MAT1341 C : Instructor Monica Nevins
Monday, March 6, 2017 : Test #3

Duration: 75 minutes

Family name: ______________________________

First name: _____SOLUTIONS_____________

Student number : ____________________________

DGD Section number : ______________

Please read the following instructions carefully.

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 and 2 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 3–5 are long answer, with point values as indicated. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Question 6 is a bonus question, worth 3 points. **The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.**
- Where it is possible to check your work, do so.
- Good luck!

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Marker's use only:

Question	Marks
1 & 2 (/2)	
3 (/6)	
4 (/6)	
5 (/6)	
6 (/3) (bonus)	
Total (/20)	

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1. (1 point) Which of the following are correct statements about sets of vectors in a vector space V?

(1) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if setting $a_1 = a_2 = a_3 = 0$ gives $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$. false: setting the coefficients to zero makes the zero vector, no matter if the vectors are LI or LD

(2) The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent if and only if the only solution to the equation $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}$ is $a_1 = a_2 = a_3 = 0$. true: this is the definition of LI

(3) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then it is a basis for $U = \mathsf{span}\{\vec{u}, \vec{v}, \vec{w}\}$. true: it is LI and every set spans its span!

(4) If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent then $\dim(V) = 3$. false: we don't know if these vectors span V; V could be a lot bigger.

A. All are true.

B. Only (2) and (4) are true.

C. Only (1) and (3) are true.

D. Only (2), (3) and (4) are true.

E. Only (2) and (3) are true.

F. Only (3) and (4) are true.

Your answer: E

2. (1 point) Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^5$ and let $U = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Suppose we also have two nonzero vectors $\vec{u}, \vec{w} \in U$ which are not collinear. Which of the following must ALWAYS be true?

A. $2 \leq \dim(U) \leq 4$

B. $1 \leq \dim(U) < 4$

C. $\dim(U) = 2$

D. $\dim(U) = 4$

E. $\dim(U) = 5$

F. $\dim(U) = 6$

Your answer: A

So U is spanned by 4 vectors, so $\dim(U) \leq 4$. And U contains at least two LI vectors, so $\dim(U) \geq 2$.

3. (6 points) Consider the following four vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

(a) Explain why $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

(b) Find at least one non-trivial linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ that results in the zero vector, by setting up and solving an appropriate linear system of equations using row reduction. Verify that your answer is correct.

(c) Express one of the vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ as a linear combination of the rest.

(d) Let $W = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Give a spanning set for W which has fewer than 4 vectors, and explain why your answer is correct.

(a) These are four vectors in a three-dimensional space, so by a theorem from class, the vectors cannot be LI, hence are LD.

(b) We set up the system $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$, which leads to the following augmented matrix:

$$\begin{bmatrix} 1 & 1 & 3 & 2 & | & 0 \\ 3 & 2 & 7 & 6 & | & 0 \\ 0 & 1 & 2 & -1 & | & 0 \end{bmatrix}$$

We row reduce:

$$-3R1 + R2 \begin{bmatrix} 1 & 1 & 3 & 2 & | & 0 \\ 0 & -1 & -2 & 0 & | & 0 \\ 0 & 1 & 2 & -1 & | & 0 \end{bmatrix}$$

$$-R2 \begin{bmatrix} 1 & 1 & 3 & 2 & | & 0 \\ 0 & 1 & 2 & 0 & | & 0 \\ 0 & 1 & 2 & -1 & | & 0 \end{bmatrix}$$

$$-R2 + R3, -R2 + R1 \begin{bmatrix} 1 & 0 & 1 & 2 & | & 0 \\ 0 & 1 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & -1 & | & 0 \end{bmatrix}$$

$$-R3 \begin{bmatrix} 1 & 0 & 1 & 2 & | & 0 \\ 0 & 1 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$-2R3 + R1 \begin{bmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

which is in RREF. Therefore the general solution is

$$a_1 + a_3 = 0, \quad a_2 + 2a_3 = 0, \quad a_4 = 0, \quad a_3 \text{ is free.}$$

Take $a_3 = 1$; this gives $a_1 = -1, a_2 = -2, a_4 = 0$, and this is the answer:

$$-1\begin{bmatrix}1\\3\\0\end{bmatrix} - 2\begin{bmatrix}1\\2\\1\end{bmatrix} + 1\begin{bmatrix}3\\7\\2\end{bmatrix} + 0\begin{bmatrix}2\\6\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

which we can see is correct.

(c) Since $-\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$, we can solve for $\vec{v}_3$ in terms of the rest:

$$\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$$

which we can check is correct.

(d) Since $\vec{v}_3 \in \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$, by a theorem from class, this vector is redundant and

$$W = \mathsf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}.$$

4. (6 points) Suppose that

$$U = \{p(x) \in P_2 \mid p(5) = 0\}$$

and consider the following two elements of U:

$$f(x) = x - 5, \quad g(x) = x^2 - 5x.$$

(a) Give a polynomial $p(x) \in P_2$ such that p is not in U.

For (b) and (c) you may use any of the following: that $\{1, x, x^2\}$ is linearly independent; that U is a subspace of P_2; that a polynomial has 5 as a root if and only if it has $(x - 5)$ as a factor; and/or theorems from class.

(b) Show that $\{f, g\}$ is linearly independent.

(c) Show that $\{f, g\}$ is a basis for U.

Now for (d) and (e) let $h(x) = x^2 - 6x + 5$, which is also in U.

(d) Express h as a linear combination of f and g.

(e) Find the coordinate vector of h with respect to the basis $\{f, g\}$.

(a) for example, $p(x) = x$ is not in U because $p(5) = 5 \neq 0$.

(b) If $af + bg = 0$ then we'd have $a(x-5)+b(x^2-5x) = bx^2+(a-5b)x-5a = 0$. Since $\{1, x, x^2\}$ is LI, the only solution is the trivial one, so $b = 0$ and $5a = 0$ and so $a = 0, b = 0$ is the only solution to $af + bg = 0$. Therefore $\{f, g\}$ is LI.

(c) Since $f, g \in U$ and $\{f, g\}$ is LI, by a theorem from class, $\dim(U) \geq 2$. On the other hand, since $U \subset P_2$ and $U \neq P_2$ and $\dim(P_2) = 3$, from a theorem in class we know that $\dim(U) \leq 2$. Therefore $\dim(U) = 2$. Therefore by a theorem from class, any LI set in U with 2 vectors is a basis. Therefore $\{f, g\}$ is a basis of U.

(d) We solve $h = af + bg$ so

$$x^2 - 6x + 5 = a(x - 5) + b(x^2 - 5x) = bx^2 + (a - 5b)x - 5a.$$

Therefore we need $b = 1$ and $-5a = 5$ so $a = -1$. Check:

$$-(x - 5) + (x^2 - 5x) = x^2 - 6x + 5$$

so yes, $h = -f + g$.

(e) By (d), the coordinate vector of h with respect to the basis $B = \{f, g\}$ is

$$[h]_B = (-1, 1).$$

5. (6 = 4 × 1.5 points) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement. Then justify your answer:

- If you say the statement may be false, you must give an explicit counterexample (with numbers or functions!).
- If you say the statement is always true, then you CANNOT use an example to justify your response — you must give a clear general explanation using the theory from the course. If you use a theorem from class, you **must state the theorem**.

(a) Let $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\,2} \;\middle|\; b + 2c + 3d = 0 \text{ and } a = 0 \right\}$. Then $\dim(U) = 3$.

FALSE: Since $a = 0$ and $b = -2c - 3d$ we have

$$U = \left\{ \begin{bmatrix} 0 & -2c-3d \\ c & d \end{bmatrix} \middle| c, d \in \mathbb{R} \right\} = \left\{ c \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix} \middle| c, d \in \mathbb{R} \right\} = \mathsf{span} \left\{ \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix} \right\}.$$

Therefore U is spanned by 2 vectors, so by a theorem from class $\dim(U) \leq 2$.

(b) If $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $\mathsf{span}\{\vec{u}, \vec{v}\} = \mathbb{R}^2$ then $\{\vec{u}, \vec{v}\}$ is a basis for $\mathbb{R}^2$.

TRUE: since $\mathsf{span}\{\vec{u}, \vec{v}\} = \mathbb{R}^2$, and $\dim(\mathbb{R}^2) = 2$, we have that the set $\{\vec{u}, \vec{v}\}$ is a spanning set for $\mathbb{R}^2$ with $\dim(\mathbb{R}^2)$ vectors. Therefore by a theorem from class, it must automatically be LI, whence it is a basis.

(c) Suppose $\vec{u}, \vec{v}, \vec{w} \in M_{2\,2}$ and $U = \mathsf{span}\{\vec{u}, \vec{v}, \vec{w}\}$. Then any spanning set of U contains at least 3 vectors.

FALSE: for example, these vectors could be the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

in which case

$$U = \mathsf{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

so U has a spanning set with fewer than 3 vectors.

(d) If $\{\vec{u}, \vec{v}\}$ is linearly independent then $\{\vec{u}, \vec{v}, \vec{w}\}$ must also be linearly independent.

FALSE: for example, we could have $\vec{u} = (1, 0)$, $\vec{v} = (0, 1)$ which are LI vectors in $\mathbb{R}^2$, but in fact absolutely every other $\vec{w}$ would make the set $\{\vec{u}, \vec{v}, \vec{w}\}$ LD.

6. (Bonus, max 3 points) Let V be a vector space. Suppose $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a basis for a subspace U of V, and suppose $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is basis for a subspace W of V, such that the only vector that lies in both U and W is the zero vector. Carefully prove that $\dim(V) \geq 7$.

We want to prove that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is LI. So suppose to the contrary that we have a dependence equation of the form

$$a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + d\vec{u}_4 + e\vec{w}_1 + f\vec{w}_2 + g\vec{w}_3 = \vec{0}.$$

Then we can put all the vectors in U on one side and all the vectors in W on the other, to get the equation

$$a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + d\vec{u}_4 = -e\vec{w}_1 - f\vec{w}_2 - g\vec{w}_3.$$

Now the left side is in U since it is in the span of the $\vec{u}_i$s; and the right side is in W since it is in the span of the $\vec{w}_i$s. Therefore since they are equal, whatever they add up to is a vector that lies in both U and W. By hypothesis, the only such vector is the zero vector. Therefore

$$a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + d\vec{u}_4 = \vec{0}$$

and

$$-e\vec{w}_1 - f\vec{w}_2 - g\vec{w}_3 = \vec{0}.$$

Now since $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is a basis for U it is LI, therefore the first equation must have $a = b = c = d = 0$. And $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is basis for W, so it is LI, so the second equation must have $-e = -f = -g = 0$. Therefore in fact all the coefficients are 0, so the seven vectors were LI. By a theorem from class, the dimension of V must be at least seven, since V contains a set with seven LI vectors.