1. Let M be an abelian group, i.e. a $\mathbb{Z}$-module, $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/75\mathbb{Z}$

- Find the annihilator of M in $\mathbb{Z}$ (1)
- Let $I = 3\mathbb{Z}$. Describe the annihilator of I in M as a direct product of cyclic groups. (1)
- Compute the tensor product $M \otimes_{\mathbb{Z}} \mathbb{Z}/9\mathbb{Z}$ (1)
- Find the torsion submodule of M (1)

2. Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$. (2)

3. Suppose R is commutative and let I and J be ideals of R. So R/I, R/J, I and J are naturally R-modules.

- Prove that there is an R-module isomorphism (2)

$$R/I \otimes_R R/J \simeq R/(I + J).$$

- Show that there is a surjective R-module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $i \otimes j$ to ij. Give an example to show that this homomorphism need not be injective. (2)

4. Let $K = \mathbb{Q}(\xi_5) \otimes_{\mathbb{Q}} \mathbb{Q}(\xi_6)$ be the tensor product of two cyclotomic field extensions.

- Determine whether K is a field or not (1)
- Find the dimension of K over $\mathbb{Q}$ (1)