On the Construction of Space-time Hamiltonian Constellations from Group Codes

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***Abstract*— Full diversity signal constellations for any numbers of transmitter antennas and for any orders which are constructed from 2×2 Hamiltonian matrices are investigated in this paper. The diversity product of a 2×2 Hamiltonian constellation equals one half of the Euclidean distance between two points in $\mathbb{C}^2$. By considering the transformation from $\mathbb{R}^4$ to $\mathbb{C}^2$, the idea of group codes[1] is used to construct a high diversity product constellation for any order L. The $(L, 4)$ cyclic group codes are considered to obtain L 4-dimensional codewords for group codes. We show that our 2×2 Hamiltonian constellations have higher diversity product than orthogonal and diagonal constellation designs. We extend our construction to the general case for any numbers of transmitter antennas $M > 2$ by using a direct sum of 2×2 Hamiltonian matrices for M even, and a direct sum of 2×2 Hamiltonian matrices with the Lth roots of unity for M odd. It is shown that these constellations outperform cyclic groups and some of those obtained using fixed-point free groups.**

I. Introduction

Consider multiple antennas in a Rayleigh flat-fading channel with M transmitter antennas and N receiver antennas. Let $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$ be an $M \times M$ unitary signal constellation, where $|\mathcal{V}| = L$. For high SNR ρ, the pairwise error probability, P_e, that the receiver antenna decodes an error from V_l to $V_{l'}$ can be computed by [3], [4], [5], [11] as

$$P_e \leq \frac{1}{2}(\frac{4\alpha}{\rho})^{MN} \frac{1}{|\det(V_l - V_{l'})|^{2N}} \tag{1}$$

where $\alpha = 1$ and 2 for known and unknown channel respectively. We define *a diversity product*, $\zeta_{\mathcal{V}}$, which is computed from a unitary signal constellation $\mathcal{V} = \{V_l\}_{l=0}^{L-1}$, using

$$\zeta_{\mathcal{V}} = \frac{1}{2} \min_{0 \leq l < l' \leq L-1} |\det(V_l - V_{l'})|^{\frac{1}{M}} \tag{2}$$

where $0 \leq \zeta_{\mathcal{V}} \leq 1$. A constellation $\mathcal{V}$ which has $\zeta_{\mathcal{V}} > 0$ is said to have *full diversity*. Clearly we want to minimize P_e in (1). Therefore a design criteria of our constellation is to find a unitary constellation $\mathcal{V}$ which has $\zeta_{\mathcal{V}}$ as large as possible.

The problem of constructing full diversity constellations with high diversity product has been studied in many prior works, including orthogonal designs [1], [10], and cyclic and dicyclic groups [4], [5], [6]. Fixed-point free groups [8] have excellent diversity products but have some design limitations. First the possible constellations are very limited when M is large and odd. Second there exist only even order for 2×2 constellations. Parametric code design [7] was proposed for only two transmitter antennas. The search for parametric codes with optimal diversity products is computationally intensive for large L as it requires exhaustively searching three parameter values over $L(L-1)/2$ distinct pairs of V_l and $V_{l'}$ in (2).

A 2×2 Hamiltonian matrix is defined in [8] to be a full diversity constellation for two transmitter antennas which can be built from points on a unit sphere in $\mathbb{R}^4$. It was suggested that a sphere packing method can be used to obtain these points. Furthermore this proposed constellation was limited only to a 2×2 case. In this paper we use the idea of Slepian's group codes to obtain maximum equidistant points in $\mathbb{R}^4$ to construct a 2×2 Hamiltonian constellation. We extend this construction to the general case of an $M \times M$ constellation for any order L, and also show that our constellations can achieve a large diversity product. The advantage of Hamiltonian constellation is the optimization of a diversity product is not computationally intensive for large L. It only requires checking $L - 1$ distinct matrices, making it comparable to those that use group constellations.

II. Hamiltonian Constellation Design

A 2×2 Hamiltonian matrix can be used to design a signal constellation for two transmitter antennas. This matrix is defined by

$$H = \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix} \tag{3}$$

where $x, y \in \mathbb{C}$ and $|x|^2 + |y|^2 = 1$. This differs from orthogonal design [1], [10] which requires $|x|^2 = |y|^2 = 1$. From (2), the diversity product $\zeta_{\mathcal{H}}$ can be computed as

$$\begin{aligned} \zeta_{\mathcal{H}} &= \frac{1}{2}|\det(H - H')|^{\frac{1}{2}} \\ &= \frac{1}{2}\sqrt{|x - x'|^2 + |y - y'|^2}. \end{aligned} \tag{4}$$

From (4), we can easily see that now $\zeta_{\mathcal{H}}$ equals one half of Euclidean distance between two points (x, y) and (x', y') in $\mathbb{C}^2$.

[1]We use the term *group codes* refered to the original definition of Slepian's group codes in [9]

Consider a transformation from $\mathbb{R}^4$ to $\mathbb{C}^2$. If $A(a_1, a_2, a_3, a_4)$ is a point on the unit sphere in $\mathbb{R}^4$ where $a_1^2+a_2^2+a_3^2+a_4^2 = 1$, then we can convert this point onto the unit sphere in $\mathbb{C}^2$ using the mapping:

$$A(a_1, a_2, a_3, a_4)_{\mathbb{R}^4} \longmapsto A(a_1 + ja_2, a_3 + ja_4)_{\mathbb{C}^2} = A(x, y)_{\mathbb{C}^2} \tag{5}$$

where $x = a_1+ja_2$ and $y = a_3+ja_4$. Here $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Consequently now the problem of constructing a Hamiltonian constellation can be reduced to finding L points on a unit sphere in $\mathbb{R}^4$ such that the minimum distance between two points as large as possible. A possible solution to this problem is to use *Slepian's group codes* [9] to get these maximum equidistant L points on a unit sphere in $\mathbb{R}^4$ when in the group codes these points are considered as a set of L codewords in four-dimensional space.

III. Group Codes

An (L, n) *group code* [9] is a set of L codewords in Euclidean space of dimension n. Basically we can think that all L codewords are on the surface of a unit sphere in n dimensional space. Let $\{O_l\}_{l=0}^{L-1}$ be a group of orthogonal $n\times n$ matrices. Then the codewords $\{X_l\}_{l=0}^{L-1}$ can be generated by

$$X_l = O_l X \tag{6}$$

where X is called *an initial vector*. The square distance, d^2, between X and X_l is computed using

$$d^2(X, X_l) = \| X - X_l \|^2 . \tag{7}$$

The main problem of Slepian's group codes is how to choose the best initial vector X to minimize the error probability, or to maximize the minimum distance of nearest codewords.

The $n \times n$ orthogonal matrix, O_l, of *an* $(L, 4)$ *cyclic group code* [2] has the block diagonal form:

$$O_l = \text{diag}(A(lk_1), A(lk_2)) \tag{8}$$

where $A(k_i)$ is defined by

$$A(k_i) = \begin{bmatrix} \cos\frac{2\pi}{L}k_i & \sin\frac{2\pi}{L}k_i \\ -\sin\frac{2\pi}{L}k_i & \cos\frac{2\pi}{L}k_i \end{bmatrix}. \tag{9}$$

Let $X = (x_1, x_2, x_3, x_4)$ be the best initial vector. We can compute the square Euclidean distance between X and $O_l X$ by

$$d_l^2 = \|X - O_l X\|^2 = 4\sum_{i=1}^{2} \mu_i \sin^2 \frac{\pi}{L} l k_i \tag{10}$$

where $\mu_1 = x_1^2 + x_2^2$ and $\mu_2 = x_3^2 + x_4^2$. We can observe that the distance does not depend on a single entry of X, but depends on the sum of a pair of entries' power. Using the property $\sin^2 \frac{\pi}{L} l = \sin^2 \frac{\pi}{L}(L - l)$ in (10), we have that

$$\|X - O_l X\|^2 = \|X - O_{L-l} X\|^2. \tag{11}$$

The problem of the best initial vector, X, which maximizes the distance in (10) is thus equivalent to finding

$$\max_X \min_{l=1,2,\ldots,\lfloor L/2 \rfloor} \|X - O_l X\|. \tag{12}$$

Let $d_{\min}^2$ be the maximum nearest distance. Clearly $d_l^2 \geq d_{\min}^2$, so

$$\sum_{i=1}^{2} \frac{4\mu_i}{d_{\min}^2} \sin^2 \frac{\pi}{L} l k_i \geq 1. \tag{13}$$

Define

$$y_1 = \frac{4\mu_1}{d_{\min}^2} \quad \text{and} \quad y_2 = \frac{4\mu_2}{d_{\min}^2}. \tag{14}$$

Since $\mu_1 + \mu_2 = 1$, we have that $z = y_1 + y_2 = 4/d_{\min}^2$. We want $d_{\min}^2$ as large as possible, therefore we need to minimize z. According to (13) and (14), μ_i is always positive and so is $d_{\min}^2$. Hence $\mathbf{y} = (y_1, y_2)$ must also satisfy

$$y_i \geq 0, \quad i = 1, 2 \tag{15}$$

$$\mathcal{A}\mathbf{y} \geq (1, 1, \ldots, 1) \tag{16}$$

where $\mathcal{A}$ is an $\lfloor L/2 \rfloor \times 2$ matrix which is defined by

$$(\mathcal{A})_{ij} = \sin^2 \frac{\pi}{L} i k_j \tag{17}$$

where $i = 1, 2, \ldots, \lfloor \frac{L}{2} \rfloor$ and $j = 1, 2$. From the method explained above, we can find a best initial vector, X, and the maximum $d_{\min}$ for any given L, k_1 and k_2 by the simplex programming method. For fixed L, different values of k_1, k_2 will give us different values of $d_{\min}$. The next step is to find k_1, k_2 which give the best $d_{\min}$ for fixed L. We observe that the number of possible choices of k_1, k_2 is $\binom{\lfloor L/2 \rfloor}{2}$ since k_i and $L - k_i$ will give us the same distance.

IV. 2×2 Constellation

From an initial vector $X = (x_1, x_2, x_3, x_4)$ and (14), we have

$$y_1 = \frac{4}{d_{\min}^2}(x_1^2 + x_2^2) \quad \text{and} \quad y_2 = \frac{4}{d_{\min}^2}(x_3^2 + x_4^2). \tag{18}$$

If we start by setting $x_2 = x_4 = 0$, then the initial vector will be

$$X = (x_1, 0, x_3, 0) = (\sqrt{\frac{d_{\min}^2 y_1}{4}}, 0, \sqrt{\frac{d_{\min}^2 y_2}{4}}, 0). \tag{19}$$

Generate the codewords by $X_l = O_l X$, where O_l is defined in (8). This gives

$$X_l = \begin{bmatrix} \sqrt{\frac{d_{\min}^2 y_1}{4}} \cos\frac{l2\pi k_1}{L} \\ -\sqrt{\frac{d_{\min}^2 y_1}{4}} \sin\frac{l2\pi k_1}{L} \\ \sqrt{\frac{d_{\min}^2 y_2}{4}} \cos\frac{l2\pi k_2}{L} \\ -\sqrt{\frac{d_{\min}^2 y_2}{4}} \sin\frac{l2\pi k_2}{L} \end{bmatrix}, \qquad l = 0, 1, 2, \ldots L-1. \tag{20}$$

Transforming these codewords to the new codewords in $\mathbb{C}^2$, $X_{l\mathbb{C}^2}$, by (5) gives

$$\begin{aligned} X_{l\mathbb{C}^2} &= \begin{bmatrix} \sqrt{\frac{d_{\min}^2 y_1}{4}}(\cos\frac{l2\pi k_1}{L} - j\sin\frac{l2\pi k_1}{L}) \\ \sqrt{\frac{d_{\min}^2 y_2}{4}}(\cos\frac{l2\pi k_2}{L} - j\sin\frac{l2\pi k_2}{L}) \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{\frac{d_{\min}^2 y_1}{4}} e^{-j\frac{l2\pi k_1}{L}} \\ \sqrt{\frac{d_{\min}^2 y_2}{4}} e^{-j\frac{l2\pi k_2}{L}} \end{bmatrix}. \end{aligned} \tag{21}$$

Substitute in a Hamiltonian matrix form in (3) to get a 2×2 Hamiltonian constellation $\mathcal{H} = \{H_l\}_{l=0}^{L-1}$ with

$$H_l = \begin{bmatrix} \sqrt{\frac{d_{\min}^2 y_1}{4}} e^{-j\frac{l2\pi k_1}{L}} & -\sqrt{\frac{d_{\min}^2 y_2}{4}} e^{j\frac{l2\pi k_2}{L}} \\ \sqrt{\frac{d_{\min}^2 y_2}{4}} e^{-j\frac{l2\pi k_2}{L}} & \sqrt{\frac{d_{\min}^2 y_1}{4}} e^{j\frac{l2\pi k_1}{L}} \end{bmatrix} \quad (22)$$

and the diversity product $\zeta_{\mathcal{H}} = d_{\min}/2$. For a given L, the 2×2 Hamiltonian constellation $\mathcal{H}$ constructed from (22) is very simple to compute by substituting the values $d_{\min}, k_1, k_2, y_1, y_2$ obtained by linear programming as explained in section III. We compare the diversity product of 2×2 Hamiltonian constellations among orthogonal constellation [10] and diagonal constellation designs [4], [5] in Table I.

L	R	ζ	Constellation designs
4	1.00	0.7071	quaternion group Q_1 [5]
4	1.00	0.7071	cyclic group $u = (1, 1)$ [5]
4	1.00	0.7071	orthogonal with 2th-roots of unity [10]
4	1.00	0.8165	$\mathcal{H}$ with $k = (1, 2), y = (1.0000, 0.5000)$
8	1.50	0.7071	quaternion group Q_2 [5]
8	1.50	0.5946	cyclic group $u = (1, 3)$ [5]
8	1.50	0.7071	$\mathcal{H}$ with $k = (1, 3), y = (1.0000, 1.0000)$
16	2.00	0.3827	quaternion group Q_3 [5]
16	2.00	0.3827	cyclic group $u = (1, 7)$ [5]
16	2.00	0.5000	orthogonal with 4th-roots of unity [10]
16	2.00	0.5098	$\mathcal{H}$ with $k = (1, 4), y = (2.0000, 1.8478)$
32	2.50	0.1951	quaternion group Q_4 [5]
32	2.50	0.2494	cyclic group $u = (1, 7)$ [5]
32	2.50	0.3827	$\mathcal{H}$ with $k = (1, 7), y = (3.3597, 3.4687)$
64	3.00	0.0980	quaternion group Q_5 [5]
64	3.00	0.1985	cyclic group $u = (1, 19)$ [5]
64	3.00	0.2706	orthogonal with 8th-roots of unity [10]
64	3.00	0.2816	$\mathcal{H}$ with $k = (1, 27), y = (7.9212, 4.6904)$
120	3.45	0.1353	cyclic group $u = (1, 43)$ [5]
121	3.46	0.1922	orthogonal with 11th-roots of unity [10]
121	3.46	0.2106	$\mathcal{H}$ with $k = (1, 22), y = (12.5987, 9.9387)$
128	3.50	0.0491	quaternion group Q_6 [5]
128	3.50	0.1498	cyclic group $u = (1, 47)$ [5]
128	3.50	0.2031	$\mathcal{H}$ with $k = (1, 35), y = (13.9707, 10.2745)$
240	3.95	0.1045	cyclic group $u = (1, 151)$ [5]
240	3.95	0.1511	$\mathcal{H}$ with $k = (1, 85), y = (18.2884, 25.5335)$
256	4.00	0.0245	quaternion group Q_7 [5]
256	4.00	0.0988	cyclic group $u = (1, 75)$ [5]
256	4.00	0.1379	orthogonal with 16th-roots of unity [10]
256	4.00	0.1477	$\mathcal{H}$ with $k = (1, 119), y = (25.3405, 20.5131)$

TABLE I

COMPARISON OF DIFFERENT CONSTELLATION DESIGNS, $M = 2$ (OUR CONSTELLATIONS ARE HIGHLIGHTED IN GREY)

V. EXTENSION TO $M \times M$ CONSTELLATION

The $M \times M$ signal constellations are constructed using the direct sum of 2×2 Hamiltonian matrices, as defined in (22), for the case where M is even, and direct sum of 2×2 Hamiltonian matrices with the Lth roots of unity for the case where M is odd. We begin with the cases of $M = 3$ and 4, then these will extend to general M odd and even respectively.

A. $M = 3$ Case

A 3×3 constellation is built using the direct sum of an Lth root of unity $\mu \in \{1, e^{j2\pi/L}, e^{j2\pi 2/L}, \ldots, e^{j2\pi(L-1)/L}\}$ and a 2×2 Hamiltonian matrix. Then a 3×3 constellation, $\mathcal{H}_{3\times3} = \{J_l\}_{l=0}^{L-1}$ has the form

$$J_l = \text{diag}(e^{j2\pi u_1 l/L}, H_{u_2 l}) \quad (23)$$

where $H_{u_2 l}$ is defined in (22) and u_1, u_2 are chosen from $\{1, 2, \ldots, L-1\}$. From the definition of diversity product, $\zeta_{\mathcal{H}}$ can be computed by

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{0 \le l < l' \le L-1} |\det(J_l - J_{l'})|^{\frac{1}{M}}. \quad (24)$$

To compute this easily, set $J_{l=0}$ and $J_{l'=l}$ (see the proof in Appendix I), then $\zeta_{\mathcal{H}} =$

$$\frac{1}{2} \min_{l=1,2,\ldots,L-1} \left\{ \left|1 - e^{j2\pi u_1 l/L}\right| \left[\frac{d_{\min}^2 y_1}{4} \left|1 - e^{j\frac{l2\pi u_2 k_1}{L}}\right|^2 + \frac{d_{\min}^2 y_2}{4} \left|1 - e^{j\frac{l2\pi u_2 k_2}{L}}\right|^2 \right] \right\}^{\frac{1}{3}}. \quad (25)$$

Using a property $|1 - e^{j\theta}| = 2 \sin \frac{\theta}{2}$, we will get the diversity product of a 3×3 constellation being computed as

$$\zeta_{\mathcal{H}} = \min_{l=1,2,\ldots,L-1} \left| \frac{d_{\min}^2}{4} \sin \frac{\pi u_1 l}{L} \sum_{i=1}^{2} y_i \sin^2 \frac{\pi u_2 k_i l}{L} \right|^{\frac{1}{3}}. \quad (26)$$

The next step is to search for $u = (u_1, u_2)$ such that the diversity product in (26) is maximum. (*Note*: for a given L, $d_{\min}, y_1, y_2, k_1$ and k_2 are known from section III.)

B. $M = 4$ Case

A 4×4 constellation is simply constructed using the direct sum of two 2×2 Hamiltonian matrices. Thus a 4×4 constellation, $\mathcal{H}_{4\times4} = \{J_l\}_{l=0}^{L-1}$ will be given by

$$J_l = \text{diag}(H_{u_1 l}, H_{u_2 l}) \quad (27)$$

where $H_{u_i l}$ is also defined in (22) and u_1, u_2 are chosen from $\{1, 2, \ldots, L-1\}$. Using a similar derivation as above, the diversity product can be computed as

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{l=1,2,\ldots,L-1} \left| d_{\min}^4 \prod_{j=1}^{2} \sum_{i=1}^{2} y_i \sin^2 \frac{\pi k_i u_j l}{L} \right|^{\frac{1}{4}}. \quad (28)$$

Then we search for the value of $u = (u_1, u_2)$ which maximizes $\zeta_{\mathcal{H}}$ in (28). A necessary condition for full diversity is that $gcd(u_i, L) = 1$ for $\forall i$. Hence we may restrict our search to choices of u such that $u_1 = 1$ and $\gcd(u_2, L) = 1$.

C. M Odd Case

The block diagonal form of an $M \times M$ constellation will be

$$J_l = \text{diag}(e^{j2\pi u_1 l/L}, H_{u_2 l}, H_{u_3 l}, \ldots, H_{u_{M+1/2} l}), \quad (29)$$

$l = 0, 1, \ldots, L-1$. The diversity product is given by $\zeta_{\mathcal{H}} =$

$$\frac{1}{2} \min_{l=1,2,\ldots,L-1} \left| d_{\min}^{M-1} 2 \sin \frac{\pi u_1 l}{L} \prod_{j=2}^{(M+1)/2} \sum_{i=1}^{2} y_i \sin^2 \frac{\pi k_i u_j l}{L} \right|^{\frac{1}{M}}. \quad (30)$$

And $u = (u_1, u_2, \ldots, u_{M+1/2})$ is found such that it maximizes $\zeta_{\mathcal{H}}$ in (30). We can restrict our search to choices of u such that $\gcd(u_i, L) = 1, i \geq 2$.

D. M Even Case

By taking a direct sum of $M/2$ 2×2 Hamiltonian matrices, an $M \times M$ constellation can be formed and will have the block diagonal form:

$$J_l = \text{diag}(H_{u_1 l}, H_{u_2 l}, \ldots, H_{u_{M/2} l}), \tag{31}$$

$l = 0, 1, \ldots, L-1$. The diversity product is given by

$$\zeta_{\mathcal{H}} = \frac{1}{2} \min_{l=1,2,\ldots,L-1} \left| d_{\min}^M \prod_{j=1}^{M/2} \sum_{i=1}^{2} y_i \sin^2 \frac{\pi k_i u_j l}{L} \right|^{\frac{1}{M}}. \tag{32}$$

Here we search again for $u = (u_1, u_2, \ldots, u_{M/2})$ which maximizes $\zeta_{\mathcal{H}}$ in (32). We can restrict our search to $u_1 = 1$ and $\gcd(u_i, L) = 1$ for $i = 2, \ldots, M/2$.

The diversity product of our $M \times M$ constellations is compared with cyclic [5] and fixed-point free group [8] designs in Table II. The value of u is found by an exhaustive search and is selected to maximize $\zeta_{\mathcal{H}}$ in (26), (28), (30) and (32) for $M = 3, 4, 5, 6$ respectively.

For the case of $M = 3$, we can see that our constellations have diversity product higher than cyclic group designs. In the case of $L = 9$, our constellation, $\mathcal{H}_{3\times 3}$ with $u = (1, 4), k = (1, 3)$ and $y = (1.3333, 1.1254)$, has excellent diversity product, $\zeta_{\mathcal{H}} = 0.6395$, which is even greater than for the fixed-point free group, $G_{9,1}$. For the case of $M = 4$, our constellation has diversity product lower than the cyclic group constellation designs at $R = 1$ but higher at $R = 2$. For $M = 5$ and 6, the diversity product of our constellation is still higher than for the cyclic group constellation design at $R = 1$. From Table II, we observe that our $M \times M$ constellation will be superior to cyclic group constellation designs for large L.

M	L	R	ζ	Constellation designs
3	8	1.00	0.5134	cyclic group $u = (1, 1, 3)$ [5]
3	8	1.00	0.5762	$\mathcal{H}$ with $u = (1, 1)$
3	9	1.06	0.6004	$G_{9,1}$ with $u = (1, 2, 5)$ [8]
3	9	1.06	0.6394	$\mathcal{H}$ with $u = (1, 4)$
3	63	1.99	0.3301	cyclic group $u = (1, 17, 26)$ [5]
3	63	1.99	0.3851	$G_{21,4}$ [8]
3	63	1.99	0.3371	$\mathcal{H}$ with $u = (1, 16)$
3	64	2.00	0.2765	cyclic group $u = (1, 11, 27)$ [5]
3	64	2.00	0.3230	$\mathcal{H}$ with $u = (1, 29)$
3	513	3.00	0.1353	$G_{171,64}(t = 19)$ [8]
3	513	3.00	0.1069	$\mathcal{H}$ with $u = (1, 176)$
4	16	1.00	0.5453	cyclic group $u = (1, 3, 5, 7)$ [5]
4	16	1.00	0.5098	$\mathcal{H}$ with $u = (1, 1)$
4	240	1.98	0.2145	cyclic group $u = (1, 31, 133, 197)$ [5]
4	240	1.98	0.5000	$K_{1,1,-1}$ [8]
4	240	1.98	0.3210	$\mathcal{H}$ with $u = (1, 91)$
4	256	2.00	0.2208	cyclic group $u = (1, 25, 97, 107)$ [5]
4	256	2.00	0.3110	$\mathcal{H}$ with $u = (1, 47)$
5	32	1.00	0.4095	cyclic group $u = (1, 5, 7, 9, 11)$ [5]
5	32	1.00	0.5384	$\mathcal{H}$ with $u = (1, 7, 11)$
6	64	1.00	0.3792	cyclic group $u = (1, 7, 15, 23, 25, 31)$ [5]
6	64	1.00	0.4953	$\mathcal{H}$ with $u = (1, 5, 21)$

TABLE II

COMPARISON OF DIFFERENT CONSTELLATION DESIGNS (OUR PROPOSED CONSTELLATIONS ARE HIGHLIGHTED IN GREY)

VI. PERFORMANCE

We consider the performance by plotting the block error rate, P_e against SNR, ρ as shown in Fig 1, 2, 3. All plots are considered in an unknown Rayleigh flat fading channel which use the differential modulation to transmit signals. The fading coefficients and additive noise are independent complex Gaussian variables with mean zero and variance one, $\mathcal{CN}(0, 1)$, and the channel matrix is assumed to be constant within two consecutive time periods [4], [5].

Fig. 1 compares the block error rate performance of different 2×2 constellations at $R \approx 3.45$ in Table I: Hamiltonian constellation with $L = 121, R = 3.46$, orthogonal design with 11th-root of unity $L = 121, R = 3.46$, cyclic group design $L = 120, R = 3.45$ and dicyclic group design with $L = 128, R = 3.50$. Our constellation outperforms the other three constellation designs. The block error rate performance at $M = 3, L = 64$ and $R = 2$ of Hamiltonian and cyclic group designs in Table II is shown in Fig. 2. Fig. 3 displays the block error rate performance at $L = 256, R = 2$ of Hamiltonian and cyclic group designs shown in Table II for $M = 4$ transmitter antennas. We can see that the performance of our $M \times M$ Hamiltonian constellations is better than cyclic group designs for both $M = 3$ and 4.

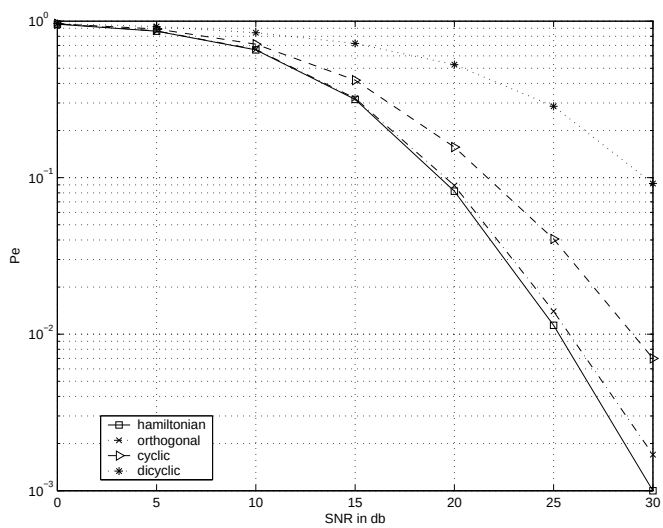

Fig. 1. Block error rate performance $M = 2, N = 1, R \approx 3.45$

VII. CONCLUSION

We have constructed an $M \times M$ full diversity unitary signal constellation for any M transmitter antennas and any order L using Hamiltonian matrices. Our constellation can be used for both the known and unknown channel. Slepian's group codes allow us construct high diversity product constellations for any order L. The Hamiltonian constellation in (22) which is obtained from a cyclic group code is shown to be very simple to construct. For the $M \times M$ case, $M > 2$, the value

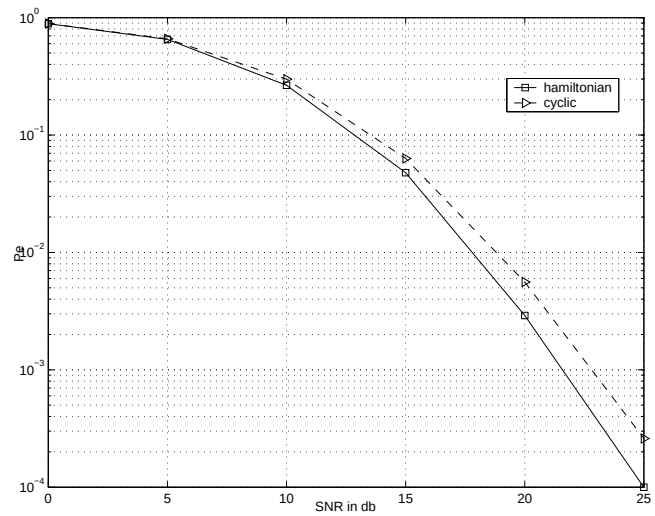

Fig. 2. Block error rate performance $M = 3, N = 1, L = 64, R = 2$

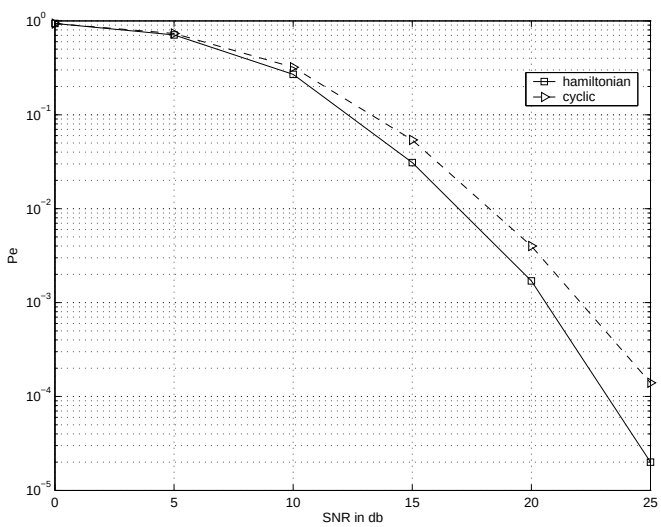

Fig. 3. Block error rate performance $M = 4, N = 1, L = 256, R = 2$

of $k = (k_1, k_2)$ of (30) and (32) is obtained from $\mathcal{H}_{2\times 2}$ in (22) which has a maximum $d_{\min}$. We have not examined for all different $k_i \in \{1, \ldots, \lfloor L/2 \rfloor\}$. In some cases, those different choices of k can make the Hamiltonian constellations of Table II have higher diversity product as shown in Table III. Another possible extension to this work is to find nongroup $M \times M$ matrices whose diversity product in (2) equals the Euclidean distance of two points in $\mathbb{C}^n$ for which we can use Slepian's group codes to obtain those maximum equidistant points.

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M	L	R	ζ	$\mathcal{H}$ with
3	8	1.00	0.6374	$k = (1, 2),\ u = (1, 3)$
4	16	1.00	0.6190	$k = (1, 5),\ u = (1, 7)$
5	32	1.00	0.5433	$k = (1, 9),\ u = (1, 13, 15)$

TABLE III

SOME CONSTELLATION $\mathcal{H}$ WITH THEIR BEST DIVERSITY PRODUCTS

APPENDIX I
PROOF OF DIVERSITY PRODUCT OF (24)

We prove that to compute the diversity product in (24) it suffices to choose $l = 0$ for J_l and arbitrary $l' = 1, \ldots, L-1$ for $J_{l'}$. The proof is shown for the 2×2 case, from which the general $M \times M$ case follows immediately. Using (22), we can write H_l in terms of H_0 as

$$H_l = e^{j\frac{2\pi l k_1}{L}} R_l H_0 T_l \tag{33}$$

where

$$R_l = \begin{bmatrix} e^{-j\frac{2\pi l k_1}{L}} & 0 \\ 0 & e^{-j\frac{2\pi l k_2}{L}} \end{bmatrix}, T_l = \begin{bmatrix} e^{-j\frac{2\pi l k_1}{L}} & 0 \\ 0 & e^{j\frac{2\pi l k_2}{L}} \end{bmatrix}.$$

Here $\mathcal{R} = \{R_l\}_{l=0}^{L-1}$ and $\mathcal{T} = \{T_l\}_{l=0}^{L-1}$ form cyclic groups of order L. We show that $|\det(H_l - H_{l'})| = |\det(H_0 - H_{l'-l})|$ for $0 \le l < l' \le L-1$.

Proof: $|\det(H_l - H_{l'})|$

$$\begin{aligned} &= |\det(e^{j\frac{2\pi l k_1}{L}} R_l H_0 T_l - e^{j\frac{2\pi l' k_1}{L}} R_{l'} H_0 T_{l'})| \\ &= |\det(H_0 - e^{j\frac{2\pi (l'-l) k_1}{L}} R_{l'-l} H_0 T_{l'-l})| \\ &= |\det(H_0 - H_{l'-l})| \end{aligned}$$

□

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