Borel-Moore Homology

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Outline

1. Construction
2. Examples
3. Properties

Definitions and Notation

Notation

- X is a topological space.
- Δ^n is the standard n-simplex.
- $C_n(X)$ is the group of all n-chains.
- $\sim$: homeomorphic
- $\simeq$: homotopy equivalent

Definition (Support)

Let $\sigma = n_1\sigma_1 + \cdots + n_k\sigma_k \in C_n(X)$. The *support* of σ is

$$\text{supp}(\sigma) = \bigcup_{i \mid n_i \neq 0} \sigma_i(\Delta^n).$$

What's Borel-Moore Homology all about?

Idea:

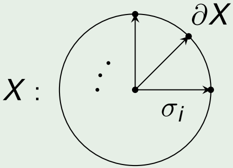
- Singular homology $\rightarrow$ (the support of) cycles are compact.
- Borel-Moore homology $\rightarrow$ want to allow non-compact cycles.
- To do this:

$$\text{Replace } \; C_n(X) = \bigoplus_{\sigma:\Delta^n \to X} \mathbb{Z}\,\sigma \; \text{ with } \; C_n'(X) = \prod_{\sigma:\Delta^n \to X} \mathbb{Z}\,\sigma.$$

But be careful!

Example (1)

Let $X = D^2$. Choose infinitely many rays $\sigma_i : \Delta^1 \to X$ from 0 to ∂X.



Let $\sigma = \sum_i \sigma_i \in C'_1(X)$.

Then the coefficient of 0 in $\partial_1(\sigma)$ is $\sum_i \partial_1(\sigma_i) = -\infty$.

$\implies$ the boundary map is not well-defined!

Constructing Borel-Moore Homology

Definition (Locally finite)

Let $\sigma = \sum_i n_i \sigma_i \in C'_n(X)$. Then σ is *locally finite* if $\forall\ x \in X$, $\exists$ a neighbourhood $U \subseteq X$ of x such that

$$\{\sigma_i \mid n_i \neq 0 \text{ and } \sigma_i(\Delta^n) \cap U \neq \emptyset\}$$

is finite. From now on, $C'_n(X) =$ locally finite n-chains.

Remark

We can naturally extend the boundary map to

$$\partial_n : C'_n(X) \to C'_{n-1}(X)$$

since every $(n-1)$-simplex appears as the face of only finitely many σ_i.

Definition of Borel-Moore Homology

Definition (Borel-Moore homology)

Consider the chain complex

$$\cdots \to C'_{n+1}(X) \xrightarrow{\partial_{n+1}} C'_n(X) \xrightarrow{\partial_n} C'_{n-1}(X) \to \cdots \to C'_0(X) \to 0.$$

The n-th *Borel-Moore homology group* is

$$H_n^{\mathrm{BM}}(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Remark

$H_n^{\mathrm{BM}}(X) = H_n(X)$ if X is compact.

Alternative Definitions

Theorem

Let X be a "nice" topological space (e.g. real or complex varieties). Then the following are equivalent definitions of Borel-Moore homology:

- *Let $\widehat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Then*
  $$H_n^{\mathrm{BM}}(X) \cong H_n(\widehat{X}, \infty).$$
- *Let $\overline{X}$ be any compactification of X such that $(\overline{X}, \overline{X} \setminus X)$ is a CW-pair (CW-complex and a subcomplex). Then*
  $$H_n^{\mathrm{BM}}(X) \cong H_n(\overline{X}, \overline{X} \setminus X).$$

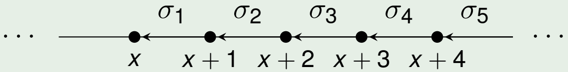
Some Examples

Example (2)

Let $X = \mathbb{R}$. Recall ordinary homology: $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for all $n \geq 1$.

Borel-Moore homology:

- $H_0^{\text{BM}}(X) = \ker \partial_0 / \operatorname{im} \partial_1 = C_0'(X) / \operatorname{im} \partial_1$. Let $x \in X$.



Let $\sigma = \sum_{i \in \mathbb{N}} \sigma_i$. Then

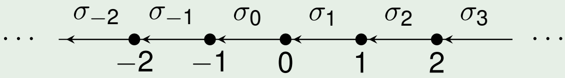
$$\begin{aligned} \partial_1(\sigma) &= \partial_1(\sigma_1) + \partial_1(\sigma_2) + \partial_1(\sigma_3) + \dots \\ &= x - (x+1) + (x+1) - (x+2) + (x+2) - (x+3) + \dots \\ &= x \end{aligned}$$

Some Examples

Example (2, continued)

$\implies \partial_1$ is surjective, so $H_0^{\mathrm{BM}}(X) = 0$.

- $H_1^{\mathrm{BM}}(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1 \quad (\text{since } \partial_2 = 0)$.



If $\sigma = \sum_{i \in \mathbb{Z}} n_i \sigma_i \in \ker \partial_1$, then

$$\sum_{i \in \mathbb{Z}} n_i \partial_1(\sigma_i) = \sum_{i \in \mathbb{Z}} n_i((i-1) - i) = 0$$

$\implies n_i = n$ for some fixed $n \in \mathbb{Z}$ for all i.
$\implies H_1^{\mathrm{BM}}(X) \cong \mathbb{Z}$.

Some Examples

Example (2, concluded)

- $H_n^{\text{BM}}(X) = 0$ for all $n \geq 2$.

Example (3)

Let $X = \mathbb{R}$ (again). Compute $H_n^{\text{BM}}(X)$ using other definitions.

- One point compactification: $\widehat{X} = X \cup \{\infty\} \sim \mathbb{S}^1$.

$$H_n^{\text{BM}}(X) \cong H_n(\widehat{X}, \infty) = H_n(\mathbb{S}^1, \{*\}) \cong \widetilde{H_n}(\mathbb{S}^1) = \begin{cases} \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

Some Examples

Example (3, continued)

- "Arbitrary" compactification: Let $\overline{X} = X \cup \{\pm\infty\} \sim [0,1]$.

  $$H_n^{\text{BM}}(X) \cong H_n(\overline{X}, \overline{X} \setminus X) = H_n\left([0,1], \{0,1\}\right)$$

  We have the exact sequences:

  $$\underbrace{H_0(\{0,1\})}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{i_*} \underbrace{H_0([0,1])}_{\mathbb{Z}} \overset{j_*}{\twoheadrightarrow} H_0([0,1],\{0,1\}) \to 0$$

  $$\implies H_0([0,1],\{0,1\}) \cong \mathbb{Z}\,/\ker j_* = \mathbb{Z}\,/\operatorname{im} i_* = \mathbb{Z}\,/\,\mathbb{Z} \cong 0.$$

  And,

  $$\underbrace{H_1([0,1])}_{0} \to H_1([0,1]),\{0,1\}) \overset{\delta}{\hookrightarrow} \underbrace{H_0(\{0,1\})}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{i_*} \underbrace{H_0([0,1])}_{\mathbb{Z}}$$

  $$\implies H_1([0,1],\{0,1\}) \cong \operatorname{im}\delta = \ker i_* \cong \mathbb{Z}.$$

  Finally, all higher homologies vanish.

Quick Observation

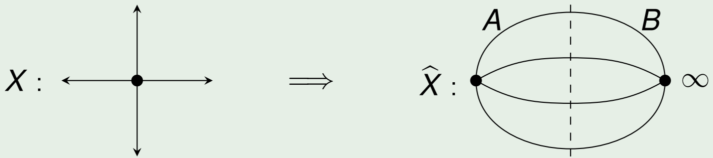
Remark

- Can generalize: $H_n^{\text{BM}}(\mathbb{R}^m) = \begin{cases} \mathbb{Z}, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$
- Borel-Moore homology is not invariant under homotopy, since $\mathbb{R}^m \simeq \{*\}$, but $H_n^{\text{BM}}(\{*\}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$

One more example

Example (4)

Let $X = \mathbb{R} \vee \mathbb{R}$. Recall, $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for all $n \geq 1$.



By Mayer–Vietoris sequence:

$$H_n^{\mathrm{BM}}(X) = H_n(\widehat{X}, \infty) = \widetilde{H_n}(\widehat{X}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

Pushforwards

Recall

- In ordinary homology, $f : X \to Y$ defines a pushforward $f_* : H_n(X) \to H_n(Y)$.
- Not (necessarily) so in Borel-Moore homology.

Example (5)

Let $X = \mathbb{R} \setminus \{0\}$, $Y = \mathbb{R}$ and $f : X \hookrightarrow Y$ inclusion. Let $\sigma_i : \Delta^1 \to X$ be a path from $1/2^i$ to $1/2^{i+1}$ for $i \in \mathbb{N}$. Let $\sigma = \sum_{i \in \mathbb{N}} \sigma_i$. Try to apply "$f_\#$":

$$f_\#(\sigma) = \sum_{i \in \mathbb{N}} f \circ \sigma_i$$

Then $f_\#(\sigma) \notin C_1'(Y)$ since it is not locally finite at 0.

To Fix This

Definition (Proper map)

A map $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for all compact $K \subseteq Y$.

Proposition

A proper map $f : X \to Y$ (of spaces that admit closed embeddings into Euclidean space) induces a pushforward $f_* : H_n^{\mathrm{BM}}(X) \to H_n^{\mathrm{BM}}(Y)$.

Cohomology

Definition (Cohomology)

Define the *group of n-cochains* to be

$$C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z}),$$

and n-th *coboundary map* to be

$$\delta^n : C^n(X) \to C^{n+1}(X), \text{ by } (\delta^n \varphi)(\sigma) = \varphi(\partial_{n+1}\sigma).$$

We obtain a *cochain complex*:

$$\cdots \leftarrow C^{n+1}(X) \xleftarrow{\delta^n} C^n(X) \xleftarrow{\delta^{n-1}} C^{n-1}(X) \leftarrow \cdots \leftarrow C^0(X) \leftarrow 0.$$

The n-th *cohomology group* of X is $H^n(X) = \ker \delta^n / \operatorname{im} \delta^{n-1}$.
Define *Borel-Moore cohomology* $H^n_{\mathrm{BM}}(X)$ in the same way.

What's the point?

Why is cohomology useful?

- Can be given a ring structure (via "cup product" $H^m \times H^n \to H^{m+n}$).
- The homology $\bigoplus_{n \in \mathbb{N}} H_n$ is a module over this ring (via "cap product" $H^m \times H_n \to H_{n-m}$).
- "Encodes" more information about X.

Poincaré duality

Theorem (Poincaré duality)

- *Let X be a closed subset of a smooth, oriented manifold M with $\dim_{\mathbb{R}} M = m$. Suppose X has a closed neighbourhood $U \subseteq M$ such that X is a proper deformation retract of U. Then*
$$H_n^{\mathrm{BM}}(X) \cong H^{m-n}(M, M \setminus X)$$
- *Let M be an oriented manifold with $\dim_{\mathbb{R}} = m$. Then*
$$H^n_{\mathrm{BM}}(M) \cong H_{m-n}(M)$$

Building Long Exact Sequences

Theorem

Let $Z \subseteq Y \subseteq X$. There is a long exact sequence of cohomology groups

$$\cdots \to H^n(X,Y) \to H^n(X,Z) \to H^n(Y,Z) \to H^{n+1}(X,Y) \to \cdots .$$

Theorem

Let X be "nice", U open in X and $Y = X \setminus U$. Then there is a long exact sequence of Borel-Moore homology groups

$$\cdots \to H_n^{\mathrm{BM}}(Y) \to H_n^{\mathrm{BM}}(X) \to H_n^{\mathrm{BM}}(U) \to H_{n-1}^{\mathrm{BM}}(Y) \to \cdots .$$

Proof: *Embed X into a smooth, oriented manifold M, take the long exact sequence of cohomology groups of $M \setminus X \subseteq M \setminus Y \subset M$ and apply Poincaré duality.*

That's all!

Thank you :-)