A Classification of p-adic Quadratic Forms

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Abstract

Two regular quadratic forms are considered to be equivalent if their matrices are congruent. There are, however, easier ways to find these equivalences, such as Witt's Chain-Equivalence Theorem, and a combination of dimension, determinant and Hasse invariant. As we will show, all p-adic quadratic forms of dimension 5 or higher are isotropic, and thus are equivalent to a lower-dimensional anisotropic quadratic form with some number of hyperbolic planes added (and possibly some totally isotropic subspace). Thus, to classify p-adic quadratic forms, it is sufficient to classify all anisotropic forms of dimension 4 or lower. We will use these methods to classify the p-adic quadratic forms, including finding explicit representatives of each equivalence class.

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1 Quadratic Forms

Before we begin, we will gather some useful definitions, notations and theorems. All results in this section will be stated without proof; they can mostly be found in Chapter 1 of [3], or in Chapter 3 of [1]. We will always be assuming that we are working over a field $\mathbb{F}$ with characteristic not equal to 2; in the p-adic case, we additionally assume that the residual characteristic is not 2, in other words $p \neq 2$.

Definition 1.1. A *quadratic form* over a field $\mathbb{F}$ is a homogeneous polynomial of degree 2 with coefficients in $\mathbb{F}$.

In other words, a quadratic form is a polynomial of the form

$$q(X_1, \dots X_n) = \sum_{i,j=1}^{n} a_{ij} X_i X_j \qquad a_{i,j} \in \mathbb{F}. \tag{1}$$

This can be written equivalently and symmetrically as

$$q(X_1, \dots X_n) = \sum_{i,j=1}^{n} a'_{ij} X_i X_j \qquad a'_{i,j} = \frac{1}{2}(a_{ij} + a_{ji}); \tag{2}$$

Using this notation, we can associate each quadratic form with a symmetric matrix $\left(a'_{ij}\right)$.

Definition 1.2. Two n-dimensional quadratic forms are considered to be *equivalent* if thier matrices are congruent; in other words, if f, g are two quadratic forms with associated matrices M_f, M_g, they are equivalent precisely when $\exists\, C \in GL(n)$ such that

$$M_f = C^t M_g C.$$

Definition 1.3. A quadratic form is called *regular* or *nondegenerate* when its associated matrix is non-singular, i.e. when its rank is equal to its dimension.

Unless otherwise stated, we will be assuming that all quadratic forms are regular; this will be justified later by Witt's Decomposition Theorem (Theorem 1.6).

There are two important properties of quadratic forms which we will use:

1. If q is an n-dimensional quadratic form over a field $\mathbb{F}$, then $\forall\, a \in \mathbb{F}, x \in \mathbb{F}^n$,
$$q(ax) = a^2 q(x).$$

2. If q is a quadratic form, there is a uniquely determined symmetric bilinear pairing on $\mathbb{F}^n$ which we associate to it, defined as
$$B_q(x, y) = [q(x + y) - q(x) - q(y)]/2.$$

Another common way to view quadratic forms is through their associated quadratic space; in other words, given an n-dimensional quadratic form q over a field $\mathbb{F}$, we can examine an n-dimensional $\mathbb{F}$-vector space equipped with the bilinear pairing B_q defined above.

Definition 1.4. Two quadratic spaces $(V, B), (V', B')$ are *isometric* if there exists a linear isomorphism $\tau : V \to V'$ which preserves the inner product on the space, i.e. $\forall\, x, y \in V$,
$$B(x, y) = B'(\tau(x), \tau(y)).$$

These two ways of viewing quadratic forms are equivalent to each other, and we will use them interchangeably. In particular, two quadratic forms are equivalent iff their associated quadratic spaces are isometric.

Notation. If two quadratic forms q, q' are equivalent, or if their associated quadratic spaces are isometric, we will write
$$q \cong q'.$$

Since every symmetric matrix is orthogonally diagonalizable, every quadratic form is equivalent to a diagonal form. It will thus be sufficient to look only at diagonal quadratic forms, i.e. those of the form $q = a_1X_1^2 + a_2X_2^2 + \cdots + a_nX_n^2$. These are equivalent to the quadratic spaces of the form $\langle a_1\rangle \perp \langle a_2\rangle \perp \ldots \perp \langle a_n\rangle$, where $\langle a_i\rangle$ is the 1-dimensional space associated to the form $q = a_iX^2$. Note that two permuted diagonal forms are equivalent; thus two permuted quadratic forms are equivalent.

Notation. If $q = a_1X_1^2 + a_2X_2^2 + \cdots + a_nX_n^2$, we will often abbreviate this as $\langle a_1, a_2, \ldots, a_n\rangle$.

Another important simplification we shall note requires the use of isotropic vectors and hyperplanes.

Definition 1.5. Let (V, B_q) be a quadratic space.

1. A non-zero vector $v \in V$ is called *isotropic* (as a vector) if $B_q(v, v) = 0$, i.e. if $q(v) = 0$. Otherwise, it is called *anisotropic.*

2. If (V, B_q) contains at least one isotropic vector, then V is called *isotropic* (as a vector space), and q is called *isotropic* (as a quadratic form). Otherwise, V and q are both called *anisotropic.*

3. If all non-zero vectors in a space are isotropic, then the space is called *totally isotropic.* (In the case of regular (i.e. nondegenerate) quadratic forms, the associated space will never be totally isotropic, but it may have a totally isotropic subspace.)

4. If a regular quadratic space (V, B) contains one isotropic vector v (i.e. V is isotropic), it must also contain a second isotropic vector w such that $B(v, w) = 1$. The span of these two vectors will be a basis for a 2-dimensional subspace called a *hyperbolic plane.*

5. A space which is the orthogonal sum of any number of hyperbolic planes is called a *hyperbolic space.*

Any hyperbolic plane is isometric to $\langle 1, -1\rangle$. This can be seen from the fact that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is a diagonalization of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which is the matrix of the form associated to a hyperbolic plane relative to the basis described above.

We mentioned Witt's Decomposition Theorem above; it also has something to say about hyperbolic planes.

Theorem 1.6 (Witt's Decomposition Theorem)**.** *Any (not necessarily regular) quadratic space (V, B) splits into an orthogonal sum*

$$(V_a, q_a) \perp (V_h, q_h) \perp (V_t, q_t) \tag{3}$$

where V_a is anisotropic, V_h is hyperbolic, and V_t is totally isotropic, and each of these spaces are uniquely determined (up to isometry).

This theorem allows us to focus our classification on the anisotropic quadratic forms; any other quadratic form can be classified as simply the appropriate anisotropic quadratic form plus some number of hyperbolic planes and some appropriately sized totally isotropic space. In other words, any quadratic form can be written as

$$q = \langle a_1, a_2, \ldots, a_k\rangle \perp m \cdot \langle 1, -1\rangle \perp \langle 0, 0, \ldots, 0\rangle \tag{4}$$

where $m \cdot \langle 1, -1\rangle$ means m copies of $\langle 1, -1\rangle$. (This also makes it clear why it's fairly safe to ignore non-regular quadratic forms; looking at $q = aX_1^2 + 0X_2^2$ doesn't really add anything to the discussion that $q = aX_1^2$ doesn't already cover.)

One last important definition is that of the determinant. It would be ideal if we could define this simply as the determinant of the associated matrix; however, it turns out that this is not an invariant. However, it *is* invariant up to a square factor, which motivates the following definition.

Definition 1.7. Let $\mathbb{F}$ be a field with multiplicative group $\dot{\mathbb{F}}$. Let q be a quadratic form with matrix representation M_q. Then the *determinant* $\det q$ is the element of $\dot{\mathbb{F}}/\dot{\mathbb{F}}^2$ such that

$$\det q = \det M_q \cdot \dot{\mathbb{F}}^2 \tag{5}$$

With this definition, two quadratic forms can only be equivalent if they have the same determinant; however, this is only a necessary condition, not a sufficient one.

Some simple matrix calculations (conjugating by diagonal matrices) also show that two diagonal forms are equivalent if their coefficients differ only by square factors. We can thus use square classes to describe both the determinant and coefficients of quadratic forms.

2 p-adics, Square Classes and 1-Dimensional Forms

2.1 Square Classes of p-adic Fields

We first restrict our attention to the case where $\mathbb{F} = \mathbb{Q}_p$, $p \neq 2$. Recall that every non-zero p-adic number x can be written as the product $x = p^n v$, where $v \in \dot{\mathbb{Z}}_p$, the multiplicative group of p-adic integers (ie those with valuation 0). In other words,

$$\dot{\mathbb{Q}}_p \cong \dot{\mathbb{Z}}_p \times \mathbb{Z}.$$

Hensel's Lemma allows us to reduce the question of whether v is a square to a question of whether it is a quadratic residue mod p, in other words whether the coefficient of p^0 (which must be non-zero since v has valuation 0) is a square in $\dot{\mathbb{F}}_p$. Meanwhile, p never has a square root in $\mathbb{Q}_p$, so p^n is a square iff $2|n$. This means that in every case,

$$\mathbb{Q}_p/\mathbb{Q}_p^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \{1, r, p, rp\} \tag{6}$$

where r is any quadratic non-residue modulo p.

The p-adic numbers may be extended to a p-adic (extension) field in one of two essential ways (or a combination thereof): by adjoining some number of roots of Teichmuller representatives or roots of the residue field, or by adjoining some number of roots of p. In the case of the former, what changes is that the coefficients are now drawn from some finite field, rather than simply some prime field; in the case of the latter, what changes is that the valuation is now relative to the generator of the prime ideal of the integer ring, also known as the uniformizing element or conductor.

Definition 2.1. If a p-adic field is extended by adjoining roots of p, the extension is called a *ramified* extension; if it is extended by ajoining roots of the residue field, it is called an *unramified* extension. If a field is extended by a combination of the two, we can talk about the ramified and unramified *parts* of the extension.

Notation. We will use the notation $\mathbb{K}_p$ to mean any p-adic field of residual characteristic p, including $\mathbb{Q}_p$ and any extensions thereof, with multiplicative group $\dot{\mathbb{K}}_p = \mathbb{K}_p - \{0\}$.

Unless otherwise stated we will be assuming that the quadratic forms we are dealing with are defined over some fixed p-adic field $\mathbb{K}_p$.

In the first case the multiplicative group of any finite field is cyclic and thus still divides into squares and non-squares, and in the second the uniformizing element, being essentially the smallest positive root of p, will by definition have no square root. Thus, we obtain essentially the same result as (6):

$$\mathbb{K}_p/\mathbb{K}_p^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \{1, \rho, \pi, \rho\pi\}\,, \tag{7}$$

where $\mathbb{K}_p$ is some field extension of $\mathbb{Q}_p$ with smallest positive root of p π, and ρ is a non-square with valuation 0. For more information about field extensions of the p-adics, see [2].

A common issue we will come across will be that some finite fields, and therefore some p-adic fields, contain -1 as a square, specifically those where $p \equiv 1 \pmod 4$ and/or the degree of the unramified part of the extension is divisible by 2. This causes some fundamental differences between the two classifications: notably, if $-1 \in \dot{\mathbb{K}}_p^2$, then $\langle 1, 1\rangle \cong \langle 1, -1\rangle$, in other words $\langle 1, 1\rangle$ is a hyperbolic plane. This is certainly not the case when $-1 \notin \dot{\mathbb{K}}_p^2$ (ie $p \equiv 3 \pmod 4$ and the degree of the unramified part of the extension is odd). Although we will be able to circumvent this with some clever representative choices, it is important to note that a look at the deeper structure will reveal a significant difference between the two cases. In fact, our representative choices may be representing very different equivalence classes in the two cases.

However, in the 1-dimensional case, the classification is very simple and does not differ between the two cases. Conjugation by a 1-dimensional matrix is equivalent to multiplication by a square scalar; so the 1-dimensional anisotropic quadratic forms are essentially equivalent to the p-adic square classes. In other words, all 1-dimensional anisotropic quadratic forms over the p-adics are equivalent to exactly one of

$$\langle 1\rangle, \langle \rho\rangle, \langle \pi\rangle, \langle \rho\pi\rangle. \tag{8}$$

2.2 Other Useful Results

The following results will prove useful in our classifications.

Lemma 2.2. *Let $\mathbb{K}_p$ be a p-adic field, $\mathbb{K}_p(\alpha)$ an unramified extension by $\alpha^2 = \rho$, ρ a non-square of valuation 0 in $\mathbb{K}_p$. Then α is a square in $\mathbb{K}_p(\alpha) \Leftrightarrow -1 \notin \dot{\mathbb{K}}_p^2$.*

Proof. Suppose -1 is not a square in the base field; without loss of generality, set $\rho = -1$, $\alpha^2 = -1$. Then in $\mathbb{K}_p(\alpha)$,

$$\left(\frac{\alpha+1}{\sqrt{2}}\right)^2 = \frac{\alpha^2 + 2\alpha + 1}{2} = \frac{-1 + 2\alpha + 1}{2} = \alpha.$$

Conversely, suppose that α is a square. Then, since every element of $\mathbb{K}_p(\alpha)$ is of the form $a + b\alpha$, $a, b \in \mathbb{K}_p$, there must exist $a, b \in \mathbb{F}$ such that

$$\begin{aligned}
&(a + b\alpha)^2 = \alpha \\
&\Leftrightarrow a^2 + 2ab\alpha + b^2\alpha^2 = \alpha \\
&\Leftrightarrow a^2 + b^2\rho + 2ab\alpha = \alpha \\
&\Rightarrow a^2 + b^2\rho = 0 \\
&\Leftrightarrow a^2 = -\rho b^2 \\
&\Leftrightarrow \rho = -1 \cdot x^2 \quad \text{for some } x \in \mathbb{K}_p \\
&\Rightarrow -1 \text{ is not a square in } \mathbb{K}_p.
\end{aligned}$$

□

Theorem 2.3. *Let $a = \pi^n v$ be an element in any p-adic field $\mathbb{K}_p$, where v has valuation 0. If v is a square, then for some $b, c \in \dot{\mathbb{K}}_p$ where b and c are non-squares with valuation 0, $a = \pi^n b + \pi^n c$. If v is a non-square, then for some $d, e \in \dot{\mathbb{K}}_p$ where d and e are squares with valuation 0, $a = \pi^n d + \pi^n e$.*

Proof. Since $v = a + b \Rightarrow \pi^n v = \pi^n a + \pi^n b$, it is sufficient to prove this result for elements with valuation zero. Therefore, using Hensel's Lemma, this result is equivalent to the same result in the case of finite fields; we will therefore proceed to prove this in the finite field case.

We begin by showing that it is sufficient to prove this in a single case. Let $\mathbb{F}$ be a finite field with multiplicative group $\dot{\mathbb{F}}$. Suppose that we have shown that for some $x \in \dot{\mathbb{F}}$, $x^2 = r + s$, where $r, s \in \dot{\mathbb{F}}$ are non-squares. Then, in $\dot{\mathbb{F}}/\dot{\mathbb{F}}^2$,

$$\begin{aligned}
&x^2 \cdot \dot{\mathbb{F}}^2 = r \cdot \dot{\mathbb{F}}^2 + s \cdot \dot{\mathbb{F}}^2 \Rightarrow 1 \cdot \dot{\mathbb{F}}^2 = r \cdot \dot{\mathbb{F}}^2 + r \cdot \dot{\mathbb{F}}^2 \\
&\Rightarrow \forall y \in \dot{\mathbb{F}}, \exists a, b \in \dot{\mathbb{F}},\ a, b \text{ non-squares, such that } y^2 = a + b,
\end{aligned}$$

and moreover by multiplying through by a non-square, say r, we get

$$\begin{aligned}
&r \cdot \dot{\mathbb{F}}^2 = r^2 \cdot \dot{\mathbb{F}}^2 + r^2 \cdot \dot{\mathbb{F}}^2 \Rightarrow s \cdot \dot{\mathbb{F}}^2 = 1 \cdot \dot{\mathbb{F}}^2 + 1 \cdot \dot{\mathbb{F}}^2 \\
&\Rightarrow \forall s \in \dot{\mathbb{F}}, \text{s a non-square}, \exists c, d \in \dot{\mathbb{F}} \text{ such that } s = c^2 + d^2.
\end{aligned}$$

So, we need only show that a single square element can be written as the sum of two non-squares to prove the theorem.

Suppose that $\mathbb{F}_p$ is a prime field. Then $\mathbb{F}_p$ is additively generated by the element 1; so there must be a non-square r that is one more than a square x^2. So, $r = x^2 + 1^2$ and we are done.

Suppose now that $\mathbb{F}_n$ is any finite field. Since the finite field of any given order is unique up to isomorphism, we can view any finite field as a field extension of some $\mathbb{F}_p$ created by adjoining roots. If the degree of the field extension is odd, then there can have been no quadratic roots adjoined, and so any squares and non-squares in the base field will still be squares and non-squares with the same sum; thus we can simply use the prime field calculation.

Suppose then that the degree of the field extension is even; say the degree is $2^m k$, where k is odd. We use $[E : F]$ to notate the degree of a field extension E over a field F. Then,

$$[\mathbb{F}_n : \mathbb{F}_p] = 2^m k = [\mathbb{F}_n : \mathbb{F}_{2^m p}][\mathbb{F}_{2^m p} : \mathbb{F}_{2^{m-1} p}] \cdots [\mathbb{F}_{2p} : \mathbb{F}_p]$$

Again we use the fact that the finite field of any given order is unique up to isomorphism to say that since it is possible to construct a finite field of order n in this way, we can treat $\mathbb{F}_n$ as having been constructed in this way. Each step of this chain except the last is a quadratic extension; the last step can be done by adjoining a k-th root to $\mathbb{F}_{2^m p}$. As argued above this last step will not add any quadratic roots, and thus if the result is true for $\mathbb{F}_{2^m p}$ we can use the same equation in $\mathbb{F}_n$ to prove it true for $\mathbb{F}_n$. Thus we can simply use an inductive argument on the length of the chain of quadratic extensions.

Suppose for some finite field $\mathbb{F}$, every non-square element ρ is the sum of two square elements. Fix some non-square ρ. Let $\mathbb{F}(\alpha)$ be the extension field where $\alpha^2 = \rho$. Every non-square in the original field is now a square; α may be a square or may not be a square.

If α is a square, then since every element of $\mathbb{F}$ is now a square and every element of $\mathbb{F}(\alpha)$ can be written as $a + b\alpha$ for some $a, b \in \mathbb{F}$, it is trivial to write any element as the sum of two squares. So, suppose α is not a square. Then 2α is not a square either. However,

$$(\alpha + 1)^2 = \alpha^2 + 1 + 2\alpha \Rightarrow (\alpha + 1)^2 - (\alpha^2 + 1) = 2\alpha;$$

since every element of F is now a square, $-(\alpha^2 + 1)$ must be a square, and therefore there is a non-square that is the sum of two squares. $\square$

3 2-Dimensional Forms

Unfortunately, dimension and determinant cease to be sufficient for a classificaion of quadratic forms when we reach the two-dimensional case. Before we can classify these forms, we will need a few more definitions and results.

3.1 Equivalence

Definition 3.1. A quadratic form $q = \langle a_1, \dots a_n \rangle$ is said to *represent* an element $c \in \dot{\mathbb{K}}_p$ if $q(v) = c$ for some vector $v \in V$, i.e. if $\exists\, x_1, \dots x_n \in \mathbb{K}_p$ such that

$$c = a_1 x_1^2 + \dots a_n x_n^2.$$

Lemma 3.2. *Let $q = \langle a, b \rangle$ be a 2-dimensional quadratic form. Suppose q represents an element $c \in \dot{\mathbb{K}}_p$. Then $\exists\, d \in \mathbb{Q}_p$ such that $q \cong \langle c, d \rangle$.*

Proof. For this proof it will be most useful to think of the quadratic forms in terms of their quadratic spaces. Let (V, B_q) be the quadratic space associated to q. Then, if q represents c, there must be a vector $v \in V$ such that $q(v) = c$.

Recall that $q(av) = a^2 q(v)$; in other words, for any vector $u = xv \in \mathrm{span}(v)$, $q(u) = x^2 \cdot c$. But then this means that $\mathrm{span}(v) \cong \langle c \rangle$, so $\langle c \rangle$ is a subspace of V.

Recall that the *radical* of a subspace W in (V, B_q), denoted $W^\perp$, is the subspace of all elements $w^\perp \in V$ such that $\forall w \in W$, $B_q(w^\perp, w) = 0$. It is well known, for example by the rank-nullity theorem, that $\dim V = \dim W + \dim W^\perp$, for any subspace W of a regular quadratic space V. So, $\langle c \rangle^\perp$ must have dimension 1, and therefore must be the span of vector u orthogonal to v. If u is isotropic, then u is orthogonal to every vector in V and V is singular, contradicting its regularity; so u must be anisotropic. This means that $\exists d \in \dot{\mathbb{K}}_p$ such that $q(u) = d$. Thus $\mathrm{span}(u) \cong \langle d \rangle$, $V \cong \langle c \rangle \perp \langle d \rangle$, and therefore $q \cong \langle c, d \rangle$, as desired. □

Theorem 3.3. *Let $q_1 = \langle a, b \rangle, q_2 = \langle c, d \rangle$ be 2-dimensional quadratic forms. Then $q_1 \cong q_2$ iff* $\det q_1 = \det q_2$ and *there exists a common element $e \in \mathbb{K}_p$ represented by both q_1 and q_2.*

Proof. This follows almost immediately from the above lemma. Clearly, if two forms are equivalent, they must have the same determinant and represent the same elements. Conversely, suppose q_1, q_2 both represent some element $e \in \mathbb{K}_p$. If $e = 0$ then they are both isotropic, and thus are both equivalent to the hyperbolic plane $\langle 1, -1 \rangle$; so we can assume $e \neq 0$, i.e. $e \in \dot{\mathbb{K}}_p$. Now by the lemma, $\exists f, f' \in \dot{\mathbb{K}}_p$ such that $q_1 \cong \langle e, f \rangle, q_2 \cong \langle e, f' \rangle$. But $\det q_1 = \det q_2 \Rightarrow ef \cdot \dot{\mathbb{Q}}_p^2 = ef' \cdot \dot{\mathbb{Q}}_p^2$. So f, f' must be in the same square class; and since quadratic forms are equivalent if their coefficients differ only by squares, this means that $q_1 \cong q_2$. □

So, the problem of classification of 2-dimensional forms comes down to looking at forms with the same determinant, and determining when they represent a common element.

3.2 Classification

We will begin by stating our proposition; to prove it, we will go through the possibilities by determinant.

Proposition 3.4. *The 2-dimensional quadratic forms can be classified as follows:*

1. *If $-1 \in \mathbb{K}_p^2$, the anisotropic 2-dimensional forms are*

$$\langle 1, \pi \rangle, \langle 1, \rho \rangle, \langle 1, \rho\pi \rangle, \langle \rho, \pi \rangle, \langle \rho, \rho\pi \rangle, \langle \pi, \rho\pi \rangle;$$

   the only isotropic form is the hyperbolic plane $\langle 1, -1 \rangle$ (or $\langle 1, 1 \rangle$).

2. *If $-1 \notin \mathbb{K}_p^2$, the anisotropic 2-dimensional forms are*

$$\langle 1, 1 \rangle, \langle 1, \pi \rangle, \langle 1, \rho\pi \rangle, \langle \rho, \pi \rangle, \langle \rho, \rho\pi \rangle, \langle \pi, \pi \rangle;$$

   the only isotropic form is the hyperbolic plane $\langle 1, -1 \rangle$ (or $\langle 1, \rho \rangle$).

3.2.1 Determinant 1

Since diagonal quadratic forms are equivalent if their coefficients differ by squares, we can essentially treat the coefficients of diagonal quadratic forms as being elements of $\dot{\mathbb{K}}_p/\dot{\mathbb{K}}_p^2$. Using this method, there are four possibilities with determinant one:

$$\langle 1,1\rangle, \langle \rho,\rho\rangle, \langle \pi,\pi\rangle, \langle \rho\pi,\rho\pi\rangle.$$

Claim. $\langle 1,1\rangle \cong \langle \rho,\rho\rangle$

Proof. By Theorem 2.3, it is always possible to write a square of valuation 0 as the sum of two non-squares of valuation 0 and vice versa. Thus $\exists r,s \in \mathbb{K}_p$, non-square and with valuation 0, such that $r+s=1$; and since $r\cdot\dot{\mathbb{K}}_p^2 = s\cdot\dot{\mathbb{K}}_p^2 = \rho\cdot\dot{\mathbb{K}}_p^2$, $\exists a,b \in \dot{\mathbb{K}}_p$ such that $\rho a^2+\rho b^2 = 1$. Meanwhile for $\langle 1,1\rangle$, clearly $1\cdot 1^2+1\cdot 0^2 = 1$. Thus these forms share a common element and are therefore equivalent. □

Similarly, $\langle \pi,\pi\rangle \cong \langle \rho\pi,\rho\pi\rangle$.

Claim. $\langle 1,1\rangle \cong \langle \pi,\pi\rangle \Leftrightarrow -1 \in \dot{\mathbb{K}}_p^2$

Proof. If $-1 \in \dot{\mathbb{K}}_p^2$, then both of these will be isotropic, and thus isometric to the hyperbolic plane. Conversely, suppose $-1 \notin \dot{\mathbb{K}}_p^2$. Now,

$$\langle 1,1\rangle \cong \langle \pi,\pi\rangle \Leftrightarrow \exists x,y,a,b \in \mathbb{K}_p \text{ such that } x^2+y^2 = \pi(a^2+b^2).$$

Since we are talking about sums of elements which may not have valuation zero, we will first simplify the situation by dividing through this equation by π^m, where m is the least valuation. The equation will now be non-trivial if we look at it modulo π; specifically, modulo π the equation will either reduce to

$$x^2+y^2 \equiv 0 \pmod{\pi} \quad \text{or} \quad 0 \equiv a^2+b^2 \pmod{\pi},$$

with the non-zero side being that which contained at least one element with lowest valuation. Without loss of generality, we assume it reduces to $x^2+y^2 \equiv 0$ or, equivalently, $y^2 \equiv -x^2 \pmod{\pi}$. This is possible only if -1 is a square in $\mathbb{F}_{p^k}$, which implies it must be a square in $\mathbb{K}_p$ as well, contradicting our assumption. □

So if $-1 \in \dot{\mathbb{K}}_p^2$, there is one equivalence class with determinant one, and it is the hyperbolic plane; otherwise, there are two inequivalent anisotropic equivalence classes with determinant one, $\langle 1,1\rangle$ and $\langle \pi,\pi\rangle$.

3.2.2 Determinant ρ

The forms with determinant ρ work, in essence, the opposite way that the forms with determinant 1 did, with respect to their behaviour in relation to whether or not $-1 \in \dot{\mathbb{K}}_p^2$. An interesting way to look at this is that it is always the 2-dimensional forms with determinant -1 which are isotropic, whereas (as we shall see) there are always two anisotropic forms with any other given determinant; the question is simply into which square class -1 falls.

After taking into account the equivalence of permutations, there are only two possibilities for forms with determinant ρ:

$$\langle 1, \rho\rangle, \langle \pi, \rho\pi\rangle$$

Claim. $\langle 1, \rho\rangle \cong \langle \pi, \rho\pi\rangle \Leftrightarrow -1 \notin \dot{\mathbb{K}}_p^2$

Proof. This proof is more or less symmetric to the proof for $\langle 1, 1\rangle$ and $\langle \pi, \pi\rangle$. If $-1 \notin \dot{\mathbb{K}}_p^2$, then without loss of generality we can assume $\rho = -1$. It is then clear that both forms are isotropic, and thus equivalent. Otherwise, the problem again reduces, this time to whether we can find a solution to

$$x^2 + \rho y^2 \equiv \pi(a^2 + \rho b^2) \equiv 0 \pmod{\pi}.$$

Since this requires that $x^2 \cong -\rho y^2 \pmod{\pi}$, $\rho \equiv -\left(\frac{x}{y}\right)^2 \pmod{\pi}$, so -1 must be a non-square. $\square$

3.2.3 Determinants π, $\rho\pi$

There are once again only two possibilities with determinant π:

$$\langle 1, \pi\rangle, \langle \rho, \rho\pi\rangle$$

Claim. $\langle 1, \pi\rangle \ncong \langle \rho, \rho\pi\rangle$

Proof. These represent a common element only if the equation $x^2 + \pi y^2 = \rho a^2 + \pi\rho b^2$ is solvable. Once again, we can simplify the situation by dividing the equation by π^m where m is the least valuation, making it non-trivial modulo π; then either

$$x^2 \equiv \rho a^2 \pmod{\pi} \quad \text{or} \quad y^2 \equiv \rho b^2 \pmod{\pi},$$

which are both absurd. $\square$

The argument for the forms with determinant $\rho\pi$ is similar; the options are

$$\langle \rho, \pi\rangle, \langle 1, \rho\pi\rangle$$

and the argument for them being distinct is almost identical to the one above. Thus, the anistropic forms are as originally claimed. $\square$

4 Witt's Chain Equivalence Theorem

The proof for Theorem 3.3 relied heavily on the fact that there were only two dimensions; as such, it is not particularly surprising that it no longer holds for higher dimensions. Luckily, Witt's Chain Equivalence Theorem allows us to essentially reduce the problem of classification to the two-dimensional case.

4.1 Witt's Chain Equivalence Theorem

As the name implies, Witt's Chain Equivalence Theorem was originally proven by Witt. Most of the results and definitions in this subsection can also be found in [3, Chapter 1].

Definition 4.1. Two diagonal forms $q = \langle a_1, \ldots, a_n \rangle, q' = \langle b_1, \ldots, b_n \rangle$ are called *simply equivalent* if, for some indices i and j,

1. $\langle a_i, a_j \rangle \cong \langle b_i, b_j \rangle$, and
2. $\forall k \neq i, j$, we have $a_k = b_k$

In other words, q_1 is simply equivalent to q_2 when one can be mapped to the other via a linear isomorphism that is an isometry on a two-dimensional subspace, and the identity everywhere else. It is clear that this implies isometry, and therefore equivalence in the normal sense.

Definition 4.2. Two diagonal forms $q = \langle a_1, \ldots, a_n \rangle, q' = \langle b_1, \ldots, b_n \rangle$ are called *chain equivalent* if there exists some sequence of forms

$$q = q_0, q_1, \ldots, q_{m-1}, q_m = q' \tag{9}$$

such that each q_i is simply-equivalent to q_{i-1}.

Notation. If two forms q, q' are chain equivalent, we will write

$$q \approx q'.$$

Once again, by transitivity, chain equivalence clearly implies normal equivalence. To prove the converse, we will need the following result, also by Witt:

Theorem 4.3 (Witt's Cancellation Theorem). *Let q, q_1, q_2 be arbitrary quadratic forms. Then $q \perp q_1 \cong q \perp q_2 \Leftrightarrow q_1 \cong q_2$.*

The proof of Witt's Cancellation Theorem is rather involved, so we will not cover it here. A proof of it can be found in Chapter 1 of [3].

Theorem 4.4 (Witt's Chain Equivalence Theorem). *Let f, g be two arbitrary diagonal quadratic forms of the same dimension. Then $f \cong g \Leftrightarrow f \approx g$.*

Proof. As stated above, $f \approx g \Rightarrow f \cong g$ follows easily from the transitivity of equivalence, so we need only prove the converse.

Any two equivalent forms will have the same totally isotropic subspace, so we can discount this and assume we are working with regular forms. Additionally, if the dimension is one or two this result is trivially true; so we will work by induction on the dimension for some $n \geqslant 3$. We can also freely permute the coefficients, since any permutation is a product of transpositions, and any two forms that are equal except for a transposition are simply-equivalent.

Now, suppose $f = \langle a_1, \ldots a_n \rangle$ and $g = \langle b_1 \ldots b_n \rangle, n \geqslant 3$. Since they are equivalent, they represent all the same elements; thus f must represent b_1, as must anything chain equivalent to f. Choose $f' = \langle c_1, \dots c_n\rangle$ such that $f' \approx f$ and $\langle c_1, \dots c_k\rangle$ represents b_1 with k the smallest possible. So there exist $e_1, \dots, e_k \in \mathbb{K}_p$ such that $b_1 = c_1 e_1^2 + \cdots + c_k e_k^2$. Since k is minimal, no subsum of this can equal zero, since then the remainder of the sum would also equal b_1 with fewer elements.

We claim $k = 1$. Suppose by way of contradiction that $k \geq 2$. Then $d = c_1 e_1^2 + c_2 e_2^2 \neq 0$, and thus (by Lemma 3.2) there exists some d' such that $\langle c_1, c_2\rangle \cong \langle d, d'\rangle$. But then

$$\begin{aligned} f \equiv f' &= \langle c_1, \dots c_k, \dots c_n\rangle \\ &\approx \langle d, d', c_3, \dots, c_k, \dots, c_n\rangle \\ &\approx \langle d, c_3, \dots c_k, \dots c_n, d'\rangle \end{aligned}$$

and $b_1 = d + c_3 e_3^2 + \cdots + c_k e_k^2$, which has $k - 1$ terms; this contradicts k's minimality. Thus, $k = 1$, as claimed. So, $f \approx \langle b_1, c_2, \dots, c_n\rangle$; we can now use Witt's Cancellation Theorem to cancel these equivalent first terms, in other words

$$\langle b_1, c_2, \dots, c_n\rangle \cong \langle b_1, b_2, \dots, b_n\rangle \Rightarrow \langle c_2, \dots, c_n\rangle \cong \langle b_2, \dots, b_n\rangle.$$

This now is a form with $n - 1$ elements; by the inductive hypothesis, it follows that $\langle c_2, \dots, c_n\rangle \approx \langle b_2 \dots, b_n\rangle$; and so, $f \approx \langle b_1, c_2, \dots, c_n\rangle \approx \langle b_1, b_2, \dots, b_n\rangle = g$, and thus $f \approx g$, as desired. □

4.2 3-Dimensional Forms

We can use a combination of Witt's Cancellation Theorem and Witt's Chain Equivalence Theorem to easily find representatives for the 3-dimensional quadratic forms. Of course, this does not necessarily give as any way to quickly tell which representative any given quadratic form is equivalent to; for that, we will need the Hasse invariant, defined in the next section.

Proposition 4.5. *The 3-dimensional quadratic forms can be classified as follows:*

1. *If $-1 \in \dot{\mathbb{K}}_p^2$, the anisotropic 3-dimensional forms are*

$$\langle 1, \rho, \pi\rangle, \langle 1, \rho, \rho\pi\rangle, \langle 1, \pi, \rho\pi\rangle, \langle \rho, \pi, \rho\pi\rangle.$$

2. *If $-1 \notin \dot{\mathbb{K}}_p^2$, the anisotropic 3-dimensional forms are*

$$\langle 1, 1, \pi\rangle, \langle 1, 1, \rho\pi\rangle, \langle 1, \pi, \pi\rangle, \langle \rho, \pi, \pi\rangle.$$

3. *In either case, the isotropic forms are*

$$\langle 1, -1, 1\rangle, \langle 1, -1, \rho\rangle, \langle 1, -1, \pi\rangle, \langle 1, -1, \rho\pi\rangle.$$

Proof. By Witt's Decomposition Theorem (Theorem 1.6) and Witt's Cancellation Theorem (Theorem 4.3), the isotropic forms must be the 1-dimensional forms with an added hyperbolic plane; in other words they must be precisely the four classes $\langle x\rangle\perp\langle 1,-1\rangle, x \in \{1, \rho, \pi, \rho\pi\}$. Thus any three-dimensional forms which are not equivalent to one of these forms must be anisotropic.

We can take any two of distinct 2-dimensional forms $q_1 = \langle a, b\rangle$, $q_2 = \langle c, d\rangle$ and construct $q_1' = \langle 1, a, b\rangle$ and $q_2' = \langle 1, c, d\rangle$. By Witt's Cancellation Theorem (Theorem 4.3), these forms will also be distinct. We can therefore use Proposition 3.4 to construct seven three-dimensional classes (not necessarily anisotropic) in this way for each case:

- $-1 \in \dot{\mathbb{K}}_p^2$: (Recall that $\langle -1\rangle \cong \langle 1\rangle$; we write hyperbolic planes as $\langle 1, -1\rangle$ for consistancy.)

$$\langle 1,-1,1\rangle, \langle 1,-1,\rho\rangle, \langle 1,-1,\pi\rangle, \langle 1,-1,\rho\pi\rangle, \langle 1,\rho,\pi\rangle, \langle 1,\rho,\rho\pi\rangle, \langle 1,\pi,\rho\pi\rangle$$

- $-1 \notin \dot{\mathbb{K}}_p^2$: (We make use of the fact that we can set $\rho = -1$, but only do so in the case of hyperbolic planes, again for consistancy.)

$$\begin{aligned}
&\langle 1,1,\rho\rangle \cong \langle 1,-1,1\rangle \\
&\langle 1,1,1\rangle \approx \langle 1,\rho,\rho\rangle \cong \langle 1,-1,\rho\rangle \\
&\langle 1,-1,\pi\rangle, \langle 1,-1,\rho\pi\rangle, \langle 1,1,\pi\rangle, \langle 1,1,\rho\pi\rangle, \langle 1,\pi,\pi\rangle
\end{aligned}$$

This necessarily covers all the forms which include $\langle 1\rangle$; thus we can find any remaining forms by looking at all the combinations of $\rho, \pi, \rho\pi$.

If $-1 \in \dot{\mathbb{K}}_p^2$:

$$\forall x, \langle \rho,\rho,x\rangle \approx \langle 1,1,x\rangle \tag{10}$$

$$\forall x, \langle \pi,\pi,x\rangle \approx \langle 1,1,x\rangle \tag{11}$$

$$\forall x, \langle \rho\pi,\rho\pi,x\rangle \approx \langle 1,1,x\rangle \tag{12}$$

We therefore need only look at the one combination which has no doubles,

$$q = \langle \rho, \pi, \rho\pi\rangle.$$

Since q has determinant 1, the only form it could be equivalent to would be $\langle 1,-1,1\rangle$, which is isotropic; however q is anisotropic, since

$$\begin{aligned}
\rho x^2 + \pi y^2 + \rho\pi z^2 = 0 &\Leftrightarrow \rho x^2 + \pi(y^2 + \rho z^2) = 0 \\
&\Rightarrow x \equiv 0, y^2 \equiv -\rho z^2 \pmod{\pi}
\end{aligned}$$

which is impossible since -1 is a square. So, q is anisotropic and thus not equivalent to $\langle 1,-1,1\rangle$.

If $-1 \notin \dot{\mathbb{K}}_p^2$:

$$\forall x, \langle \rho,\rho,x\rangle \approx \langle 1,1,x\rangle \tag{13}$$

$$\forall x, \langle \pi,\rho\pi,x\rangle \approx \langle 1,\rho,x\rangle \tag{14}$$

$$\forall x, \langle \rho\pi,\rho\pi,x\rangle \approx \langle \pi,\pi,x\rangle \tag{15}$$

There are now only two forms which have not been shown to be equivalent to a form containing a 1, $\langle \pi, \pi, \pi \rangle$ and $\langle \rho, \pi, \pi \rangle$. However,

$$\langle \pi, \pi, \pi \rangle \approx \langle \pi, \rho\pi, \rho\pi \rangle \approx \langle 1, \rho, \rho\pi \rangle, \tag{16}$$

so we need only look at $\langle \rho, \pi, \pi \rangle$. This form has determinant ρ, and thus again could only be equivalent to $\langle 1, -1, 1 \rangle$, an isotropic form; however it is anisotropic since

$$\begin{aligned} \rho x^2 + \pi y^2 + \pi z^2 = 0 &\Leftrightarrow \rho x^2 + \pi(y^2 + z^2) = 0 \\ &\Rightarrow x \equiv 0, z^2 \equiv -y^2 \pmod{\pi} \end{aligned}$$

and since -1 is not a square, $-y^2$ is not a square. Thus, this form is not anisotropic, and thus not equivalent to $\langle 1, -1, 1 \rangle$. □

4.3 Forms with Dimension $\geqslant 4$

Interestingly, things actually get simpler after three dimensions, as shown by the following result.

Theorem 4.6. *There is only one anisotropic 4-dimensional form, and it is the form $\langle 1, \rho, \pi, \rho\pi \rangle$ when $-1 \in \dot{\mathbb{K}}_p^2$ and $\langle 1, 1, \pi, \pi \rangle$ when $-1 \notin \dot{\mathbb{K}}_p^2$. There are no anisotropic forms of dimension 5 or higher.*

Proof. When $-1 \in \dot{\mathbb{K}}_p^2$, any four-dimensional form other than the one listed, and any five or higher dimensional form, must have at least one square class repeated twice in the coefficients. Since

$$\langle \rho, \rho \rangle \cong \langle \pi, \pi \rangle \cong \langle \rho\pi, \rho\pi \rangle \cong \langle 1, 1 \rangle \cong \langle 1, -1 \rangle,$$

it follows that any such form represents zero and thus is, by definition, isotropic. To see that $\langle 1, \rho, \pi, \rho\pi \rangle$ is anisotropic, note that for it to be otherwise requires that $\exists w, x, y, z \in \mathbb{K}_p$ such that

$$w^2 + \rho x^2 + \pi y^2 + \rho\pi z^2 = 0.$$

Following our usual method, we divide out by of π^m where m is the least valuation among the terms of this equation, making it non-trivial and yet necessarily true modulo π; this leaves us with either $w^2 = -\rho x^2$ or $y^2 = -\rho z^2$, both of which are impossible since $-1 \in \dot{\mathbb{K}}_p^2$. Thus, $\langle 1, \rho, \pi, \rho\pi \rangle$ is anisotropic.

For the case when $-1 \notin \dot{\mathbb{K}}_p^2$, first note that

$$\langle 1, 1 \rangle \cong \langle \rho, \rho \rangle \text{ and } \langle \pi, \pi \rangle \cong \langle \rho\pi, \rho\pi \rangle; \tag{17}$$

so we can use chain equivalence to freely multiply pairs of identical coefficents by ρ.

Next, note that any form q which can be written as $q = \langle a, \rho a, b, c \rangle$ for some $a, b, c \in \dot{\mathbb{K}}_p / \dot{\mathbb{K}}_p^2$ will be isotropic. For example: (we assume without loss of generality that $\rho = -1$)

$$a \cdot 1^2 - a \cdot 1^2 + b \cdot 0^2 + c \cdot 0^2 = 0. \tag{18}$$

Moreover, if a form q' has a triple of identical coefficients, i.e. it is of the form $\langle a, a, a, b\rangle$, then by (17) it is equivalent to $\langle a, \rho a, \rho a, b\rangle$, which by the previous statement is also isotropic.

To summarize, when $-1 \notin \dot{\mathbb{K}}_p^2$, an anisotropic 4-dimensional form may not have both 1 and ρ as coefficients, nor may it have both π and $\rho\pi$; it also can have each of these coefficients at most twice. Since it is 4-dimensional and has only two options for coefficients, it must therefore have both of these coefficients twice; by chain-equivalence and (17), all such forms are therefore equivalent to $\langle 1, 1, \pi, \pi\rangle$.

We again check that this is in fact an anisotropic form by looking at the possibility the existence of $w, x, y, z \in \mathbb{K}_p$ such that

$$w^2 + x^2 + \pi(y^2 + z^2) = 0;$$

this again reduces to either $w^2 \equiv -x^2$ or $y^2 \equiv -z^2 \pmod{\pi}$, depending on the parity of the least valuation, which are both impossible since $-1 \notin \dot{\mathbb{K}}_p^2$. Thus, $\langle 1, 1, \pi, \pi\rangle$ is the only anisotropic form of dimension 4 or higher. □

4.4 Combined Representatives

We were able, when talking about the 3-dimensional isotropic forms, to find a representative that was accurate for both forms, even though the forms were structurally different (i.e. they had the same determinant). It turns out that we can extend this trick, using the observation that

$$\langle -1\rangle \cong \begin{cases} \langle 1\rangle & \text{when } -1 \in \dot{\mathbb{K}}_p^2 \\ \langle \rho\rangle & \text{when } -1 \notin \dot{\mathbb{K}}_p^2 \end{cases} \tag{19}$$

$$\langle -\rho\rangle \cong \begin{cases} \langle \rho\rangle & \text{when } -1 \in \dot{\mathbb{K}}_p^2 \\ \langle 1\rangle & \text{when } -1 \notin \dot{\mathbb{K}}_p^2 \end{cases} \tag{20}$$

and similarly for $-\pi, -\rho\pi$.

Table 1 shows what the representatives look like up to this point, both using this combined representation and the original representations we used previously. Be aware that some of the characteristics of the combined representations are different; for example, their determinant is usually different, and as we will see in the next section, their Hasse invariant may be as well.

5 Hasse Invariant

The general definition of the Hasse invariant for any field is quite complicated, takes up a few chapters of [3], and requires knowledge of the Brauer Group. Luckily, there is a simpler, but equivalent, construction for the p-adics, which can be found in [4]. This construction allows us to look at the Hasse invariant in terms of the Hilbert 2-symbol.

$-1 \in \dot{\mathbb{K}}_p^2$	$-1 \notin \dot{\mathbb{K}}_p^2$	Combined
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1\rangle$
$\langle \rho\rangle$	$\langle \rho\rangle$	$\langle \rho\rangle$
$\langle \pi\rangle$	$\langle \pi\rangle$	$\langle \pi\rangle$
$\langle \rho\pi\rangle$	$\langle \rho\pi\rangle$	$\langle \rho\pi\rangle$
$\langle 1,\rho\rangle$	$\langle 1,1\rangle$	$\langle 1,-\rho\rangle$
$\langle 1,\pi\rangle$	$\langle 1,\pi\rangle$	$\langle 1,\pi\rangle$
$\langle 1,\rho\pi\rangle$	$\langle 1,\rho\pi\rangle$	$\langle 1,\rho\pi\rangle$
$\langle \rho,\pi\rangle$	$\langle \rho,\pi\rangle$	$\langle \rho,\pi\rangle$
$\langle \rho,\rho\pi\rangle$	$\langle \rho,\rho\pi\rangle$	$\langle \rho,\rho\pi\rangle$
$\langle \pi,\rho\pi\rangle$	$\langle \pi,\pi\rangle$	$\langle \pi,-\rho\pi\rangle$
$\langle 1,\rho,\pi\rangle$	$\langle 1,1,\rho\pi\rangle$	$\langle 1,-\rho,-\pi\rangle$
$\langle 1,\rho,\rho\pi\rangle$	$\langle 1,1,\pi\rangle$	$\langle 1,-\rho,-\rho\pi\rangle$
$\langle 1,\pi,\rho\pi\rangle$	$\langle 1,\pi,\pi\rangle$	$\langle 1,\pi,-\rho\pi\rangle$
$\langle \rho,\pi,\rho\pi\rangle$	$\langle \rho,\pi,\pi\rangle$	$\langle \rho,\pi,-\rho\pi\rangle$
$\langle 1,\rho,\pi,\rho\pi\rangle$	$\langle 1,1,\pi,\pi\rangle$	$\langle 1,-1,\pi,-\rho\pi\rangle$

Table 1: Representatives of Anisotropic Quadratic Forms, Dimensions 1-4

5.1 The Hilbert Symbol

Definition 5.1. The Hilbert 2-symbol (a, b) for two elements a, b in a field $\mathbb{F}$ is defined as

$$(a,b) := \begin{cases} 1 & \exists x, y, z \in \mathbb{F}, \text{ not all zero, such that } ax^2 + by^2 = z^2 \\ -1 & \text{otherwise.} \end{cases} \tag{21}$$

Looking at this in terms of square classes for the p-adics, we know that for any such z, there will be some w such that $w^2z^2 = 1 = a(xw)^2 + b(yw)^2$. Therefore, we can define the Hilbert Symbol equivalently as

$$(a,b) := \begin{cases} 1 & \exists x, y \in \mathbb{K}_p \text{ such that } ax^2 + by^2 = 1 \\ -1 & \text{otherwise.} \end{cases} \tag{22}$$

Using the equivalence classes of 2-dimensional quadratic forms in Proposition 3.4 will yield the Hilbert symbol for elements of each pair of square classes; the Hilbert symbol is simply asking whether the form in question represents 1. The results can be found in Tables 2 and 3. Since the Hilbert Symbol is symmetric, I have only included the first instance of each pair.

5.2 The Hasse Invariant

Definition 5.2. Let $q = \langle a_1, \ldots, a_n\rangle$ be a quadratic form of dimension 2 or higher. The *Hasse invariant* of q is defined over the p-adics as

$$H(q) = \prod_{i<j} (a_i, a_j).$$

	1	ρ	π	$\rho\pi$
1	1	1	1	1
ρ		1	-1	-1
π			1	-1
$\rho\pi$				1

Table 2: Hilbert symbol results when $-1 \in \dot{\mathbb{K}}_p^2$

	1	ρ	π	$\rho\pi$
1	1	1	1	1
ρ		1	-1	-1
π			-1	-1
$\rho\pi$				-1

Table 3: Hilbert symbol results when $-1 \notin \dot{\mathbb{K}}_p^2$

The usual proofs of the Hasse invariant's properties are rather involved; we will use a bit of brute force to provide a simplified version of the proofs here. For more elegant proofs of the following results, we refer the reader to Chapters 3 and 4 of [4] or Chapters 4 and 5 of [3].

Lemma 5.3. *If two quadratic forms are equivalent, they have the same Hasse invariant.*

Proof. Since equivalence implies chain equivalence, it is sufficient to prove this result for two simply equivalent forms. Permutation will not change the Hasse invariant, since multiplication is commutative and the Hilbert symbol is symmetric. Thus we may assume that the coefficients of the quadratic forms are ordered so that $q_1 = \langle a_1, a_2, a_3, \ldots, a_n \rangle$ and $q_2 = \langle a_1', a_2', a_3, \ldots, a_n \rangle$, where $\langle a_1, a_2 \rangle \cong \langle a_1', a_2' \rangle$. Note that since the Hasse invariant is based on the Hilbert symbol, which is in turn based on 2-dimensional quadratic forms and thus cares only about square classes, it is legitimate for us to continue to speak about coefficients of the form in terms only of their square classes.

First observe that the Hasse invariant can be broken up as follows:

$$H(q) = (a_1, a_2) \cdot \left(\prod_{i=3}^{n} (a_1, a_i)(a_2, a_i) \right) \left(\prod_{3 \leqslant j < k} (a_j, a_k) \right) \tag{23}$$

Since $\langle a_1, a_2 \rangle \cong \langle a_1', a_2' \rangle$, these pairs must have the same Hilbert symbol, so the first term of equation (23) is equal. Similarly, since all the coefficients involved in the last term are the same in both cases, the last segment of equation (23) will also be equal. It is now sufficient to prove that the middle term is equal.

Note that $(a, c)(b, c) = (ab, c)$; the proof of this is quite complex and technical, however it can be seen explicitly by inspection of Tables 2 and 3. Now, since $\langle a_1, a_2\rangle \cong \langle a_1', a_2'\rangle$,

$$\det\langle a_1, a_2\rangle = \det\langle a_1', a_2'\rangle \Rightarrow a_1a_2 \cdot \dot{\mathbb{K}}_p^2 = a_1'a_2' \cdot \dot{\mathbb{K}}_p^2,$$

in other words a_1a_2 and $a_1'a_2'$ are in the same square class; therefore, for any other element $b \in \mathbb{K}_p$,

$$(a_1, b)(a_2, b) = (a_1a_2, b) = (a_1'a_2', b) = (a_1', b)(a_2', b),$$

thus each term of the middle product of equation (23) will be equal for q_1 and q_2. Therefore every part of the equation is equal in both cases; thus, the Hasse invariant is indeed invariant for any equivalent quadratic forms. □

Theorem 5.4. *Two p-adic quadratic forms of dimension $\geqslant 2$ are equivalent iff they have the same dimension, the same determinant* and *the same Hasse invariant.*

Proof. Since these are all invariants, we already know that if two forms are equivalent, they will have the same dimension, determinant and Hasse invariant; we need only prove the converse.

In the 1-dimensional case, determinants are enough to classify the forms, so this is clearly true. In the 2-dimensional case, equivalence was additionally based on whether the forms shared a common element. By looking at Proposition 3.4, we can see that there were only two distinct forms for any given determinant, and exactly one of those forms always represented 1. Thus the Hasse invariant, which in the 2-dimensional case is just the Hilbert symbol, is enough to classify the 2-dimensional forms of the same determinant.

For the higher dimensions, we will prove the result by induction. Suppose we know this is true for all forms of dimension less than some $n \geq 3$. Let $q_1 = \langle a_1, \dots, a_n\rangle$, $q_2 = \langle b_1, \dots, b_n\rangle$ be two forms with the same dimension, determinant and Hasse invariant. We will first show that these two forms represent a common element, in other words that there is some $d \in \mathbb{K}_p$ and some $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{K}_p$ such that

$$\sum_{i=1}^{n} a_i x_i^2 = d = \sum_{j=1}^{n} b_i y_i^2$$

This is equivalent to asking whether there is a $d \in \dot{\mathbb{K}}_p/\dot{\mathbb{K}}_p^2$ such that both $\langle a_1, \dots, a_n, -d\rangle$ and $\langle b_1, \dots, b_n, -d\rangle$ are isotropic. As proven in Theorem 4.6, there is only one anisotropic 4-dimensional form, and no anisotropic forms of dimension 5 or higher, so this is always possible (in the 4-dimensional case, just choose d such that $-d \cdot \det q \neq 1$, since the only 4-dimensional anisotropic form has determinant 1).

Now, we need an extension Lemma 3.2; we want to say that we can write, for some $a_1', \dots a_{n-1}', b_1', \dots, b_{n-1}'$,

$$q_1 \cong \langle d, a_1', \dots, a_{n-1}'\rangle, \tag{24}$$

$$q_2 \cong \langle d, b_1', \dots, b_{n-1}'\rangle. \tag{25}$$

To show this, we will look at the corresponding quadratic spaces, (V, B_{q_1}) and (W, B_{q_2}). Take $v \in V$ and $w \in W$ to be any vectors such that $q_1(v) = d = q_2(w)$; now expand these vectors to orthogonal bases for their respective spaces using the standard methods. The coefficients of the (diagonal) matrix representations of the pairing with respect to these new bases will be as above.

So now we have $q_1 \cong \langle d\rangle \perp \langle a'_1, \ldots, a'_{n-1}\rangle$, $q_2 \cong \langle d\rangle \perp \langle b'_1, \ldots, b'_{n-1}\rangle$. By Witt's Cancellation Theorem (Theorem 4.3), these are equivalent iff $\langle a'_1, \ldots, a'_{n-1}\rangle \cong \langle b'_1, \ldots, b'_{n-1}\rangle$. Let's call these subforms $q'_1 = \langle a'_1, \ldots, a'_{n-1}\rangle$ and $q'_2 = \langle b'_1, \ldots, b'_{n-1}\rangle$. Then,

$$\begin{aligned}
\det q_1 = \det q_2 \qquad & \Rightarrow \qquad d \cdot \prod_{i=1}^{n-1} a'_i = d \cdot \prod_{j=1}^{n-1} b'_i \\
& \Rightarrow \qquad \prod_{i=1}^{n-1} a'_i = \prod_{j=1}^{n-1} b'_i \\
& \Rightarrow \qquad \det q'_1 = \det q'_2
\end{aligned}$$

and

$$\begin{aligned}
& H(q_1) = H(q_2) \\
\Rightarrow \quad & \left(\prod_{i=1}^{n-1}(d, a_i)\right)\left(\prod_{j<k}(a'_j, a'_k)\right) = \left(\prod_{i'=1}^{n-1}(d, b_{i'})\right)\left(\prod_{j'<k'}(b'_{j'}, b'_{k'})\right) \\
\Rightarrow \quad & (d, a_1 a_2 \cdots a_{n-1})\left(\prod_{j<k}(a'_j, a'_k)\right) = (d, b_1 b_2 \cdots b_{n-1})\left(\prod_{j'<k'}(b'_{j'}, b'_{k'})\right) \\
\Rightarrow \quad & (d, \det q'_1) H(q'_1) = (d, \det q'_2) H(q'_2) \\
\Rightarrow \quad & H(q'_1) = H(q'_2).
\end{aligned}$$

We now have two forms of degree $n-1$ with the same determinant and Hasse invariant; by the induction hypothesis, they are equivalent, and thus our original forms are as well. Therefore, this information completely classifies the p-adic quadratic forms of any dimension. $\square$

We can now use the Hasse invariant to provide an alternate (and possibly more satisfying) proof of the classification of three-dimensional forms, by simply multiplying the Hilbert Symbols as found in Tables 2 and 3 of the appropriate elements. Table 4 contains the results. For the sake of interest, I have also included the results for dimensions 2 and 4. I have denoted the isotropic forms by $\langle a_1, \ldots, a_n\rangle^*$; for the sake of visual simplicity, I have used the combined representatives.

6 The Full Classification

We are now able to completely classify the quadratic forms over any p-adic field.

Representative $\langle a_1, \ldots, a_n \rangle$	$-1 \in \dot{\mathbb{K}}_p^2$ $(\det, H)$	$-1 \notin \dot{\mathbb{K}}_p^2$ $(\det, H)$
$\langle 1, \pi \rangle$	$(\pi, +)$	$(\pi, +)$
$\langle 1, -\rho \rangle$	$(\rho, +)$	$(1, +)$
$\langle 1, \rho\pi \rangle$	$(\rho\pi, +)$	$(\rho\pi, +)$
$\langle \rho, \pi \rangle$	$(\rho\pi, -)$	$(\rho\pi, -)$
$\langle \rho, \rho\pi \rangle$	$(\pi, -)$	$(\pi, -)$
$\langle \pi, -\rho\pi \rangle$	$(\rho, -)$	$(1, -)$
$\langle 1, -1 \rangle^*$	$(1, +)$	$(\rho, +)$
$\langle 1, -\rho, -\rho\pi \rangle$	$(\pi, -)$	$(\pi, +)$
$\langle 1, \pi, -\rho\pi \rangle$	$(\rho, -)$	$(1, -)$
$\langle 1, -\rho, -\pi \rangle$	$(\rho\pi, -)$	$(\rho\pi, +)$
$\langle \rho, \pi, -\rho\pi \rangle$	$(1, -)$	$(\rho, -)$
$\langle 1, -1, 1 \rangle^*$	$(1, +)$	$(1, +)$
$\langle 1, -1, \rho \rangle^*$	$(\rho, +)$	$(\rho, +)$
$\langle 1, -1, \pi \rangle^*$	$(\pi, +)$	$(\pi, -)$
$\langle 1, -1, \rho\pi \rangle^*$	$(\rho\pi, +)$	$(\rho\pi, +)$
$\langle 1, -\rho, \pi, -\rho\pi \rangle$	$(1, -)$	$(1, -)$
$\langle 1, -1, 1, -1 \rangle^*$	$(1, +)$	$(1, +)$
$\langle 1, -1, 1, \pi \rangle^*$	$(\pi, +)$	$(\rho\pi, -)$
$\langle 1, -1, 1, -\rho \rangle^*$	$(\rho, +)$	$(\rho, +)$
$\langle 1, -1, 1, \rho\pi \rangle^*$	$(\rho\pi, +)$	$(\pi, -)$
$\langle 1, -1, \rho, \pi \rangle^*$	$(\rho\pi, -)$	$(\pi, +)$
$\langle 1, -1, \rho, \rho\pi \rangle^*$	$(\pi, -)$	$(\rho\pi, +)$
$\langle 1, -1, \pi, -\rho\pi \rangle^*$	$(\rho, -)$	$(\rho, -)$

Table 4: Determinants and Hasse Invariants, Dimensions 2-4

Theorem 6.1 (Classification of the Quadratic Forms over the p-adics). *Every quadratic form over the p-adics is the orthogonal sum of some completely isotropic space, some hyperbolic space, and at most one of the following anisotropic forms:*

$$\begin{array}{lllll}
\langle 1\rangle & \langle\rho\rangle & \langle\pi\rangle & \langle\rho\pi\rangle & \langle 1,\pi\rangle \\
\langle 1,-\rho\rangle & \langle 1,\rho\pi\rangle & \langle\rho,\pi\rangle & \langle\rho,\rho\pi\rangle & \langle\pi,-\rho\pi\rangle \\
\langle 1,-\rho,-\rho\pi\rangle & \langle 1,\pi,-\rho\pi\rangle & \langle 1,-\rho,-\pi\rangle & \langle\rho,\pi,-\rho\pi\rangle & \langle 1,-\rho,\pi,-\rho\pi\rangle.
\end{array}$$

□

References

[1] Emil Artin, *Geometric Algebra*, Interscience Tracts in Pure and Applied Mathematics Vol. 3, Interscience Publishers Inc, New York-London, 1957.

[2] Fernando Q. Gouvêa, *p-adic Numbers: An Introduction (Second Edition)*, Universitext, Springer-Verlag Berlin Heiddberg, New York, 1997.

[3] T. Y. Lam, *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics Vol. 67, American Mathematical Society, Providence-Rhode Island, 2005.

[4] Jean-Pierre Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics Vol. 8, Springer, New York, 1973.