Mar 23: Complex numbers

We are used to the real numbers $\mathbb{R}$, which are good numbers to use for modelling the physical world. Complex numbers make more sense for modelling signals, electrical impulses, and quantum mechanics (among many other things).

So you know that x^2+1 has no real roots.

So we just make one up:

$$i = \sqrt{-1}$$

(i stands for "imaginary".)

Then $(i)^2+1 = -1+1 = 0$

since $i = \sqrt{-1}$
$\Rightarrow i^2 = -1$

so i is a root of x^2+1.

Also $(-i)^2 = (-1)^2(i)^2+1 = 1(-1)+1 = 0$,

so $-i$ is the other root.

Using i we can define the square root of <u>any</u> negative number:

$$\sqrt{-a} = \sqrt{a}\, i \quad \text{if } a>0.$$

{So we don't need to invent more numbers!

so for example

$$\sqrt{-9} = 3i.$$

In fact, we can solve all quadratic polynomials now.

(eg) Find the roots of $x^2-6x+13$.

Use the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(13)}}{2}$$

$$= 3 \pm \frac{\sqrt{36-52}}{2} = 3 \pm \frac{\sqrt{-16}}{2}$$

$$= 3 \pm \frac{4i}{2}$$

$$= 3 \pm 2i$$

So the roots are $3+2i$ and $3-2i$.

Check: (plugging in the root $3+2i$)

$$(3+2i)^2 - 6(3+2i) + 13$$

$$= 9 + 12i + \underbrace{4i^2}_{-4} - 18 - 12i + 13$$

$$= 0 \quad \checkmark$$

We can also check that:

$$(x-(3+2i))(x-(3-2i)) = x^2 - (3+2i)x - (3-2i)x + (3+2i)(3-2i)$$

$$= x^2 - 6x + (9 - 6i + 6i - 4i^2)$$

$$= x^2 - 6x + (9+4)$$

$$= x^2 - 6x + 13 \quad \checkmark$$

□

Definition: The complex numbers are the elements of the set

$$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$$

(eg) $3+2i$, 0, i, $\sqrt{2}i$, $\pi + ei$, $6 \in \mathbb{C}$

We often use z for an element of $\mathbb{C}$.

If $z = a+bi \in \mathbb{C}$

then a is the <u>real part of z</u>: $Re(z) = a$

b is the <u>imaginary part of z</u>: $Im(z) = b$

↑ the real number — no "i".

(eg) $7i$ is a <u>purely imaginary number</u>

all real numbers are complex: $\mathbb{R} \subseteq \mathbb{C}$.

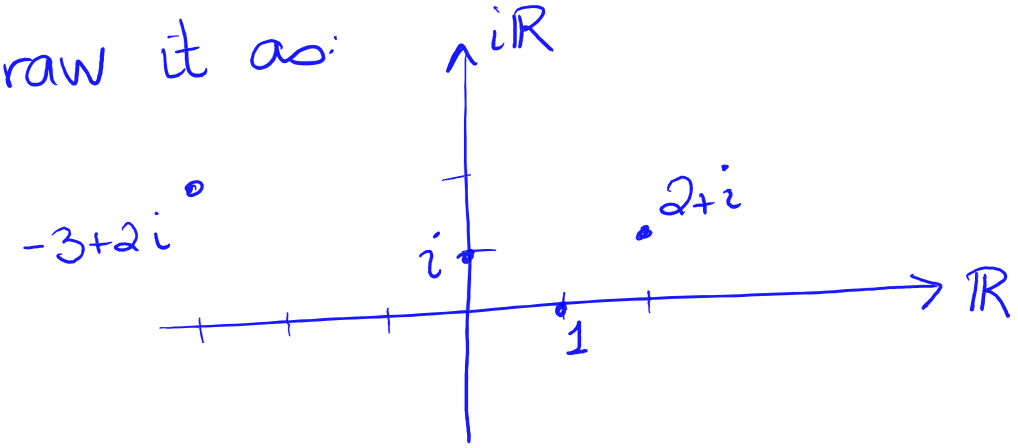
From the definition of $\mathbb{C}$:

∴ $\mathbb{C} = \text{span}\{1, i\}$ is a 2-dimensional <u>vector space</u>

(↑ LI)

called "the complex plane".

We draw it as:



So we know how to add and subtract complex numbers, and to multiply a complex number by a real scalar:

$$a+bi + c+di = (a+c) + (b+d)i$$

$$c(a+bi) = ca + cbi$$

(just the usual vector space operations)

Multiplication formula:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

Proof: $= ac + adi + bci + \underbrace{adi^2}_{=-1}$ ∎

FACTS: $z_1 z_2 = z_2 z_1$, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ { multiplication is commutative and associative.

for all $z_1, z_2, z_3 \in \mathbb{C}$.

eg) $(2+4i)(3-2i) = (6+8) + (-4+12)i = 14+8i$

$(3-2i)(2+4i) = (6+8) + (12-4)i = 14+8i$

Division is a challenge:

$$\frac{1}{3+4i}$$

$\frac{1}{3+4i} \neq \frac{1}{3} + \frac{1}{4i}$

$\frac{1}{1+2} \neq \frac{1}{1} + \frac{1}{2}$.

think of as

$$\frac{1}{3+4\sqrt{-1}} \times \frac{3-4\sqrt{-1}}{3-4\sqrt{-1}}$$

rationalize!

$$\frac{1}{3+4i} \times \frac{3-4i}{3-4i} = \frac{3-4i}{9+16} = \underbrace{\frac{3}{25} - \frac{4}{25}i}_{\text{correct form} \; \therefore \in \mathbb{C}}$$

Check: $\frac{1}{a} \cdot a = 1$

$$\left(\frac{3}{25} - \frac{4}{25}i\right)(3+4i) = \left(\frac{9}{25} + \frac{16}{25}\right) + \left(\frac{12}{25} - \frac{12}{25}\right)i$$

$$= 1 \quad \checkmark$$

Definition: If $z = a+bi \in \mathbb{C}$ then:

- $\overline{z} = a - bi$ is the complex conjugate of z
- $|z| = \sqrt{a^2+b^2}$ is the norm or absolute value or modulus of z.

$\in \mathbb{R}$

a real number equal to the length of the vector $a+bi$ in the complex plane.

Note:

- $|z|^2 = z\overline{z}$
- $\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{1}{|z|^2}\overline{z}$

$(a+bi)(a-bi) = a^2+b^2$

eg) Simplify $\frac{4+2i}{1-i}$.

Solution: $\frac{4+2i}{1-i} \cdot \frac{(1+i)}{(1+i)} = \frac{(4-2)+(4+2)i}{1^2+1^2} = \frac{2}{2} + \frac{6}{2}i$

$= 1+3i.$ □

Properties of the complex conjugate:

$$\overline{z+w} = \overline{z} + \overline{w}$$

$$\overline{zw} = \overline{z}\,\overline{w}$$

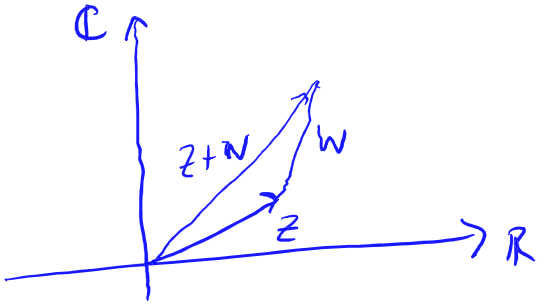
$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

$$z = \overline{z} \iff z \in \mathbb{R}$$

$$z = -\overline{z} \iff z \text{ is purely imaginary}$$

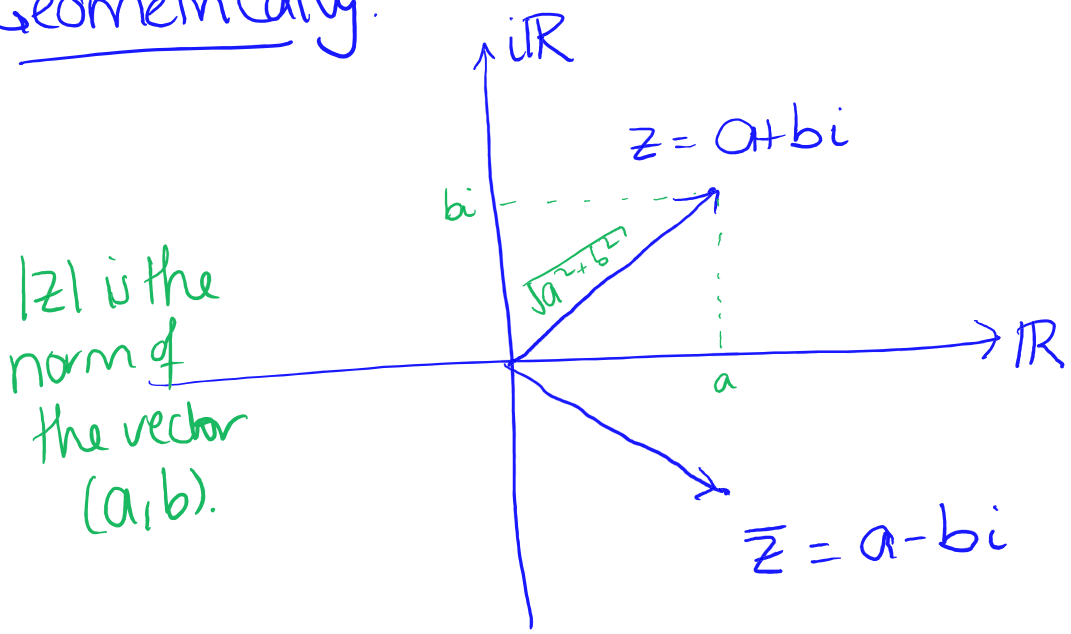
Properties of the norm:

- $|z| \in \mathbb{R}$ and $|z| \geq 0$ $\quad (|z| = 0 \iff z = 0)$
- $|z| = |\overline{z}|$
- $|zw| = |z|\,|w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
- $|z+w| \leq |z| + |w|$ triangle inequality



- if $a \in \mathbb{R}$ then $|a| = \sqrt{a^2+0^2} = |a|$ (it's still just the absolute value)

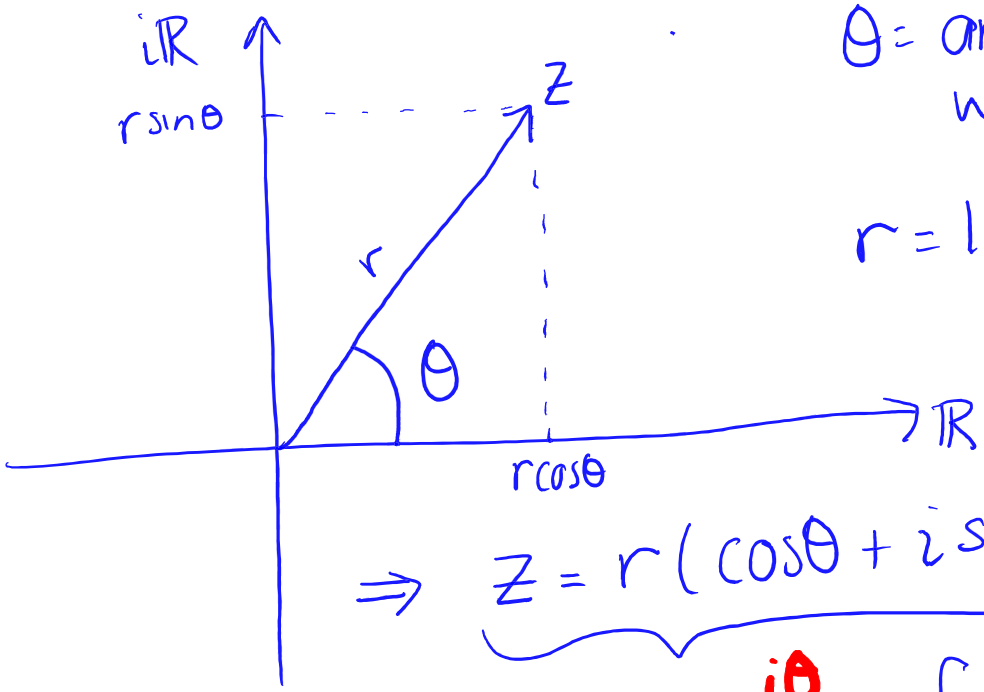
Geometrically:



$|z|$ is the norm of the vector (a,b).

Complex conjugation is a reflection in the $\mathbb{R}$ axis.

But what is the geometric interpretation of multiplication?
For that we need **polar coordinates**.



θ = angle z makes with positive $\mathbb{R}$ axis

$r = |z|$ the norm of z

$$\Rightarrow \underbrace{z = r(\cos\theta + i\sin\theta)}$$

$$z = re^{i\theta} \quad \text{for short}$$

So if $z = re^{i\theta}$ then $z = r\cos\theta + r\sin\theta\, i$ in rectangular coordinates.

If $z = a+bi$ then $r = |z|$ and $\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$.
solve for $-\pi < \theta \leq \pi$.

In MAT1322 we see that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

Plug in $x = i\theta$:

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{1}{3!}\theta^3 i + \frac{1}{4!}\theta^4 + \dots$$

$$= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right)$$

$$= \cos\theta + i(\sin\theta) \quad \square$$

Polar coordinates are weird:

- $re^{i\theta} = r'e^{i\theta'} \iff r = r'$ and, if $r \neq 0$, $\theta = \theta' + 2n\pi$ for some integer n.

but nice, too:

- $\overline{re^{i\theta}} = re^{-i\theta}$
- $|e^{i\theta}| = 1$.

eg) $e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$\therefore \boxed{e^{i\pi} + 1 = 0}$$

voted #1 most beautiful equation ever (Wiki)

The real reason polar coordinates are wonderful: multiplication

$$\boxed{(re^{i\theta})(r'e^{i\theta'}) = rr'e^{i(\theta+\theta')}}$$

and $$\frac{re^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}e^{i(\theta-\theta')}$$

We can check this by multiplying out both sides.

(eg) $i = e^{i\frac{\pi}{2}}$ $\quad 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

$$\therefore \frac{i}{1+i} = \frac{e^{i\pi/2}}{\sqrt{2}e^{i\pi/4}} = \frac{1}{\sqrt{2}}e^{i\pi/4}$$

Check: $\frac{i}{1+i}\cdot\frac{1-i}{1-i} = \frac{i+1}{1^2+1^2} = \frac{1}{2}(1+i) = \frac{1}{2}(\sqrt{2}e^{i\pi/4})$ ✓

But

eg) $e^{i\pi/2} + \sqrt{2}e^{i\pi/4} = ??$

$= i + (1+i) = 1+2i$

$= \sqrt{5}\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i\right) = \sqrt{5}e^{i\arccos(1/\sqrt{5})}$ (yuck)

Point: rectangular coordinates are good for addition
polar coordinates are good for multiplication

Geometric interpretation of multiplication:

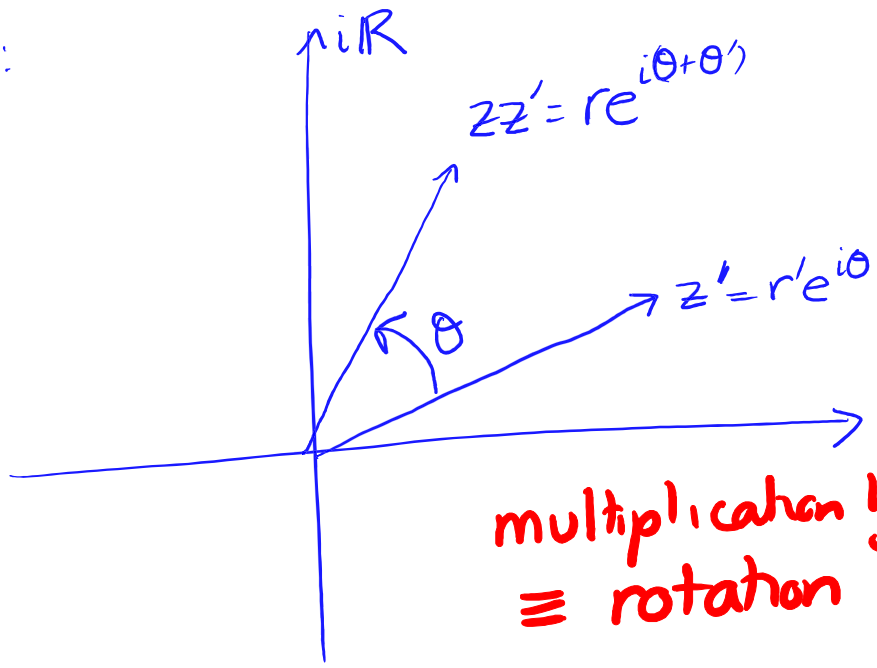
$z = re^{i\theta}$ → If $\theta = 0$, $z = r$ and multiplication is scalar multiplication.

↓

Otherwise:

say $r = 1$

$z = e^{i\theta}$



multiplication by $e^{i\theta}$ ≡ rotation by θ.

cool fact:

Notice that if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then if $w = a+bi$

$A\begin{bmatrix} a \\ b \end{bmatrix}$ gives the coordinates of $e^{i\theta}w$.

"matrices give linear transformations of vector spaces".

Final remark: back to polynomials.

So if $ax^2+bx+c=0$, then

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

and now this makes sense (and gives 2 roots — counting roots with multiplicity!) for any choices of a, b or c.

Even better, we have:

The Fundamental Theorem of Algebra

Every polynomial with coefficients in the complex numbers factors completely into linear factors.

eg) $x^2+i = (x-e^{-i\pi/4})(x+e^{-i\pi/4})$

since

$$= x^2 - e^{-i\pi/4}x + e^{-i\pi/4}x - e^{-i\pi/2}$$

$$= x^2 - (-i)$$

$$= x^2+i.$$

That is: we are not missing any more square root in $\mathbb{C}$ (or roots of any kind): $\mathbb{C}$ is **algebraically closed.**