

Equivariant pretheories and invariants of torsors

Kirill Zaynullin

Department of Mathematics and Statistics
University of Ottawa

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This is a report on the joint paper with Stefan Gille
(<http://arxiv.org/abs/1007.3780>).

- All schemes/varieties are defined over the base field k .
- By a scheme over a field k (k -scheme) we mean a reduced separated Noetherian scheme over k .
- By a variety over a field k (k -variety) we mean a quasi-projective scheme of finite type over k .
- By pt we denote $\text{Spec } k$.
- If I/k is a field extension and X is a k -scheme, we define $X_I = X \times_{\text{pt}} \text{Spec } I$ to be the respective base change.
- By an algebraic group we mean an *affine smooth group scheme* over k .
- By an action of an algebraic group G on a scheme X we mean a morphism $G \times_{\text{pt}} X \rightarrow X$ of schemes over k (all group actions are assumed to be on the left), subject to the usual axioms).
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Part I. Equivariant pretheories.

Examples:

- equivariant Chow- and K-theory
- equivariant algebraic cobordism
- equivariant cycle (co)homology
- spectral sequence for equivariant cycle homology

Equivariant pretheories. Definition.

Let G be an algebraic group over a field k .

Consider a contravariant functor from the category of smooth G -varieties over k to the category of abelian groups

$$h_G: G\text{-Sm}_k \longrightarrow \mathfrak{Ab}, \quad X \mapsto h_G(X).$$

Given $X, Y \in G\text{-Sm}_k$ and a G -equivariant map $f: X \rightarrow Y$ the induced functorial map $h_G(Y) \rightarrow h_G(X)$ is called a *pull-back* and is denoted by f_G^* .

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Equivariant pretheories. Definition.

The functor $\mathfrak{h}_G: G\text{-Sm}_k \rightarrow \mathfrak{Ab}$ is called a *G-equivariant pretheory* over k if it satisfies the following two axioms:

- H. (homotopy invariance) For a G -equivariant map $p: \mathbb{A}_k^n \rightarrow \text{pt}$ (where G acts trivially on pt) the induced pull-back

$$p_G^*: \mathfrak{h}_G(\text{pt}) \longrightarrow \mathfrak{h}_G(\mathbb{A}_k^n)$$

is an isomorphism.

- L. (localization) For a smooth G -variety X and a G -equivariant open embedding $\iota: U \hookrightarrow X$ the induced pull-back

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A G -equivariant pretheory \mathfrak{h}_G is called *essential* if it can be extended to the category of essentially smooth varieties $G\text{-Ess}_k$ with G -equivariant flat morphisms, i.e.

$$\mathfrak{h}_G: G\text{-Ess}_k \longrightarrow \mathfrak{Ab},$$

in such a way that

- C. For every $f: X \rightarrow Y$ and $U \subseteq Y$ as above the canonical morphism induced by the pull-backs

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is an isomorphism.

Here X_K denotes the generic fiber of f and $\bar{\mathfrak{h}}_G(X_K)$ denotes the colimit $\varinjlim_{U \subseteq Y} \mathfrak{h}_G(f^{-1}(U))$, where U ranges over the set of non-empty open G -equivariant subsets of Y .

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Examples: Equivariant Chow-theory and K -theory

Let G be an algebraic group over k and let X be a smooth G -variety. The functor

$$h_G : X \longmapsto \mathrm{CH}_G^*(X) = \bigoplus_{i \in \mathbb{Z}} \mathrm{CH}_G^i(X)$$

where $\mathrm{CH}_G^*(X)$ is the equivariant Chow-theory of Totaro, Edidin and Graham, provides an example of an essential G -equivariant pretheory.

Consider the category $\mathcal{P}(G, X)$ of locally free G -modules on X . It is an exact category and following Thomason one defines the i -th G -equivariant K -group $K_i(G, X)$ as Quillen's i -th K -group of $\mathcal{P}(G, X)$. The functor

$$h_G(X) := K(X, G)$$

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Example: Equivariant algebraic cobordism

Assume that $\text{char}(k) = 0$. Consider a family of pairs $(V_i, U_i)_{i \in \mathbb{N}}$ of vector spaces with $U_i \subseteq V_i$ endowed with an action of an algebraic group G such that

- (i) G acts freely on U_i and $U_i \rightarrow U_i/G$ is a G -torsor,
- (ii) $V_{i+1} = V_i \oplus W_i$ for some k -subspace W_i , such that $U_i \oplus W_i \subseteq U_{i+1}$,
- (iii) $\sup \dim V_i = \infty$, and
- (iv) $\text{codim}_{V_i}(V_i \setminus U_i) < \text{codim}_{V_{i+1}}(V_{i+1} \setminus U_{i+1})$, where we consider V_i as an affine space over k .

Observe that assumption (i) ensures that the quotient $X \times^G U := (X \times_k U)/G$ is a quasi-projective variety over k . Moreover, it is smooth over k by the descent.

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Examples: Equivariant algebraic cobordism

Let G be connected. Then the n -th *equivariant cobordism group* of a smooth G -variety X is defined [Heller and Malagón-López] by

$$\Omega_n^G(X) := \varprojlim_i \Omega_{n-\dim G + \dim U_i}(X \times^G U_i),$$

where $\Omega_*(X)$ denotes the ring of algebraic cobordism of Levine-Morel.

The functor

$$h_G: X \mapsto \bigoplus_{n \in \mathbb{Z}} \Omega_n^G(X)$$

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Equivariant cycle cohomology

Following Rost we consider a *cycle module* over the field k that is a (covariant) functor M_* from the category of field extensions of k to the category of graded abelian groups satisfying several axioms.

The prototype of such a functor is *Milnor K-theory* K_*^M , and by the very definition $M_*(E) = \bigoplus_{i \in \mathbb{Z}} M_i(E)$ is a graded $K_*^M(E)$ -module for all field extensions $E \supseteq k$.

Given a quasi-projective k -variety X and a cycle module M_* Rost had defined a complex, the so called *cycle complex* (generalizing a construction of Kato for Milnor K -theory):

$$\dots \rightarrow \bigoplus_{x \in X_{(2)}} M_{n+2}(k(x)) \xrightarrow{d_2} \bigoplus_{x \in X_{(1)}} M_{n+1}(k(x)) \xrightarrow{d_1} \bigoplus_{x \in X_{(0)}} M_n(k(x)),$$

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Equivariant cycle cohomology

We denote this complex by $C_\bullet(X, M_n)$ and consider it as a homological complex with the direct sum $\bigoplus_{x \in X_{(i)}} M_{n+i}(k(x))$ in degree i .

The i -th cycle homology group $H_i(X, M_n)$ of M_n over X is then defined as $H_i(C_\bullet(X, M_n))$.

Example. Note that there is a natural isomorphism

$$H_i(X, K_{-i}^M) \simeq CH_i(X)$$

for all $i \geq 0$, where we have set $K_{-i}^M \equiv 0$ for $i < 0$.

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Equivariant cycle cohomology

To introduce the equivariant cycle homology we adapt the definition of equivariant Chow groups due to Edidin and Graham (see also works by Guillot and Totaro).

Let G be an algebraic group over k of dimension s and X a G -variety. To define an equivariant i -th cycle homology group with coefficients in the cycle module \mathbb{M}_* we chose a linear representation V of G , such that there is an open subscheme $U \hookrightarrow V$ with $\text{codim}_V(V \setminus U) \geq c = \dim X$ on which G acts freely.

By shrinking U we can assume that $U \rightarrow U/G$ is a principal bundle. The later assures that $X \times^G U := (X \times_k U)/G$ exists in the category of k -varieties.

Equivariant cycle cohomology

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Let G be an algebraic group over k of dimension s and X a G -variety. To define an equivariant i -th cycle homology group with coefficients in the cycle module \mathbb{M}_* we chose a linear representation V of G , such that there is an open subscheme $U \hookrightarrow V$ with $\text{codim}_V(V \setminus U) \geq c = \dim X$ on which G acts freely.

By shrinking U we can assume that $U \rightarrow U/G$ is a principal bundle. The later assures that $X \times^G U := (X \times_k U)/G$ exists in the category of k -varieties.

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Equivariant cycle cohomology

We call the pair (U, V) an (X, G) -admissible pair for the G -variety X . Note that for a finite number of G -varieties there always exist a pair (U, V) which is admissible for all of them.

Definition. The i -th G -equivariant cycle homology group with values in the cycle module M_* over k is defined as

$$H_i^G(X, M_*) := H_{i+l-s}(X \times^G U, M_{*(l-s)}),$$

where $s = \dim G$ and (U, V) is a (X, G) -admissible pair with $\dim V = l$ and $\operatorname{codim}_V V \setminus U \geq \dim X$. If X is smooth, we define

$$H_G^i(X, M_n) := H_{\dim X - i}^G(X, M_{n - \dim X}) = H^i(X \times^G U, M_n)$$

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Theorem (Gille, Z.) Definitions of the cyclic (co)homology don't depend on the choice of an admissible pair and the functor

$$h_G : X \mapsto H_G^*(X, M_*)$$

provides an example of a (graded) essential G -equivariant pretheory.

Spectral sequence

We provide now a version of Merkurjev's equivariant K -theory spectral sequence for the equivariant cycle homology.

Let X be a G -variety, where G is an algebraic group over k , and $T \subseteq G$ a split torus of rank m . Let χ_1, \dots, χ_m be a basis of the character group $T^* = \text{Hom}(T, \mathbb{G}_m)$.

Let T act on the affine space $\mathbb{A}_k^m = \text{Spec } k[x_1, \dots, x_m]$ by

$$t \cdot (a_1, \dots, a_m) \mapsto (\chi_1(t) \cdot a_1, \dots, \chi_m(t) \cdot a_m),$$

and on $\mathbb{A}_X^m = X \times_k \mathbb{A}_k^m$ diagonally.

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Let $Z_i \subset \mathbb{A}_k^m$ be the hyperplane defined by $x_i = 0$ for $i = 1, \dots, m$. Then $X \times_k Z_i$ are T -subvarieties of \mathbb{A}_X^m and, therefore, we have closed subschemes

$$(X \times_k Z_1) \times^G U, \dots, (X \times_k Z_m) \times^G U$$

of $\mathbb{A}_X \times^G U$, where (U, V) is a (\mathbb{A}_X^m, T) -admissible pair.

Since $U \rightarrow U/T$ is a T -torsor we have

$$\bigcap_{j \notin I} (X \times_k Z_j) \times^G U = (X \times_k Z_I) \times^G U$$

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Spectral sequence

Following Levine's construction of spectral sequence for Quillen K -theory we obtain then a convergent spectral sequence

$$\tilde{E}_1^{p,q} = \bigoplus_{|I|=p} H_{-q-m}^T(X \times_k Z_I, M_n) \implies H_{-p-q}^T(X \times_k T, M_n).$$

Using $H_{-p-q}^T(X \times_k T, M_n) \simeq H_{-p-q-m}^T(X, M_{n+m})$ and the fact that $Z_I = \mathbb{A}_k^{|I|} = \text{Spec } k[x_i, i \in I] \hookrightarrow \mathbb{A}_k^m$ is a T -equivariant vector bundle over k , we obtain that the pull-back

$$\pi_I^*_{T}: H_{-q-m-|I|}^T(X, M_{n+|I|}) \longrightarrow H_{-q-m}^T(X \times_k Z_I, M_n)$$

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A version of Merkurjev's spectral sequence

Theorem (Gille, Z.) There is a convergent spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H_{-q-p}^T(X, M_{n+p}) \implies H_{-p-q}(X, M_{n+m}).$$

Part II. Torsors and equivariant maps

Let $S = \mathrm{GL}(V)$ be the group of automorphisms of a finite dimensional k -vector space V . Let H be an algebraic subgroup of S . Consider S as a (left) H -variety.

Let \mathfrak{h}_H be a H -equivariant pretheory over k . We embed S into the affine space $\mathrm{End}_k(V)$ as a S -equivariant (and, hence, H -equivariant) open subset.

Let $\phi: S \rightarrow \mathrm{pt}$ denote the structure map. The induced pull-back ϕ_H^* factors as the composite of pull-backs

$$\mathfrak{h}_H(\mathrm{pt}) \xrightarrow{\cong} \mathfrak{h}_H(\mathrm{End}_k(V)) \twoheadrightarrow \mathfrak{h}_H(S),$$

where the first map is an isomorphism by homotopy invariance and the second map is surjective by the localization property.

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Therefore we obtain the following

Lemma 1. The induced pull-back ϕ_H^* is surjective.

Let $\mu_s: S \rightarrow S$ denote the right multiplication by $s \in S(k)$. Since $\phi \circ \mu_s = \phi$ as morphisms over k and μ_s is H -equivariant, we have $(\mu_s)_H^* \circ \phi_H^* = \phi_H^*$. Since ϕ_H^* is surjective by Lemma 1, this proves that

Lemma 2. The induced pull-back $(\mu_s)_H^*: \mathfrak{h}_H(S) \rightarrow \mathfrak{h}_H(S)$ is the identity.

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Let G be an algebraic subgroup of S such that $H \subseteq G \subseteq S$ so that G is considered as a (left) H -variety.

Let E be a (left) G -variety over k and let $\eta_E: \operatorname{Spec} K \rightarrow E$ denote its generic point, where $K = k(E)$.

Consider the G -equivariant (and, hence, H -equivariant) map

$$\psi_E: G_K = G \times_{\operatorname{Spec} k} \operatorname{Spec} K \xrightarrow{(\operatorname{id}, \eta_E)} G \times_{\operatorname{Spec} k} E \longrightarrow E$$

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Suppose that there is a G -equivariant map $\rho: E \rightarrow S$ over k .
 Then there is a commutative diagram of H -equivariant maps

$$\begin{array}{ccccc}
 G_K & \xrightarrow{\psi_E} & E & \xrightarrow{\rho} & S \\
 i \downarrow & & & & \uparrow \rho \\
 S_K & \longrightarrow & \cdot \rho(\eta_E) & \longrightarrow & S_K
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where the map i is the embedding, ρ is the projection
 $S_K = S \times_{\text{Spec } k} \text{Spec } K \rightarrow S$ to the first factor and the bottom
 horizontal map is the multiplication by $\rho(\eta_E)$.

By the diagram $(\psi_E)_H^* \circ \rho_H^* = (\rho \circ \psi_E)_H^*$ coincides with the pull
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Lemma 3. Let E be a G -variety together with a G -equivariant map $\rho: E \rightarrow S$. Then we have

$$(\psi_E)_H^* \circ \rho_H^* = i_H^* \circ p_H^*: \mathfrak{h}_H(S) \rightarrow \bar{\mathfrak{h}}_H(G_K).$$

We are now in position to prove the main result of this part

Theorem (Gille, Z.) Let $H \subset G$ be algebraic groups and let $\mathfrak{h}_H(-)$ be a H -equivariant pretheory. Then for any G -torsor E with $K = k(E)$ we have

$$\mathrm{Im}(\varphi_H^*) \subseteq \mathrm{Im}((\psi_E)_H^*) \text{ in } \bar{\mathfrak{h}}_H(G_K),$$

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Theorem then follows from Lemma 3 and the fact that there exists a finite dimensional k -vector space V and a G -equivariant map $E \rightarrow S = \mathrm{GL}(V)$.

Corollary. Let $H \subset G$ be algebraic groups over k and let $\mathfrak{h}_H(-)$ be an essential H -equivariant pretheory. Then there exists a field extension l/k and a G -torsor E over l with $L = l(E)$ such that

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Such a torsor E will be called generic.

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Theorem then follows from Lemma 3 and the fact that there exists a finite dimensional k -vector space V and a G -equivariant map $E \rightarrow S = \mathrm{GL}(V)$.

Corollary. Let $H \subset G$ be algebraic groups over k and let $\mathfrak{h}_H(-)$ be an essential H -equivariant pretheory. Then there exists a field extension l/k and a G -torsor E over l with $L = l(E)$ such that

$$\mathrm{Im}(\varphi_{H_l}^*) = \mathrm{Im}((\psi_E)_{H_l}^*) \text{ in } \bar{\mathfrak{h}}_{H_l}(G_L)$$

Such a torsor E will be called generic.

Proof of the corollary. We fix an embedding $G \rightarrow S = \mathrm{GL}(V)$ for some finite dimensional k -vector space V . The quotient $S \rightarrow G \backslash S$ (for the right action of G on S) is a (left) G -torsor. Let I be its function field and consider the cartesian square

$$\begin{array}{ccc} E & \xrightarrow{\rho} & S \\ \downarrow & & \downarrow \\ \mathrm{Spec} I & \longrightarrow & G \backslash S \end{array}$$

Since $S \rightarrow G \backslash S$ is a G -torsor, the map $E \rightarrow \mathrm{Spec} I$ is a G -torsor. The I -scheme E is a localization of S and, therefore, by (C) and (L) the pull-back $\rho_H^*: \mathfrak{h}_H(S) \rightarrow \bar{\mathfrak{h}}_H(E) \rightarrow \mathfrak{h}_H(E)$ is surjective. This implies that the pull-back $\rho_{H_I}^*: \mathfrak{h}_{H_I}(S_I) \rightarrow \mathfrak{h}_{H_I}(E)$ is surjective. It remains to apply the proof of the theorem over I and to observe that $\mathrm{Im}(\varphi_{H_I}^*) = \mathrm{Im}((\psi_E)_{H_I}^* \circ \rho_{H_I}^*) = \mathrm{Im}((\psi_E)_{H_I}^*)$.

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The latter theorem and the corollary can be viewed as a generalization of the following result

Theorem (Karpenko-Merkurjev). Consider the characteristic map for the Chow theory

$$c: S^*(T^*) \rightarrow CH(G/B),$$

where G/B is the variety of Borel subgroups. Let ${}_{\xi}G/B$ be a twisted form by means of a G -torsor ξ . Then

$$\mathrm{Im}(res) \supseteq \mathrm{Im}(c),$$

where $res: CH({}_{\xi}G/B) \rightarrow CH(G/B)$ is the restriction map. Moreover, there exists a torsor over a field extension of k such that the equality $\mathrm{Im}(res) = \mathrm{Im}(c)$ holds.

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Indeed, we have

$$T = H, G = G, S^*(T^*) = CH_T(pt), CH(G/B) = \overline{CH}_T(G)$$

$$c = \varphi_H^* \text{ and } res = (\psi_E)_H^*.$$

Note that the result by Karpenko-Merkurjev plays a fundamental role in computations of canonical/essential dimensions, discrete motivic invariants of G and in the study of splitting properties of G -torsors.

To measure the difference between $\bar{h}_H(G_K)$ and the image $\text{Im}((\psi_E)_H^*)$ we introduce the following

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Part III. Applications to equivariant oriented cohomology.

In this part we investigate the case of a B -equivariant oriented cohomology, where B is a Borel subgroup of a split semisimple linear algebraic group.

Let G be a split semisimple linear algebraic group of rank n over a field k and let T be a split maximal torus of G . Following the construction of the spectral sequence we consider the action of T on the affine space \mathbb{A}_k^n with weights χ_1, \dots, χ_n together with an action of T on G by left multiplication. Then T embeds into $\mathbb{A}_k^n = \text{Spec } k[x_1, \dots, x_n]$ as the complement of the coordinates hyperplanes Z_i , $i = 1, \dots, n$.

Let $V = \mathbb{A}_k^n \times^T G$ be the associated vector bundle over G/T . By definition $V = L_{G/T}(\chi_1) \oplus \dots \oplus L_{G/T}(\chi_n)$, where $L_{G/T}$ is the associated line bundle, and $G = T \times^T G$ embeds into V as the complement of the union of zero-sections

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Let now $\mathfrak{h}(-)$ be an oriented cohomology theory in the sense of Levine-Morel, i.e. a contravariant functor from the category of smooth varieties over k to the category of graded commutative rings satisfying certain axioms.

In particular, if X is a k -variety with an open subvariety $\iota: U \hookrightarrow X$ there is an exact sequence

$$\mathfrak{h}(Z) \xrightarrow{j_*} \mathfrak{h}(X) \xrightarrow{\iota^*} \mathfrak{h}(U) \rightarrow 0$$

where $j: Z = X \setminus U \hookrightarrow X$ is the closed complement of U , and there is also a first Chern class which we denote by $c_1^{\mathfrak{h}}$.

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$$\bigoplus_{j=1}^n \mathfrak{h}(V_j) \xrightarrow{\bigoplus_j (e_j)_*} \mathfrak{h}(V) \rightarrow \mathfrak{h}(G) \rightarrow 0$$

By the properties of the first Chern class we have

$$(e_j)_*(1_{V_j}) = c_1^{\mathfrak{h}}(L_V(\chi_j))$$

which implies that the image of $\bigoplus_j (e_j)_*$ is an ideal generated by the first Chern classes $c_1^{\mathfrak{h}}(L_V(\chi_j))$, $j = 1, \dots, n$.

Oriented cohomology of a group

Let B be a Borel subgroup of G containing T and let G/B be the variety of Borel subgroups. The composite of projections $V \rightarrow G/T \rightarrow G/B$ is a chain of affine bundles. Therefore, by the homotopy invariance there is an isomorphism $\mathfrak{h}(G/B) \xrightarrow{\cong} \mathfrak{h}(V)$ compatible with the Chern classes and we obtain the following

Proposition (Gille, Z.) There is an isomorphism of rings

$$\mathfrak{h}(G) \simeq \mathfrak{h}(G/B) / (c_1^{\mathfrak{h}}(L_{G/B}(\chi_1)), \dots, c_1^{\mathfrak{h}}(L_{G/B}(\chi_n))),$$

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Let $\mathfrak{h}_B(-)$ be an B -equivariant pretheory to the category of commutative rings such that

- (i) $\mathfrak{h}_B(E) = \mathfrak{h}(E/B)$ for every G -torsor E , where $\mathfrak{h}(-)$ is an oriented cohomology in the sense of Levine-Morel.
- (ii) $\bar{\mathfrak{h}}_B(B_K) = \mathfrak{h}(\text{pt})$ and $\bar{\mathfrak{h}}_B(G_K) \simeq \mathfrak{h}(G/B)$.

Then the ring $\hat{\mathfrak{h}}_B(E) = \bar{\mathfrak{h}}_B(G_K) \otimes_{\mathfrak{h}_B(E)} \bar{\mathfrak{h}}_B(B_K)$ can be identified with a quotient of $\bar{\mathfrak{h}}_B(G_K) \simeq \mathfrak{h}(G/B)$ modulo the ideal generated by non-constant elements from the image of the restriction $(\psi_E)_B^*: \mathfrak{h}(E/B) \rightarrow \mathfrak{h}(G/B)$.

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Then by the proposition we obtain the following

Corollary (Gille, Z.). Assume that the image of φ_B^* is generated by the Chern classes $c_1^{\mathfrak{h}}(L_{G/B}(\chi_i))$ of line bundles associated to the characters $\chi_i \in T^*$ ($i = 1 \dots n$).

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Moreover, if $\mathfrak{h}_B(-)$ is essential and E is generic, then

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Examples: Equivariant Chow-theory

Consider the equivariant Chow groups $\mathfrak{h}_B(-) = \mathrm{CH}^B(-)$.

Let E be a G -torsor.

The ring $\mathfrak{h}_B(\mathrm{pt})$ can be identified with the symmetric algebra $S(T^*)$ and the map

$$\varphi_B^*: S(T^*) = \mathfrak{h}_B(\mathrm{pt}) \longrightarrow \bar{\mathfrak{h}}_B(G_K) = \mathrm{CH}(G/B)$$

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$$\text{res}: \text{CH}(E/B) \longrightarrow \text{CH}(G/B),$$

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is the augmentation map. If E is generic, then we have

$$\widehat{\mathfrak{h}}_B(E) \simeq \text{CH}(G/B) \otimes_{S(T^*)} \mathbb{Z} \simeq \text{CH}(G).$$

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Therefore, for an arbitrary G -torsor E the ring

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is a quotient ring of $\mathrm{CH}(G/B)$ modulo the ideal generated by non-constant elements from the image of the restriction $\mathrm{CH}(E/B) \rightarrow \mathrm{CH}(G/B)$.

Examples: Equivariant Chow-theory

Observe that the characteristic map φ_B^* is not surjective in general. However, its image is a subgroup of finite index in $\text{CH}(G/B)$ measured by the torsion index τ of G . This implies that for a G -torsor E we have $\widehat{h}_B(E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

If $p \mid \tau$, then there is an isomorphism

$$\widehat{h}_B(E) \otimes_{\mathbb{Z}} \mathbb{Z}/p \simeq \frac{\mathbb{Z}/p[x_1, \dots, x_r]}{(x_1^{p^{j_1}}, \dots, x_r^{p^{j_r}})},$$

where (j_1, \dots, j_r) is the J -invariant of G twisted by E [Petrov, Semenov, Z.]. Observe that $j_i \leq k_i$, $i = 1 \dots r$, where k_i are defined via the p -exceptional degrees introduced by Kac, and for a generic torsor E we have equalities $j_i = k_i$ for each i .

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Examples: Equivariant K_0

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Let E be a G -torsor.

The ring $\mathfrak{h}_B(\text{pt})$ can be identified with the integral group ring $\mathbb{Z}[T^*]$ and with the representation ring $\text{Rep } T$ of T , i.e.

$$\mathfrak{h}_B(\text{pt}) = \mathbb{Z}[T^*] = \text{Rep } T.$$

The map

$$\varphi_B^*: \mathbb{Z}[T^*] = \mathfrak{h}_B(\text{pt}) \longrightarrow \bar{\mathfrak{h}}_B(G_K) \simeq K_0(G/B)$$

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As before the map $(\psi_E)_B^*$ coincides with the restriction map

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Corollary (Gille, Z.) Let E be a G -torsor over k and let E/B be a twisted form of G/B by E .

Then

- (i) $\mathfrak{c}(\mathbb{Z}[T^*]) \subseteq \text{res}(K_0(E/B))$;
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According to a result of Panin the image of the restriction map is given by the sublattice

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where W is the Weyl group of G , $\{g_w\}_{w \in W}$ is the Steinberg basis of $K_0(G/B)$ and $\{i_{w,E}\}$ are indexes of the respective Tits algebras.

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Then by the corollary proven in the last part there exists a maximal set of indexes $\{m_w\}_{w \in W}$ such that

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The indexes m_w are called the *maximal Tits indexes*. They have been extensively studied by Merkurjev, Panin and Wadsworth. They are closely related to the dimensions of irreducible representations of G . Comparing with the case of Chow groups one observes that

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Since the map $\mathbb{Z}[T^*] = \mathfrak{h}_B(\text{pt}) \longrightarrow \bar{\mathfrak{h}}_B(B_K) = K_0(\text{pt}) = \mathbb{Z}$ is the augmentation map, for a generic torsor E we have

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where the last isomorphism follows by the corollary.

Hence, for an arbitrary G -torsor E

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Consider the equivariant algebraic cobordism $\mathfrak{h}_B(-) = \Omega^B(-)$.

Let E be a G -torsor.

The completion $\mathfrak{h}_B(\text{pt})^\wedge$ of $\mathfrak{h}_B(\text{pt})$ at the augmentation ideal, (the kernel of $\mathfrak{h}_B(\text{pt}) \rightarrow \mathfrak{h}_B(B)$) can be identified with the formal group ring $\mathbb{L}[[T^*]]_U$ introduced by [Calmes, Petrov, Z.], where \mathbb{L} is the Lazard ring and U denotes the universal formal group law.

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- What is the analogue of p -exceptional degrees/maximal Tits indices for algebraic Morava K -theories/cobordisms ? The same question for cycle (co)homology theories.
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