 receiver antennas. The analytical curves in Figure 3 are the function $\min(P_{\text{PEP 1}}, P_{\text{PEP 2}})$, as defined in (12) and (18)[2].

Using the upper bounds of the Q-function which arises in the expression for the exact conditional PEP yields a gap between the analytical curves and the simulations. However we can see that the proposed PEP will give particularly tight upper bound on the simulation at high SNR.

6   Conclusion and Future Work

We have derived a closed form solution for the average PEP of space-time codes over a keyhole channel, as shown in (12) and (18). The block error rate performance obtained from our proposed PEP gives a good tight upper bound compared to simulation.

This PEP is an improvement on that given in [9], both in its simplicity and in that it applies even when the eigenvalues are not distinct (which arises, for example, in the Alamouti space-time code). On the other hand, note that the key upper bound (10) would not be tight in the case that there is a large disparity in the sizes of the eigenvalues. Hence future work should consider tightening the bound in this case, as well as extending to rank deficient codes.

¿From the proposed average PEPs presented in (12) and (18) and given that the generalized hypergeometric function $_2F_0$ is an increasing function, a design criterion of robust codes for a keyhole channel might be to maximize the $\lambda_{\min}$ of $\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger$ for all distinct codewords $\boldsymbol{c}$ and $\boldsymbol{e}$. An

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[2]The generalized hypergeometric function $_2F_0(a, b; ; -z)$ is computed using the MATLAB with the built-in command `hypergeom([a,b],[],-z)`.

extension work would be to find a method to construct space-time codes which meet this design criterion; this is a max-min eigenvalue problem that can be done by the power method using the POWER software [3].

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Appendix A: Hypergeometric Functions

All definitions of hypergeometric functions are from [5]. The generalized hypergeometric function is given by

$$_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

The degenerate hypergeometric function is defined as

$$\Psi(a, b; z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt, \quad \text{Re}\{a\} > 0$$

12

where $\Gamma(\cdot)$ denotes a gamma function.

On particular interest, the relationship of $\Psi(a, b; z)$ and ${}_pF_q$ is

$$ {}_2F_0(a, b; ; -z^{-1}) = z^a \Psi(a, a - b + 1; z). $$

Appendix B: 2×2 Orthogonal Space-time Code

The transmitted signal matrix of 2×2 orthogonal space-time block code is given in (24). Given two codewords $\boldsymbol{c} = \begin{bmatrix} x & y \\ -y^* & x^* \end{bmatrix}$ and $\boldsymbol{e} = \begin{bmatrix} x' & y' \\ -y'^* & x'^* \end{bmatrix}$, we have

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})\boldsymbol{B}(\boldsymbol{c}, \boldsymbol{e})^\dagger &= \begin{bmatrix} x - x' & y - y' \\ -(y - y')^* & (x - x')^* \end{bmatrix} \begin{bmatrix} (x - x')^* & -(y - y') \\ (y - y')^* & x - x' \end{bmatrix} \\
&= \begin{bmatrix} ||x - x'||^2 + ||y - y'||^2 & 0 \\ 0 & ||x - x'||^2 + ||y - y'||^2 \end{bmatrix}
\end{aligned}
$$

whose eigenvalues are $\lambda_1 = \lambda_2 = ||x - x'||^2 + ||y - y'||^2$. Hence the eigenvalues are not distinct, so the PEP and design criterion of equation (11) and (12) in [9] can not be applied. $\square$

Figure 1: Keyhole channel

Figure 2: The Q-function

Figure 3: Block-error rate of 2×2 orthogonal space-time block code at 1 bit/s/Hz for $N = 1, 2$ and 3 receiver antennas