MAT1341 C : Instructor Monica Nevins
Monday, March 6, 2017 : Test #3

Duration: 75 minutes

Family name: ______________________________

First name: Solutions

Student number : ______________________________

DGD Section number : ______________

Faculté des sciences
Mathématiques et statistique

Faculty of Science
Mathematics and Statistics

613–562–5864
613–562–5776
www.uOttawa.ca

585 King Edward
Ottawa
ON K1N 6N5

Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 and 2 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 3–5 are long answer, with point values as indicated. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Question 6 is a bonus question, worth 3 points. **The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.**
- Where it is possible to check your work, do so.
- Good luck!

Marker's use only:

Question	Marks
1 & 2 (/2)	
3 (/6)	
4 (/6)	
5 (/6)	
6 (/3) (bonus)	
Total (/20)	

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1. (1 point) Which of the following are correct statements about sets of vectors in a vector space V?

(1) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent if and only if setting $a_1 = a_2 = a_3 = 0$ gives $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$.

(2) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent if and only if there is a solution to the equation $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$ different from $a_1 = a_2 = a_3 = 0$.

(3) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then $\dim(V) \geq 3$.

(4) If $U = \mathsf{span}\{\vec{u}, \vec{v}, \vec{w}\}$ and $\dim(U) = 3$ then $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

A. Only (2) and (4) are true.
B. Only (3) and (4) are true.
C. All are true.
D. Only (1) and (3) are true.
E. Only (2) and (3) are true.
F. Only (2), (3) and (4) are true.

Your answer: E

see other 4pm test

2. (1 point) Suppose X is a subspace of $\mathbb{R}^6$ such that $X \neq \{\vec{0}\}$ and $X \neq \mathbb{R}^6$. Which of the following are always true?

(1) X has a spanning set consisting of 6 vectors.

(2) X contains a linearly independent set of 6 vectors.

(3) X contains fewer than 6 vectors.

(4) There is a basis of X which spans $\mathbb{R}^6$.

(5) $1 \leq \dim(X) \leq 5$.

A. Only (2) and (5) are true.
B. Only (1) and (5) are true.
C. Only (1) and (4) are true.
D. Only (1), (3) and (5) are true
E. Only (2) and (4) are true.
F. Only (3) and (5) are true.

Your answer: B

3. (6 points) Consider the subspace $W = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y+z+t=0\}$ of $\mathbb{R}^4$. Consider the following four vectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -4 \end{bmatrix}.$$

(a) Each of these vectors are elements of W. Explain why $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

(b) Find at least one non-trivial linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ that results in the zero vector, by setting up and solving an appropriate linear system of equations using row reduction. Verify that your answer is correct.

(c) Express one of the vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ as a linear combination of the rest.

(d) Assuming the remaining vectors are linearly independent, give a basis for W and explain why your answer is correct.

(a) Since $W \subseteq \mathbb{R}^4$ but $W \neq \mathbb{R}^4$

$\dim W \leq 3$

$\therefore$ a set of 4 vectors in W is LD, by a theorem from class.

(b) $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 = \vec{0} \iff$ augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 2 & 0 \\ -1 & -3 & -1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & -2 & -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \therefore \begin{aligned} a + d &= 0 \\ b &= 0 \\ c + 3d &= 0 \\ d &\text{ free.} \end{aligned}$$

(Continue your answer on the next page if necessary.)

(Continue your answer to Q3 on this page if needed.)

We can take $d=1$: $a=-1$, $b=0$, $c=-3$

$$\Rightarrow -1\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}+0\begin{bmatrix}1\\2\\0\\-3\end{bmatrix}-3\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}+1\begin{bmatrix}1\\1\\2\\-4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \checkmark$$

(c) We have $-\vec{v}_1-3\vec{v}_3+\vec{v}_4=\vec{0}$

$$\therefore \vec{v}_4=\vec{v}_1+3\vec{v}_3$$

(d) If $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is an LI set in W, then $\dim W \geq 3$.

We noticed in (a) that $W \neq \mathbb{R}^4$ (since, for example, $(1,0,0,0) \notin W$)

so $\dim W \leq 3$.

$\therefore$ $\dim W = 3$ and $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is an LI set in W with 3 vectors

$\therefore$ by thm in class, it is a basis of W.

4. (6 points) Consider the following elements of P_2:

$$f(x) = x - 5, \quad g(x) = x^2 - 2x \quad \text{and} \quad h(x) = x^2 - 10.$$

(a) Let $U = \mathsf{span}\{f, g\}$. Show that $h \in U$.

(b) Show that $\{f, g\}$ is linearly independent. (You may use the fact that $\{1, x, x^2\}$ is linearly independent.)

In parts (c) and (d) you may use your results from (a) and (b).

(c) Let $W = \mathsf{span}\{f, g, h\}$. Show that $\{f, g\}$ is a basis for W.

(d) Find the coordinate vector of h with respect to the basis $\{f, g\}$.

(a) We solve (or notice) that

$$x^2 - 10 = x^2 - 2x + 2(x-5)$$

$$\therefore \; h = 2f + g \in \operatorname{span}\{f,g\} = U.$$

(b) Since f and g are nonzero, and every scalar multiple of g has a quadratic term but every scalar multiple of f is linear, f and g are not scalar multiples of each other $\therefore$ LI.

(c) Since $h \in \operatorname{span}\{f,g\}$, by a theorem from class, $W = \operatorname{span}\{f,g,h\} = \operatorname{span}\{f,g\}$. Since $\{f,g\}$ is LI by (b), it is a basis for its span. $\therefore$ $\{f,g\}$ is a basis for W.

(Continue your answer on the next page if necessary.)

(Continue your answer to Q4 on this page if needed.)

(d) By (a), $h = 2f + g$ $\therefore$ the coordinates of h with respect to the basis $B = \{f, g\}$ are $[h]_B = (2, 1)$.

5. (6 = 4 × 1.5 points) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement. Then justify your answer:

- If you say the statement may be false, you must give an explicit counterexample (with numbers or functions!).
- If you say the statement is always true, then you CANNOT use an example to justify your response — you must give a clear general explanation using the theory from the course. If you use a theorem from class, you **must state the theorem**.

(a) Suppose $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^5$. If $2\vec{u} + 3\vec{v} + 0\vec{w} = \vec{0}$ then $\{\vec{u}, \vec{v}\}$ is linearly dependent.

True or false: TRUE

Justification:

We have $2\vec{u} + 3\vec{v} = \vec{0}$ and this is a nontrivial dependence relation.

$\therefore$ LD.

(b) Let $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \;\middle|\; a + 2b + 3c = 0 \quad \text{and} \quad d = 0 \right\}$. Then $\dim(U) = 3$.

True or false: FALSE

Justification:

$U = \left\{ \begin{bmatrix} -2b-3c & b \\ c & 0 \end{bmatrix} \;\middle|\; b, c \in \mathbb{R} \right\}$ since $a = -2b-3c$, $d = 0$

$= \left\{ b\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} \;\middle|\; b, c \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Since $\dim U \leq$ # vectors in a spanning set for U (thm from class) $\dim U \leq 2$

(c) Suppose $\vec{u}, \vec{v} \in \mathbb{R}^3$ and $\dim(\mathsf{span}\{\vec{u}, \vec{v}\}) = 2$. Then it is always possible to find a vector $\vec{w} \in \mathbb{R}^3$ such that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$.

True or false: TRUE

Justification:

If $\dim(\overbrace{\text{span}\{\vec{u},\vec{v}\}}^{U}) = 2$ then $\{\vec{u},\vec{v}\}$ is a spanning set with $\dim U$ elements so is a basis for U. $\therefore$ $\{\vec{u},\vec{v}\}$ is LI.

$\therefore$ $\{\vec{u},\vec{v}\}$ can be extended to a basis for the ambient vector space (here, $\mathbb{R}^3$) by a theorem in class. $\dim \mathbb{R}^3 = 3$ so the basis you get has 3 vectors.

(d) If $\{\vec{u}, \vec{v}\}$ is linearly dependent then $\{\vec{u}, \vec{v}, \vec{w}\}$ must also be linearly dependent.

True or false: TRUE

Justification:

If $\{\vec{u},\vec{v},\vec{w}\}$ were LI, then by a thm from class, $\{\vec{u},\vec{v}\}$ would be LI. But it is LD, so $\{\vec{u},\vec{v},\vec{w}\}$ must be LD.

6. (Bonus, max 3 points) Let V be a vector space. Suppose $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a basis for a subspace U of V, and suppose $\{\vec{w}_1, \vec{w}_2\}$ is basis for a subspace W of V. Suppose further that $\vec{v}$ is a nonzero vector lying in both U and W. Carefully prove that if

$$X = \mathsf{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \overset{u_4\ w_1\ w_2}{\cancel{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{v}}}\} \quad \vec{v}$$

then $\dim(X) \leq 5$.

See other version for a solution.