The Grothendieck γ-filtration on projective homogeneous varieties

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2012

Introduction

Let G be a split semi-simple linear algebraic group over an arbitrary field k. Here split means Chevalley, so there is a root system, Weyl group W, etc.

Let $H^1(k, G)$ denote the pointed set of G-torsors/bundles.

One of the key problems in the theory of torsors and linear algebraic groups is to construct an invariant, i.e. a computable non-trivial map

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- Cohomological invariants in the sense of Serre

$$\mathcal{F}\colon H^1(k,G) \to H^n_{\mathrm{Gal}}(k,C) \text{ or } K_n^M(k)/p$$

- Usual cohomology rings: for every $\xi \in H^1(k,G)$

$$\mathcal{F}\colon \xi \mapsto h({}_\xi G) \text{ or } h({}_\xi\mathcal{B}),$$

where h is a cohomology theory, e.g. Chow groups, motivic cohomology, Grothendieck's K_0, Levine-Morel's Ω^* etc.,
${}_\xi G$ is an algebraic group (non-split),
${}_\xi\mathcal{B} = {}_\xi G/B$ is the associated variety of Borel subgroups.

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$$\mathcal{F}\colon \xi \mapsto \gamma^*({}_\xi\mathcal{B}),$$

where γ^* is the graded commutative ring associated to the γ-filtration.

Observe that γ^* is not a cohomology theory in the usual sense (no push-forwards).

By the Riemann-Roch theorem $\gamma^*(-)\otimes\mathbb{Q} \simeq CH^*(-)\otimes\mathbb{Q}$ which is a free Abelian group in the case of ${}_\xi G/B$, therefore, it is a question about the torsion part $\gamma^*_t(-) := Tors\,\gamma^*(-)$ only.

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Definition [SGA6, Manin]

Let X be a smooth projective variety over a field k. The i-th term of γ-filtration on X is defined to be an ideal generated by products

$$\gamma_{\geq i} := \left\{ \gamma_{n_1}(x_1)\gamma_{n_2}(x_2)\cdot\ldots\cdot\gamma_{n_m}(x_m) \mid \begin{array}{c} x_1, x_2, \ldots, x_m \in K_0(X), \\ n_1 + n_2 + \ldots + n_m \geq i \end{array} \right\},$$

where γ_n is the n-th characteristic class in K_0 which satisfies usual axioms, e.g. Whitney sum formula. For example, for a line bundle $\mathcal{L}$ over X we have

$$\gamma_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] \text{ and } \gamma_2([\mathcal{L}]) = 0.$$

We define $\gamma^i(X) := \gamma_{\geq i}/\gamma_{\geq i+1}$ and $\gamma^*(X) := \bigoplus_{i\geq 0} \gamma^i(X)$.

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Examples of computations: upper bounds

We have $\gamma^1(X) = Pic(X)$. Hence, $\gamma^1_t({}_\xi\mathcal{B}) = 0$.

for groups of type A_n and C_n $\gamma^i_t(\mathcal{B}) = 0$;
for groups of type B_n and D_n there exists a non-trivial torsion in $\gamma^i_t(\mathcal{B})$ for some $i \geq 2$ [Grothendieck, SGA6, Exp.14].

$\gamma^2_t(SB(D))$ was computed by [Karpenko, 96].

Theorem [Garibaldi, Z., 2010] If G is simply connected, then

$$\gamma^2_t({}_\xi\mathcal{B}) \simeq \bigoplus_{i\ finite} \mathbb{Z}/n_i\mathbb{Z}, \quad \text{and } \gamma^3_t({}_\xi\mathcal{B}) \simeq \bigoplus_{i\ finite} \mathbb{Z}/n_i\mathbb{Z}$$

where n_i divides the Dynkin index N of G. Moreover, for a simple G there exists a G-torsor ξ such that $\gamma^2_t({}_\xi G/B) \simeq \mathbb{Z}/N\mathbb{Z}$.

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In general, we expect that for a simply connected G

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Low bounds

Essentially nothing is known about $\gamma_t^i({}_\xi\mathcal{B})$, $i \geq 2$, if G is not simply connected (we recall that $\xi \in H^1(k, G)$).

The following result provides a uniform low bound for $\gamma_t^i({}_\xi\mathcal{B})$:

Theorem [Z., 2011] For every $i \geq 0$ there is a surjective group homomorphism

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The twisted γ-filtration

Consider the canonical surjection

$$q\colon K_0(\mathcal{B}) \simeq \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} \twoheadrightarrow \mathbb{Z}[C^*] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} \simeq K_0(G),$$

where Λ is the weight lattice, T^* is the group of characters of a split maximal torus and $C^* = \Lambda/T^*$ is the group of characters of the center of G.

Given a torsor ξ we define a twisted filtration on $K_0(G)$ to be the image of the γ-filtration on $K_0({}_\xi\mathcal{B})$ via the composite

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Computation of the twisted filtration

According to [Steinberg] $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with a basis $\{e^{\rho_w}\}_{w\in W}$, where the weights ρ_w are defined as follows:

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Then using Panin's computation of K-theory of twisted flag varieties, the twisted filtration can be computed as follows

$$(\gamma_\xi)_{\geq i} = \langle \prod_{j=1}^{m} \binom{ind(\beta(\bar{\rho}_{w_j}))}{n_j} (1-e^{\bar{\rho}_{w_j}})^{n_j} \mid n_1+\ldots+n_m \geq i,\ w_j \in W \rangle$$

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Examples

Assume that $C^* = \langle\sigma\rangle$ is cyclic of order 2, e.g. G is a half-spin group.

Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ.

Let ξ be a G-torsor and let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma,\xi}$.

$$\text{if } i_A = 1, \text{ then } \gamma^2_\xi = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

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Corollary (Z.) If $v_2(d) > i_A \geq 3$, then for every $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}({}_\xi\mathcal{B})$. Moreover, its image in $\gamma_\xi^{2/3} = \mathbb{Z}/2\mathbb{Z}$ is non-trivial and in $\gamma^{2/3}(\mathcal{B})$ (via *res*) is trivial.

Example. Let $G = HSpin_{2n}$. Then for any algebra with orthogonal involution (A, δ) where $8 \mid ind(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^{2/3}({}_\xi\mathcal{B})$ which vanishes over a splitting field of (A, δ).

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Applications of γ_t^2

The group $\gamma_t^2({}_\xi\mathcal{B})$ describes/provides information about

- the torsion of $CH^2({}_\xi\mathcal{B})$, i.e. the torsion of the Chow group of a twisted flag variety [Garibaldi, Z., 2010].
- the torsion of the Grothendieck-Chow motive associated to ξ (generalized Rost motives) and, therefore, is related to the discrete motivic invariant of a torsor (the J-invariant) [Queguiner-Mathieu, Semenov, Z., 2011], [Junkins, 2011].
- Cohomological invariants in degree 3:
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Related publications:

- Baek, S.; Neher, E.; Zainoulline, K. *Basic polynomial invariants, fundamental representations and the Chern class map.* arxiv.org:1106.4332 (2011), 13pp.
- Garibaldi, S.; Zainoulline, K. *The gamma-filtration and the Rost invariant.* arXiv:1007.3482 (2010), 19pp.
- Junkins, C. *The J-invariant and Tits indexes for groups of inner type E_6.* arXiv:1112.1454 (2011), 12pp.
- Queguiner-Mathieu A.; Semenov, N.; Zainoulline, K. *The J-invariant, Tits algebras and Triality.* arXiv:1104.1096 (2011), 28pp.
- Petrov, V.; Semenov, N.; Zainoulline, K. *J-invariant of linear algebraic groups.* Ann. Sci. Ec. Norm. Sup. (4) 41 (2008), no.6, 1023–1053.
- Zainoulline, K. *Twisted γ-filtration of a linear algebraic group* to appear in Compositio Math. (2012), 10pp.