

# MAT1341 C : Instructor Monica Nevins

## Monday, February 13, 2017 : Test #2

Duration: 75 minutes

Family name: \_\_\_\_\_Solutions\_\_\_\_\_

First name: \_\_\_\_\_

Student number : \_\_\_\_\_

DGD Section number : \_\_\_\_\_

**Please read the following instructions carefully.**

- You have 75 minutes to complete this exam.
- This is a closed book exam. No notes, calculators, cell phones or related devices of any kind are permitted. All such devices, including cell phones, must be stored in your bag under your desk for the duration of the exam.
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1–4 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions into the table below.**
- Questions 5–7 are long answer, with point values as indicated. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Question 8 is a bonus question, worth 3 points. **The bonus question is more difficult; do not attempt it until you are satisfied that you have completed the rest of the test to the best of your ability.**
- Where it is possible to check your work, do so.
- Good luck!

Question	Your answer to the Multiple choice question
1	
2	
3	
4	

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**Marker's use only:**

Question	Marks
1–4 (/4)	
5 (/3)	
6 (/6)	
7 (/6)	
8 (/3) (bonus)	
<b>Total (/19)</b>	

1. Which of the following are **axioms of a vector space**  $V$ ?

- (1) If  $u, v \in V$  then  $u + v = v + u$ .
- (2) If  $u, v \in V$  then  $uv \in V$ .
- (3) If  $c \in \mathbb{R}$  and  $u \in V$  then  $c + u \in V$ .
- (4) For every  $u \in V$ ,  $\|u\| = \sqrt{u \cdot u}$ .
- (5) If  $u, v, w \in V$  then  $u \in \text{span}\{v, w\}$ .
- (6) If  $u, v \in V$  and  $c \in \mathbb{R}$  then  $c(u + v) = cu + cv$ .

- A. (1) and (6) **correct**
- B. (2) and (3)
- C. (4) and (5)
- D. (4) and (6)
- E. (1) and (5)
- F. (2) only

2. Which of the following are subspaces of  $\mathbb{R}^3$ ?

$$U = \{(x, x, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + y = z\}$$

$$V = \{(x, x, 2) \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$$

$$Y = \{(x, y, z) \in \mathbb{R}^3 \mid 3xy = z\}$$

- A.  $U$  and  $V$
- B.  $U$  and  $X$  **correct**
- C.  $X$  and  $V$
- D.  $X$  and  $Y$
- E.  $U$  and  $Y$
- F.  $V$  and  $Y$

$U = \text{span}\{(1, 1, 0), (0, 0, 1)\}$  so is a subspace;  $X$  is a plane through the origin with equation  $3x + y - z = 0$  so is a subspace;  $V$  does not contain  $(0, 0, 0)$  so is not a subspace;  $Y$  is not closed under addition since  $(1, 0, 0)$  and  $(0, 1, 0)$  are in  $Y$  but their sum  $(1, 1, 0)$  is not.

3. Let  $U = \{f \in F(\mathbb{R}) \mid f(0)f(1) = 0\}$  be a subset of the vector space of all functions on the real line. Let  $z$  denote the zero vector (function) of  $F(\mathbb{R})$ . Which of the following is/are true?

- (1)  $U$  contains  $z$
- (2)  $U$  is closed under vector addition
- (3)  $U$  is closed under scalar multiplication

- A. (1), (2) and (3)
- B. none of them
- C. (1) and (2) only
- D. (1) and (3) only **correct**
- E. (2) and (3) only
- F. (1) only

(1) and (3) are true but (2) is false. For (1) note that  $z(x) = 0$  satisfies  $z(0)z(1) = 0$ . For (2) note that if  $c \in \mathbb{R}$  then  $cf(0)cf(1) = c^2f(0)f(1)$  so is  $= 0$  if  $f(0)f(1) = 0$ , so  $U$  is closed under scalar multiplication. But for example  $f(x) = x$  and  $g(x) = 1 - x$  are in  $U$  but their sum is not.

4. For which value(s) of  $t$  is  $(1, 0, 8, t)$  in the span of  $(2, -3, 4, -1)$  and  $(1, -2, 0, -3)$ ?

- A.  $t = 0$  and  $t = 1$  only
- B.  $t = 2$  only
- C.  $t \geq 0$  only
- D. all  $t$
- E.  $t = 1$  and  $t = 7$  only
- F.  $t = 7$  only **correct**

Solving  $a(2, -3, 4, -1) - b(1, -2, 0, -3) = (1, 0, 8, t)$  gives  $4a = 8$  so  $a = 2$  and so  $b = -3$ . We have  $2(2, -3, 4, -1) - 3(1, -2, 0, -3) = (1, 0, 8, 7)$ .

5. (3 = 2 × 1.5 points) State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement. Then justify your answer:

- If you say the statement may be false, you must give an explicit counterexample (with numbers or functions!).
- If you say the statement is always true, then you CANNOT use an example to justify your response — you must give a clear explanation that works in all cases.

(a) If  $U$  is a subspace of  $\mathbb{R}^3$ , and  $u, v \in \mathbb{R}^3$  are two vectors such that

$$u \in U \quad \text{and} \quad u - v \in U,$$

then  $v$  is also in  $U$ .

True or false:

Justification:

True. Since  $v = u - (u - v)$ ,  $v$  is in the span of  $u$  and  $u - v$ , so by our theorem on spans,  $v$  is in  $U$ , since  $U$  is a subspace that also contains  $u$  and  $u - v$ .

(b) The set  $W = \{c + bx + (b + c)x^2 \mid b, c \in \mathbb{R}\}$  is a subspace of  $F(\mathbb{R})$ .

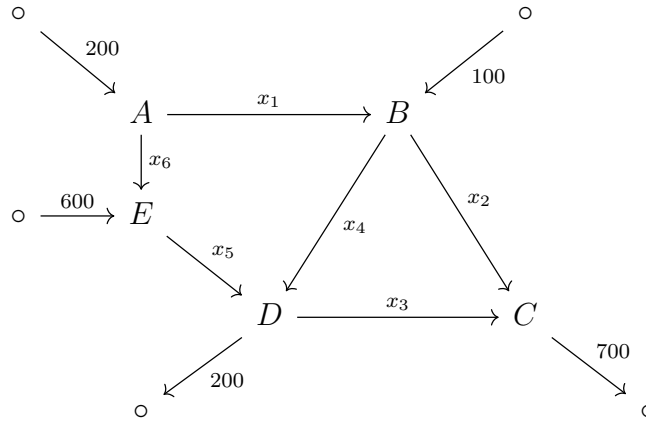
True or false:

Justification:

True:  $W = \text{span}\{x + x^2, 1 + x^2\}$  so it is a subspace.

6. (6 = 2+2+2 points) Consider the following network of storm-water pipes in a city, which meet at intersections A, B, C, D and E below. Civil engineers designed valves at each of the intersections to allow operators to regulate the flow of water. The arrows indicate the direction of the flow along each pipe (corresponding to downhill).

The numbers refer to the constant flow rate, measured in thousands of litres per hour, along the given sections of pipe, at a given time. Each  $x_i$  in the diagram denotes the unknown flow rate along that section of pipe during the same time period.



(a) Write down the linear system which describes the water flow, **together with the constraints** on the variables  $x_i$ , for  $i \in \{1, 2, 3, 4, 5, 6\}$ . **Do not solve this linear system; this is done for you in part (b).** Note: no credit is given for copying the equations implicit in (b).

By comparing flow in to flow out (Kirchoff's laws) we have

$$\begin{aligned} 200 &= x_1 + x_6 \\ x_1 + 100 &= x_2 + x_4 \\ x_2 + x_3 &= 700 \\ x_4 + x_5 &= x_3 + 200 \\ x_6 + 600 &= x_5 \end{aligned}$$

Moreover, all of the variables must satisfy  $x_i \geq 0$ , since water should flow in the given direction.

(b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[ \begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 200 \\ 0 & 1 & 0 & 1 & 0 & 1 & 300 \\ 0 & 0 & 1 & -1 & 0 & -1 & 400 \\ 0 & 0 & 0 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Find the general solution. (Ignore the constraints at this point.)

Reading the augmented matrix gives

$$\begin{aligned} x_1 + x_6 &= 200 \\ x_2 + x_4 + x_6 &= 300 \\ x_3 - x_4 - x_6 &= 400 \\ x_5 - x_6 &= 600 \end{aligned}$$

We see that  $x_4$  and  $x_6$  are non-leading variables, so we replace them by parameters  $s$  and  $t$  respectively. This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 200 - t \\ 300 - s - t \\ 400 + s + t \\ s \\ 600 + t \\ t \end{bmatrix}$$

(c) Now incorporating the constraints, suppose  $BD$  were closed due to blockage. Find the minimum flow along  $DC$ , **using your results from (b)**.

Blockage at  $BD$  means  $x_4 = 0$ , so  $s = 0$ . Flow along  $DC$  is then  $x_3 = 400 + t$ , and since  $t \geq 0$ , this means  $x_3 \geq 400$ , which makes sense from the diagram.

7. (6 points). Let  $\mathbf{M}_{2,2}$  denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2,2} \mid a + b - 2d = 0, \text{ and } c = 0 \right\}.$$

- (a) Either check that  $U$  is closed under addition, or express  $U$  in another form so you can simply state a theorem that guarantees that  $U$  is a subspace of  $\mathbf{M}_{2,2}$ .  
 (For parts (b) and (c) you may assume that  $U$  is a subspace of  $\mathbf{M}_{2,2}$ .)
- (b) Find a spanning set for  $U$ .
- (c) Give a matrix  $A \in \mathbf{M}_{2,2}$  such that  $A \notin U$ .

We note that  $a = -b + 2d$  so

$$\begin{aligned} U &= \left\{ \begin{bmatrix} -b + 2d & b \\ 0 & d \end{bmatrix} \mid b, d \in \mathbb{R} \right\} \\ &= \left\{ b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mid b, d \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\} \end{aligned}$$

and thus  $U$  is the span of two vectors, whence it is a subspace.

Alternatively, to show that  $U$  is closed under addition, we begin with two elements of  $U$ :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in U$$

which means that  $a + b - 2d = 0$  and  $c = 0$  and  $a' + b' + 2d' = 0$  and  $c' = 0$ . Now consider their sum:

$$\begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix}.$$

Is this in  $U$ ? We compute

$$(a + a') + (b + b') + 2(d + d') = (a + b + 2d) + (a' + b' + 2d') = 0 + 0 = 0$$

and

$$c + c' = 0 + 0 = 0.$$

Therefore the sum satisfies the two conditions, therefore lies in  $U$ .

(b) We found a spanning set above.

(c) We just need to choose a matrix which does not satisfy the given conditions. So for example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin U$$

since  $c \neq 0$ .

**8. (bonus, max 3 points) Do not attempt this question until you have completed the rest of the exam to the best of your ability.**

Consider the set

$$V = \{(x, x^2) \in \mathbb{R}^2 \mid x > 0\}$$

equipped with the following non-standard operations: vector addition is given by the formula

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1x_2, y_1y_2)$$

and scalar multiplication is given by the formula, for each  $c \in \mathbb{R}$ ,

$$c * (x, y) = (x^c, y^c).$$

- (a) Show that  $V$  is closed under this vector addition  $\oplus$ .  
 (b) Identify the “zero vector” of  $V$ , and show that it satisfies the axiom of being a zero vector.

- Let  $(x_1, x_1^2)$  and  $(x_2, x_2^2)$  be two elements of  $V$ , for some  $x_1, x_2 > 0$ .
- Then  $(x_1, x_1^2) \oplus (x_2, x_2^2) = (x_1x_2, x_1^2x_2^2)$  by the formula
- Since  $x_1^2x_2^2 = (x_1x_2)^2$ , this answer is in  $V$

Therefore the set is closed under vector addition.

- We have  $0 * (x, x^2) = (x^0, (x^2)^0) = (1, 1)$  so our candidate for the zero vector is  $(1, 1)$
- Let  $(x, x^2) \in V$ , for some  $x > 0$
- Then  $(1, 1) \oplus (x, x^2) = (1 \cdot x, 1 \cdot x^2) = (x, x^2)$ , as required.

Therefore the set  $V$  with these operations satisfies axiom 3.