Chapter 3

Curve Sketching

How much metal would be required to make a 400-mL soup can? What is the least amount of cardboard needed to build a box that holds 3000 cm³ of cereal? The answers to questions like these are of great interest to corporations that process and package food and other goods. In this chapter, you will investigate and apply the relationship between the derivative of a function and the shape of its graph. You will use derivatives to determine key features of a graph, and you will find optimal values in real situations.

By the end of this chapter, you will

- determine numerically and graphically the intervals over which the instantaneous rate of change is positive, negative, or zero for a function that is smooth over these intervals, and describe the behaviour of the instantaneous rate of change at and between local maxima and minima
- solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions, radical functions, and other simple combinations of functions
- sketch the graph of a derivative function, given the graph of a function that is continuous over an interval, and recognize points of inflection of the given function
- recognize the second derivative as the rate of change of the rate of change, and sketch the graphs of the first and second derivatives, given the graph of a smooth function
- determine algebraically the equation of the second derivative \( f''(x) \) of a polynomial or simple rational function \( f(x) \), and make connections, through investigation using technology, between the key features of the graph of the function and corresponding features of the graphs of its first and second derivatives
- describe key features of a polynomial function, given information about its first and/or second derivatives, sketch two or more possible graphs of the function that are consistent with the given information, and explain why an infinite number of graphs is possible
- sketch the graph of a polynomial function, given its equation, by using a variety of strategies to determine its key features, and verify using technology
- solve optimization problems involving polynomial, simple rational, and exponential functions drawn from a variety of applications, including those arising from real-world situations
- solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results.
Prerequisite Skills

Factoring Polynomials

1. Factor each polynomial fully.
   
a) \(x^3 + 2x^2 + 2x + 1\)
   
b) \(z^3 - 6z - 4\)
   
c) \(t^3 + 6t^2 - 7t - 60\)
   
d) \(b^3 + 8b^2 + 19b + 12\)
   
e) \(3n^3 - n^2 - 3n + 1\)
   
f) \(2p^3 - 9p^2 + 10p - 3\)
   
g) \(4k^3 + 3k^2 - 4k - 3\)
   
h) \(6w^3 - 11w^2 - 26w + 15\)

Equations and Inequalities

2. Solve each equation. State any restrictions on the variable.
   
a) \(x^2 - 7x + 12 = 0\)
   
b) \(4x^2 - 9 = 0\)
   
c) \(18\nu^2 = 36\nu\)
   
d) \(a^2 + 5a = 3a + 35\)
   
e) \(4.9t^2 - 19.6t + 2.5 = 0\)
   
f) \(x^3 + 6x^2 + 3x - 10 = 0\)
   
g) \(\frac{x^2 - 5x - 14}{x^2 - 1} = 0\)

   
a) \(2x - 10 > 0\)
   
b) \(x(x + 5) < 0\)
   
c) \(x^2(x - 4) > 0\)
   
d) \(x^2 + 5x - 14 < 0\)
   
e) \((x - 3)(x + 2)(x - 1) > 0\)
   
f) \(\frac{x}{x^2 - 1} > 0\)

4. Determine the \(x\)-intercepts of each function.
   
a) \(f(x) = 5x - 15\)
   
b) \(g(x) = x^2 - 3x - 28\)
   
c) \(h(x) = x^3 + 6x^2 + 11x + 6\)
   
d) \(y = \frac{x^2 - 9}{x^2 + 1}\)

Polynomial and Simple Rational Functions

5. State the domain and range of each function using set notation.
   
a) \(y = 2x + 1\)
   
b) \(f(x) = x^2 - 9\)
   
c) \(f(x) = x^3 - 5x^2 + 2\)
   
d) \(g(x) = \frac{1}{x + 1}\)
   
e) \(f(x) = \frac{x^2 - 4}{x - 2}\)
   
f) \(k(x) = \frac{3}{x^2 - 9}\)
   
g) \(p(x) = \frac{x}{x^2 + 1}\)

6. For each function in question 5, determine whether the function has any asymptotes. Write the equations of any asymptotes.

7. State the intervals of increase and decrease for each function.
   
a) 
   ![Graph](image)
   
b) 
   ![Graph](image)
8. Determine the derivative of each function.
   a) \( f(x) = 5x^2 - 7x + 12 \)
   b) \( y = x^3 - 2x^2 + 4x - 8 \)
   c) \( f(x) = \frac{1}{x} \)
   d) \( y = \frac{x^2 - 9}{x^2 + 1} \)

9. Modelling Algebraically
   A 40-cm by 60-cm piece of tin has squares cut from each corner as shown in the diagram. The sides are then folded up to make a box with no top. Let \( x \) represent the side length of the squares. Write an expression for the volume of the box.

10. A right cylinder has a volume of 1000 cm\(^3\). Express the surface area of the cylinder in terms of its radius. Recall that the formula for the volume of a cylinder is \( V = \pi r^2 h \), and the formula for the surface area of a cylinder is \( SA = 2\pi r^2 + 2\pi rh \).

11. State whether each function is even, odd, or neither.
   a) \( y = 2x \)
   b) \( r(x) = x^2 + 2x + 1 \)
   c) \( f(x) = -x^2 + 8 \)
   d) \( s(t) = t^3 - 27 \)
   e) \( h(x) = x + \frac{1}{x} \)
   f) \( f(x) = \frac{x^2}{x^2 - 1} \)

Symmetry
An even function \( f(x) \) is symmetrical about the \( y \)-axis: \( f(x) = f(-x) \) for all values of \( x \). An odd function \( f(x) \) is symmetrical about the origin: \( f(-x) = -f(x) \) for all values of \( x \).

PROBLEM

Naveen bought 20 m of flexible garden edging. He plans to put two gardens in the back corners of his property: one square and one in the shape of a quarter circle. He will use the edging on the interior edges (shown in green on the diagram). How should Naveen split the edging into two pieces in order to maximize the total area of the two gardens? Assume that each border piece must be at least 5 m long.
Increasing and Decreasing Functions

In many situations, it is useful to know how quantities are increasing or decreasing. A company might be interested in which factors result in increases in productivity or decreases in cost. By studying population increases and decreases, governments can predict the need for essential services, such as health care.

**Investigate** How can you identify intervals over which a continuous function is increasing or decreasing?

**Method 1: Use The Geometer’s Sketchpad®**

1. Open The Geometer’s Sketchpad®. Graph the function \( f(x) = x^3 - 4x \).
2. Click the function \( f(x) = x^3 - 4x \). From the Graph menu, choose Derivative. Graph the derivative on the same set of axes as the function.

3. a) Over which values of \( x \) is \( f(x) \) increasing?
   b) Over which values of \( x \) is \( f(x) \) decreasing?
4. **Reflect** Refer to the graphs of \( f(x) \) and \( f'(x) \).
   a) What is true about the graph of \( f'(x) \) when \( f(x) \) is increasing?
   b) What is true about the graph of \( f'(x) \) when \( f(x) \) is decreasing?
   c) What do your answers to parts a) and b) mean in terms of the slope of the tangent to \( f(x) \)?
5. Copy or print the graphs on the same grid. For \( f(x) \), colour the increasing parts blue and the decreasing parts red.
6. a) Draw a vertical dotted line through the points on \( f(x) \) at which the slope of the tangent is zero. Compare the graphs of \( f(x) \) and \( f'(x) \).
   b) **Reflect** What is the behaviour of the graph of \( f'(x) \) when \( f(x) \) is increasing? when \( f(x) \) is decreasing?
Method 2: Use a Graphing Calculator

1. Graph the function \( f(x) = x^3 - 4x \).

2. a) Over which values of \( x \) is \( f(x) \) increasing?
   b) Over which values of \( x \) is \( f(x) \) decreasing?

3. Follow these steps to calculate the first differences in list L3.
   - Clear the lists. In list \( L1 \), enter \(-5 \) to \(+5\), in increments of \( 0.5 \).
   - Highlight the heading of list \( L2 \). Press \( \text{ALPHA} \ ["] \ 2\text{ND} \ [L1] \ 3 \ 4 \ \text{ALPHA} \ ["] \ \text{ENTER} \).
   - Highlight the heading of list \( L3 \). Press \( 2\text{ND} \ [\text{LIST}] \ 7: \Delta \text{List} \ [\text{2ND} \ [L2] \ \text{ENTER} \).

4. a) For which values of \( x \) are the first differences positive?
   b) For which values of \( x \) are the first differences negative?

5. Reflect Refer to your answers in steps 2 and 4. Describe the relationship between the first differences and the intervals over which a function is increasing or decreasing.

6. How can the TRACe function be used to determine when the function is increasing or decreasing?

The first derivative of a continuous function \( f(x) \) can be used to determine the intervals over which the function is increasing or decreasing. The function is increasing when \( f'(x) > 0 \) and decreasing when \( f'(x) < 0 \).

Example 1 Find Intervals

Find the intervals of increase and decrease for the function defined by \( f(x) = 2x^3 + 3x^2 - 36x + 5 \).

Solution

Method 1: Use Algebra

Determine \( f'(x) \).
\[
f'(x) = 6x^2 + 6x - 36
\]
The function \( f(x) \) is increasing when \( 6x^2 + 6x - 36 > 0 \).

To solve an inequality, first solve the corresponding equation.

\[
6x^2 + 6x - 36 = 0
\]
\[
x^2 + x - 6 = 0
\]
\[
(x + 3)(x - 2) = 0
\]
\[
x = -3 \text{ or } x = 2
\]

So, \( f'(x) = 0 \) when \( x = -3 \) or \( x = 2 \).
The values of $x$ at which the slope of the tangent, $f'(x)$, is zero divide the domain into three intervals: $x < -3$, $-3 < x < 2$, and $x > 2$. Test any number in each interval to determine whether the derivative is positive or negative on the entire interval.

<table>
<thead>
<tr>
<th>Test Value</th>
<th>$x &lt; -3$</th>
<th>$x = -3$</th>
<th>$-3 &lt; x &lt; 2$</th>
<th>$x = 2$</th>
<th>$x &gt; 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>Positive</td>
<td>0</td>
<td>Negative</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the table, the function is increasing on the intervals $x < -3$ and $x > 2$, and decreasing on the interval $-3 < x < 2$. This can be confirmed by graphing the function.

**Method 2: Use the Graph of $f'(x)$**

Determine the derivative, $f'(x)$, then graph it.

$$f'(x) = 6x^2 + 6x - 36$$

Use the graph of $f'(x)$ to determine the intervals on which the derivative is positive or negative.

The graph of $f'(x)$ is above the $x$-axis when $x < -3$ and $x > 2$, so $f'(x) > 0$ when $x < -3$ and when $x > 2$.

The graph of $f'(x)$ is below the $x$-axis on the interval $-3 < x < 2$, so $f'(x) < 0$ when $-3 < x < 2$.

The function $f(x) = 2x^3 + 3x^2 - 36x + 5$ is increasing on the intervals $x < -3$ and $x > 2$ and decreasing on the interval $-3 < x < 2$. 

![Graph](image)
Example 2 Use the First Derivative to Sketch a Function

For each function, use the graph of $f'(x)$ to sketch a possible function $f(x)$.

a) The derivative $f'(x)$ is constant at $-2$. So, $f(x)$ has a constant slope of $-2$. Graph any line with slope $-2$.

b) The derivative $f'(x)$ is positive when $x < 2$ and negative when $x > 2$. So, $f(x)$ is increasing when $x < 2$ and decreasing when $x > 2$.

c) The derivative $f'(x)$ is never negative. From left to right, it is large and positive, decreases to zero at $x = 1$, and then increases again.

d) The derivative $f'(x)$ is negative when $x < 1$ and when $x > 3$. It is positive when $1 < x < 3$. So, $f(x)$ is decreasing when $x < 1$, increasing when $1 < x < 3$, and decreasing when $x > 3$. 
Example 3  Interval of Increasing Temperature

The temperature of a person with a certain strain of flu can be approximated by the function \( T(d) = -\frac{5}{18} d^2 + \frac{15}{9} d + 37 \), where \( 0 < d < 6 \),

\( T \) represents the person’s temperature, in degrees Celsius, and \( d \) is the number of days after the person first shows symptoms. During what interval will the person’s temperature be increasing?

Solution

\[ T(d) = -\frac{5}{18} d^2 + \frac{15}{9} d + 37 \]

\[ T'(d) = -\frac{5}{9} d + \frac{15}{9} \]

Method 1: Use Algebra

Solve for \( T'(d) > 0 \).

Set \( T'(d) = 0 \).

\[ -\frac{5}{9} d + \frac{15}{9} = 0 \]

\[ d = 3 \quad d = 3 \text{ divides the domain into two parts: } 0 < d < 3 \text{ and } 3 < d < 6. \]

Test any \( x \)-value from each interval:

<table>
<thead>
<tr>
<th>Test Value</th>
<th>( 0 &lt; d &lt; 3 )</th>
<th>( d = 3 )</th>
<th>( 3 &lt; d &lt; 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T'(d) )</td>
<td>( T'(1) = \frac{10}{9} )</td>
<td>0</td>
<td>( T'(4) = -\frac{5}{9} )</td>
</tr>
<tr>
<td>Positive</td>
<td>Positive</td>
<td>Negative</td>
<td></td>
</tr>
</tbody>
</table>

The function is increasing on the interval \( 0 < d < 3 \). So, the person’s temperature increases over the first 3 days.

Method 2: Use the Graph of \( T'(d) \)

From the graph, \( T'(d) \) is positive when \( d < 3 \). So, \( T(d) \) is increasing when \( d < 3 \).

The person’s temperature increases over the first 3 days.
A function is increasing on an interval if the slope of the tangent is positive over the entire interval.

A function is decreasing on an interval if the slope of the tangent is negative over the entire interval.

Intervals over which a function $f(x)$ is increasing or decreasing can be determined by finding the derivative, $f'(x)$, and then solving the inequalities $f'(x) > 0$ and $f'(x) < 0$.

When the graph of $f'(x)$ is positive, or above the $x$-axis, on an interval, then the function $f(x)$ increases over that interval. Similarly, when the graph of $f'(x)$ is negative, or below the $x$-axis, on an interval, then the function $f(x)$ decreases over that interval.
Communicate Your Understanding

C1 A function increases when $0 < x < 10$. Which is greater: $f(3)$ or $f(8)$? Explain your reasoning.

C2 How can you use the derivative of a function to find intervals over which the function is increasing or decreasing?

C3 A linear function is either increasing or decreasing. Is this statement always true, sometimes true, or always false? Explain.

A) Practise

1. Determine algebraically the values of $x$ for which each derivative equals zero.
   
   a) $f'(x) = 15 - 5x$
   b) $b'(x) = x^2 + 8x - 9$
   c) $g'(x) = 3x^2 - 12$
   d) $f'(x) = x^3 - 6x^2$
   e) $d'(x) = x^2 + 2x - 4$
   f) $k'(x) = x^3 - 3x^2 - 18x + 40$
   g) $b'(x) = x^3 + 3x^2 - 4x - 12$
   h) $f'(x) = x^4 - x^3 - x^2 + x$

2. For each derivative in question 1, find the intervals of increase and decrease for the function.

3. For each function, do the following.
   
   a) Find the derivative.
   b) Use a graphing calculator or other graphing technology to graph the derivative.
   c) Use the graph to determine the intervals of increase and decrease for the function.
   d) Verify your response by graphing the function $f(x)$ on the same set of axes.

   i) $f(x) = 6x - 15$
   ii) $f(x) = (x + 5)^2$
   iii) $f(x) = x^3 - 3x^2 - 9x + 6$
   iv) $f(x) = (x^2 - 4)^2$
   v) $f(x) = 2x - x^2$
   vi) $f(x) = 5x^3 - 6x^2 + 2x$
   vii) $f(x) = \frac{1}{3}x^3 - 6x$
   viii) $f(x) = \frac{1}{x} - 3x^3$

4. Given each graph of $f'(x)$, state the intervals of increase and decrease for the function $f(x)$.

   a) 
   b) 
   c) 
   d) 
   e) 
   f) 
   g) 
   h) 

5. Sketch a possible graph of $y = f(x)$ for each graph of $y = f'(x)$ in question 4.
6. Sketch a continuous graph that satisfies each set of conditions.
   a) \( f'(x) > 0 \) when \( x < 3 \), \( f'(x) < 0 \) when \( x > 3 \)
      \( f(3) = 5 \)
   b) \( f'(x) > 0 \) when \( -1 < x < 3 \), \( f'(x) < 0 \) when \( x < -1 \) and when \( x > 3 \)
      \( f(-1) = 0 \), \( f(3) = 4 \)
   c) \( f'(x) > 0 \) when \( x \neq 2 \), \( f(2) = 1 \)
   d) \( f'(x) = 1 \) when \( x < -2 \), \( f'(x) = -1 \) when \( x < -2 \), \( f(-2) = -4 \)

7. Given the graph of \( k'(x) \), determine which value of \( x \) in each pair gives the greater value of \( k(x) \). Explain your reasoning.

![Graph of y = k'(x)](image)

a) \( k(3), k(5) \)
   b) \( k(8), k(12) \)
   c) \( k(5), k(9) \)
   d) \( k(-2), k(10) \)

8. Use each method below to show that the function \( g(x) = x^3 + x \) is always increasing.
   a) Find \( g'(x) \), then sketch and examine its graph.
   b) Use algebra to show that \( g'(x) > 0 \) for all \( x \).

9. Given \( f(x) = x^2 + 2x - 3 \) and \( g(x) = x + 5 \), determine the intervals of increase and decrease of \( h(x) \) in each case.
   a) \( h(x) = f(x) + g(x) \)
   b) \( h(x) = f(g(x)) \)
   c) \( h(x) = f(x) - g(x) + 2 \)
   d) \( h(x) = f(x)g(x) \)

10. The derivative of a function \( f(x) \) is \( f'(x) = x(x - 1)(x + 2) \).
    a) Find the intervals of increase and decrease of \( f(x) \).
    b) Explain how your answer in part a) would change if \( f'(x) = x^2(x - 1)(x + 2) \).

11. The table shows the intervals of increase and decrease for a function \( h(x) \).

<table>
<thead>
<tr>
<th>( x &lt; -3 )</th>
<th>( x = -3 )</th>
<th>( -3 &lt; x &lt; 3 )</th>
<th>( x = 3 )</th>
<th>( x &gt; 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h'(x) )</td>
<td>Negative</td>
<td>0</td>
<td>Positive</td>
<td>0</td>
</tr>
<tr>
<td>( h(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) Sketch the graph of the function.
   b) Compare your graph to that of a classmate. Explain why there are an infinite number of graphs possible.
   c) Write the equation for a function with these properties.

12. Chapter Problem
    Naveen needs to cut 20 m of garden edging into two pieces, each at least 5 m long: one for the quarter circle and one for the square. The total area of Naveen’s gardens can be modelled by the function
    \( A(x) = \left( \frac{4 + \pi}{4\pi} \right) x^2 - 10x + 100 \), where \( x \)
    represents the length of edging to be used for the quarter circle.
    a) Evaluate \( A(0) \). Explain what \( A(0) \) represents and why your answer makes sense.
    b) Find \( A'(x) \). Determine the intervals on which \( A(x) \) is increasing and decreasing.
    c) Verify your result by graphing \( A(x) \) using graphing technology.

13. In an experiment, the number of a certain type of bacteria is given by
    \( n(t) = 100 + 32t^2 - t^4 \),
    where \( t \) is the time, in days, since the experiment began, and \( 0 < t < 5 \).
    a) Find the intervals of increase and decrease of the number of bacteria.
    b) Describe how your answer would be different if no interval were specified.
14. The range, $R$, of a small aircraft, in miles, at engine speed $s$, in revolutions per minute, is modelled by the function

$$R = -\frac{1}{2000}s^2 + 2s - 1500.$$  

a) Determine the range at engine speed 2100 r/min.

b) The engine speed is restricted to values from 500 r/min to 3100 r/min. Within these values, determine intervals on which the function is increasing and decreasing.

c) Verify your answer to part b) using graphing technology.

C Extend and Challenge

15. Explain why the function defined by $f(x) = 3x^2 + bx + c$ cannot be strictly decreasing when $a < x < \infty$, where $a$ is any number.

16. For the function defined by $f(x) = x^3 + bx^2 + 12x - 3$, find the values of $b$ that result in $f(x)$ increasing for all values of $x$.

17. Math Contest Which of these functions is increasing for all positive integers $n$?

A $y = x^{2n} + x^n + 1$

B $y = x^n + x^{n-1} + \ldots + x + 1$

C $y = x^{2n} + x^{2n-2} + \ldots + x^2 + 1$

D $y = x^{2n+1} + x^{2n-1} + \ldots + x^3 + x$

E $y = x^{2n} + x^{2n-1} + \ldots + x^n$

18. Math Contest A function $f(x)$ is even if $f(-x) = f(x)$ for all $x$; $f(x)$ is odd if $f(-x) = -f(x)$ for all $x$. Which of these statements is true?

i) The derivative of an even function is always even.

ii) The derivative of an odd function is always odd.

iii) The derivative of an even function is always odd.

iv) The derivative of an odd function is always even.

A i) and iii) only  
B i) and iv) only  
C ii) and iii) only  
D ii) and iv) only  
E none of the above

CAREER CONNECTION

Aisha studied applied and industrial math at University of Ontario Institute of Technology for 5 years. She now works in the field of mathematical modelling, by helping an aircraft manufacturer to design faster, safer, and environmentally cleaner airplanes. With her knowledge of fluid mechanics and software programs that can, for example, model a wind tunnel, Aisha will run experiments. Data from these tests help her to translate physical phenomena into equations. She then analyzes and solves these complex equations and interprets the solutions. As computers become even more powerful, Aisha will be able to tackle more complex problems and get answers in less time, thereby reducing research and development costs.
A favourite act at the circus is the famous Human Cannonball. Shot from a platform 5 m above the ground, the Human Cannonball is propelled high into the air before landing safely in a net. Although guaranteed a safe landing, the feat is not without risk. Launched at the same speed and angle each time, the Human Cannonball knows the maximum height he will reach. The stunt works best when his maximum height is less than the height of the ceiling where he performs!

**Investigate**  
**How can you find maximum or minimum values?**

**Method 1: Compare the Derivative of a Function to the Graph of the Function**

1. Consider the function \( f(x) = 2x^3 - 3x^2 \).
   a) Determine the intervals of increase and decrease for the function.
   b) For each interval, determine the values of \( f'(x) \) at the endpoints of the interval.
   c) Graph the function using a graphing calculator. In each interval, determine if there is a maximum or a minimum. If so, determine the maximum or minimum value.

2. Repeat step 1 for each function.
   a) \( f(x) = -x^3 + 6x \)  
   b) \( f(x) = 3x^4 - 6x^2 \)
   c) \( f(x) = 2x^3 - 18x^2 + 48x \)  
   d) \( f(x) = x^4 + \frac{4}{3}x^3 - 12x^2 \)

3. **Reflect** Refer to your answers to steps 1 and 2. Describe how you can use \( f'(x) \) to determine the local maximum and minimum values of \( f(x) \).

**Method 2: Use The Geometer’s Sketchpad®**

1. Open *The Geometer’s Sketchpad®*. Go to [www.mcgrawhill.ca/links/calculus12](http://www.mcgrawhill.ca/links/calculus12) and follow the links to 3.2. Download the file 3.2 SlidingTangent.gsp, which shows the function \( f(x) = 2x^3 - 3x^2 \) and a tangent that can be dragged along the curve.

2. Drag the tangent, from left to right, through the highest point on the graph. As you drag the tangent, notice what happens to the magnitude and sign of the slope.

3. **Reflect** Describe what happens to the slope of the tangent as it moves from left to right through each of the following points.
   a) the highest point on the graph
   b) the lowest point on the graph
Given the graph of a function \( f(x) \), a point is a **local maximum** if the y-coordinates of all the points in the vicinity are less than the y-coordinate of the point. Algebraically, if \( f'(x) \) changes from positive to zero to negative as \( x \) increases from \( x < a \) to \( x > a \), then \( (a, f(a)) \) is a local maximum and \( a \) is a **local maximum value**.

Similarly, a point is a **local minimum** if the y-coordinates of all the points in the vicinity are greater than the y-coordinate of the point. If \( f'(x) \) changes from negative to zero to positive as \( x \) increases from \( x < a \) to \( x > a \), then \( (a, f(a)) \) is a local minimum and \( a \) is a **local minimum value**.

Local maximum and minimum values of a function are also called local extreme values, **local extrema**, or turning points.

A function has an **absolute maximum** at \( a \) if \( f(a) \geq f(x) \) for all \( x \) in the domain. The maximum value of the function is \( f(a) \). The function has an **absolute minimum** at \( a \) if \( f(a) \leq f(x) \) for all \( x \) in the domain. The minimum value of the function is \( f(a) \).

**Example 1**  
**Local Versus Absolute Maxima and Minima**

Consider this graph of a function on the interval \( 0 < x < 10 \).

- **a)** Identify the local maximum points.
- **b)** Identify the local minimum points.
- **c)** What do all the points identified in parts a) and b) have in common?
- **d)** Identify the absolute maximum and minimum values in the interval \( 0 < x < 10 \).

**Solution**

- **a)** The local maxima are at points A and C.
- **b)** The local minimum is at point B.
- **c)** At each of the local extreme points, A, B, and C, the tangent is horizontal.
- **d)** The absolute maximum value occurs at the highest point on the graph. In this case, the absolute maximum is 8 and occurs at the local maximum at C.

The absolute minimum value occurs at the lowest point on the graph. In this case, the absolute minimum is 3 and occurs at D.
A **critical number** of a function is a value \( a \) in the domain of the function for which either \( f'(a) = 0 \) or \( f'(a) \) does not exist. If \( a \) is a critical number, the point \((a, f(a))\) is a **critical point**. To determine the absolute maximum and minimum values of a function in an interval, find the critical numbers, then substitute the critical numbers and also the \( x \)-coordinates of the endpoints of the interval into the function.

### Example 2
**Use Critical Numbers to Find the Absolute Maximum and Minimum**

Find the absolute maximum and minimum of the function 
\[ f(x) = x^3 - 12x - 3 \]
on the interval \(-3 \leq x \leq 4\).

**Solution**

Find the critical numbers.
\[ f'(x) = 3x^2 - 12 \]
\[ 3x^2 - 12 = 0 \]
\[ 3(x^2 - 4) = 0 \]
\[ 3(x + 2)(x - 2) = 0 \]
\[ x = -2 \text{ or } x = 2 \]

Examine the local extrema that occur at \( x = -2 \) and \( x = 2 \), and also the endpoints of the interval at \( x = -3 \) and \( x = 4 \). Evaluate \( f(x) \) for each of these values.

\[ f(-3) = (-3)^3 - 12(-3) - 3 = 6 \]
\[ f(-2) = (-2)^3 - 12(-2) - 3 = 13 \]
\[ f(2) = (2)^3 - 12(2) - 3 = -19 \]
\[ f(4) = (4)^3 - 12(4) - 3 = 13 \]

The absolute maximum value is 13. It occurs twice, at a local maximum point when \( x = -2 \) and at the right endpoint. The absolute minimum value is \(-19\). It occurs at a local minimum point when \( x = 2 \).

### Example 3
**Maximum Volume**

The surface area of a cylindrical container is to be 100 cm\(^2\). Its volume is given by the function \( V = 50r - \pi r^3 \), where \( r \) represents the radius, in centimetres, of the cylinder. Find the maximum volume of the cylinder in each case.

**a)** The radius cannot exceed 3 cm.

**b)** The radius cannot exceed 2 cm.
Solution

a) The radius cannot be less than zero and cannot exceed 3 cm. This means the interval will be $0 \leq r \leq 3$. Find the critical numbers on this interval.

\[ V = 50r - \pi r^3 \]

\[ V' = 50 - 3\pi r^2 \]

\[ 0 = 50 - 3\pi r^2 \]

\[ 50 = 3\pi r^2 \]

\[ r^2 = \frac{50}{3\pi} \quad r \geq 0 \text{ since } V \text{ cannot be negative.} \]

\[ r \approx 2.3 \]

There is a critical point when the radius is approximately 2.3 cm.

Substitute $r = 2.3$ and the endpoints, $r = 0$ and $r = 3$, into the volume formula, $V = 50r - \pi r^3$.

\[ V(0) = 50(0) - \pi(0)^3 \]

\[ V(2.3) = 50(2.3) - \pi(2.3)^3 \quad V(3) = 50(3) - \pi(3)^3 \]

\[ = 0 \quad \approx 76.8 \quad \approx 65.2 \]

If the radius cannot exceed 3 cm, the maximum volume is approximately 76.8 cm$^3$. The radius of the cylinder with maximum volume is approximately 2.3 cm.

b) Find the critical numbers on $0 \leq r \leq 2$.

From part a), there are no critical points between $r = 0$ and $r = 2$. If there are no critical points, and therefore no local extrema, then the maximum volume must be found at one of the endpoints.

Test $r = 0$ and $r = 2$.

\[ V(0) = 50(0) - \pi(0)^3 \]

\[ V(2) = 50(2) - \pi(2)^3 \]

\[ = 0 \quad \approx 74.9 \]

If the radius cannot exceed 2 cm, the maximum volume is approximately 74.9 cm$^3$.

The results are displayed on the graph. The vertical line marks the endpoint of the interval. The absolute maximum occurs at the intersection of the function and the vertical line.
KEY CONCEPTS

- If \( f'(x) \) changes from positive to zero to negative as \( x \) increases from \( x < a \) to \( x > a \), then \((a, f(a))\) is a local maximum value.
- If \( f'(x) \) changes from negative to zero to positive as \( x \) increases from \( x < a \) to \( x > a \), then \((a, f(a))\) is a local minimum value.
- The absolute maximum and minimum values are found at local extrema or at the endpoints of the interval.
- A critical number of a function is a number \( a \) in the domain of the function for which either \( f'(a) = 0 \) or \( f'(a) \) does not exist.

Communicate Your Understanding

C1 If \( f'(x) = 0 \), then there must be a local maximum or minimum. Is this statement true or false? Explain.

C2 Does the maximum value in an interval always occur when \( f'(x) = 0 \)? Explain.

C3 Local extrema are often called turning points. Explain why this is the case. Refer to the slope of the tangent in your explanation.

C4 A function is increasing on the interval \(-2 \leq x \leq 5\). Where would you find the absolute maximum and minimum values? Explain your reasoning.

Practise

1. Determine the absolute maximum and minimum values of each function.
   a) \[ y = -x + 7, \quad -10 \leq x \leq 10 \]
   b) \[ f(x) = 3x^2 - 12x + 7, \quad 0 \leq x \leq 4 \]
   c) \[ g(x) = 2x^3 - 3x^2 - 12x + 2, \quad -3 \leq x \leq 3 \]
   d) \[ f(x) = x^3 + x, \quad 0 \leq x \leq 10 \]
   e) \[ y = (x - 3)^2 - 9, \quad -8 \leq x \leq -3 \]

2. Determine the absolute and local extreme values of each function on the given interval.
   a) \[ y = -x^2 + 6x + 2 \]
   b) \[ f(x) = x^3 - 2x^2 + 3x \]
   c) \[ y = x^3 - 3x^3 + 5 \]
   d) \[ g(x) = 2x^3 - 3x^2 - 12x + 5 \]
   e) \[ y = x - \sqrt{x} \]
4. Find and classify the critical points of each function. Determine whether the critical points are local maxima, local minima, or neither.
   a) \( y = 4x - x^2 \)  
   b) \( f(x) = (x - 1)^4 \)  
   c) \( g(x) = 2x^3 - 24x + 5 \)  
   d) \( h(x) = x^5 + x^3 \)

5. Suppose that the function \( f(t) \) represents your elevation after riding for \( t \) hours on your mountain bike. If you stop to rest, explain why \( f'(t) = 0 \) at that time. Under what circumstances would you be at a local maximum, a local minimum, or neither?

6. a) Find the critical numbers of \( f(x) = 2x^3 - 3x^2 - 12x + 5 \).
   b) Find any local extrema of \( f(x) \).
   c) Find the absolute extrema of \( f(x) \) in the interval \([-2, 4]\).

B Connect and Apply

7. Use the critical points to sketch each function.
   a) \( f(x) = 7 + 6x - x^2 \)
   b) \( g(x) = x^4 - 8x^2 - 10 \)
   c) \( y = x(x + 2)^2 \)
   d) \( h(x) = 27x - x^3 \)

8. On the interval \( a \leq x \leq b \), the absolute minimum of a function, \( f(x) \), occurs when \( x = b \). The absolute maximum of \( f(x) \) occurs when \( x = a \). Do you agree with the following statement? Explain.
   \( f(x) \) is decreasing and there cannot be any extrema on the interval \( a \leq x \leq b \).

9. For a particular function \( f'(x) = (x - 3)^2 \).
   a) State the coordinates of the vertex and the direction of opening.
   b) Find the maximum and minimum values of \( f(x) \) on the interval \( 3 \leq x \leq 6 \).
   c) Explain how you could answer part b) without finding the derivative.

10. For a particular function \( f'(x) = x^3 - 2x^2 \).
    a) For which values of \( x \) does \( f'(x) = 0 \)?
    b) Find the intervals of increase and decrease for \( f(x) \).
    c) How can you tell by examining \( f'(x) \) that there would be only one turning point for \( f(x) \)?

11. Consider the function \( y = x^3 - 6x^2 + 11x \).
    a) Find the critical numbers.
    b) Find the absolute maximum and minimum values on the interval \( 0 \leq x \leq 4 \).

12. Chapter Problem Recall that the equation representing the total area of Naveen’s garden is \( A(x) = \left( \frac{4 + \pi}{4\pi} \right)x^2 - 10x + 100 \), where \( x \) represents the length of the edging to be used for the quarter circle.
    a) What are the critical numbers of \( A(x) \)?
    b) Make a table showing the behaviour of the derivative in the vicinity of the critical value.
    c) Is the critical point a local maximum or a local minimum? How do you know?
    d) Find the maximum area on the interval \( 5 \leq x \leq 15 \).

13. A section of roller coaster is in the shape of \( f(x) = -x^3 - 2x^2 + x + 15 \) where \( x \) is between \(-2 \) and \(+2 \).
    a) Find all local extrema and explain what portions of the roller coaster they represent.
    b) Is the highest point of this section of the ride at the beginning, the end, or neither?

14. Use Technology The height of the Human Cannonball is given by \( h(t) = -4.9t^2 + 9.8t + 5 \), where \( h \) is the height, in metres, \( t \) seconds after the cannon is fired. Graph the function on a graphing calculator.
    a) Find the maximum and minimum heights during the first 2 s of flight.
    b) How many different ways can you find the answer to part a) with a graphing calculator? Describe each way.
    c) Describe techniques, other than using derivatives or graphing technology, that could be used to answer part a).
15. The distance, \( d \), in metres, that a scuba diver can swim at a depth of 10 m and a speed of \( v \) metres per second before her air runs out can be modelled by \( d = 4.8v^3 - 28.8v^2 + 52.8v \) for \( 0 < v < 2 \).

a) Determine the speed that results in the maximum distance.

b) Verify your result using graphing technology.

c) Why does this model not apply if \( v > 2 \)?

16. The height, \( h \) metres, of a ski ramp over a horizontal distance, \( x \) metres, is given by \( h(x) = 0.01x^3 - 0.3x^2 + 60 \) for the interval \( 0 \leq x \leq 22 \).

a) Use graphing technology to draw the graph.

b) Find the minimum height of the ramp.

c) Find the vertical drop from the top of the ramp to the lowest point on the ramp.

d) Find the vertical rise from the lowest point to the end of the ramp.

**Achievement Check**

16. For the quartic function defined by \( f(x) = ax^4 + bx^3 + cx + d \), find the values of \( a, b, c, \) and \( d \) such that there is a local maximum at \((0, -6)\) and a local minimum at \((1, -8)\).

18. For the cubic function defined by \( f(x) = ax^3 + bx^2 + cx + d \), find the relationship between \( a, b, \) and \( c \) in each case.

a) There are no extrema.

b) There are exactly two extrema.

19. Explain why a cubic function has either exactly zero or exactly two extrema.

20. Consider the function \( g(x) = |x^2 - 9| \).

a) Graph \( g(x) \). How can you use \( y = x^2 - 9 \) to help?

b) Find and classify the critical points.

c) How could you find \( g'(x) \)?

21. **Math Contest** Which statement is true for the graph of \( y = x^n - nx \), for all integers \( n \), where \( n \geq 2 \)

A There is a local maximum at \( x = 1 \).

B There is a local minimum at \( x = 1 \).

C There is a local maximum at \( x = -1 \).

D There is a local minimum at \( x = -1 \).

E There are local extrema at \( x = 1 \) and \( x = -1 \).

22. **Math Contest** Which statement is true for the function \( f(x) \) at \( x = a \)?

A \( f(x) \) is increasing at \( x = a \).

B \( f(x) \) is decreasing at \( x = a \).

C \( f(x) \) has a local maximum at \( x = a \).

D \( f(x) \) has a local minimum at \( x = a \).

E None of the statements are true.

23. **Math Contest** Which statement is true for the function \( f(x) \) at \( x = b \)?

A \( f(x) \) has a local maximum at \( x = b \).

B \( f(x) \) has a local minimum at \( x = b \).

C \( f(x) \) is undefined at \( x = b \).

D \( f'(x) \) is undefined at \( x = b \).

E \( f(x) \) has a horizontal tangent at \( x = b \).
Concavity and the Second Derivative Test

Two cars are travelling side by side. The cars are going in the same direction, both at 80 km/h. Then, one driver decelerates while the other driver accelerates. How would the graphs that model the paths of the two cars differ? How would they be the same? In this section, you will explore what it means when the slope of the tangent is increasing or decreasing and relate it to the shape of a graph.

Investigate A  How can you determine the shape of a function?

Method 1: Use The Geometer’s Sketchpad®

1. Open The Geometer’s Sketchpad®. Go to mcgrawhill.ca/links/calculus12 and follow the links to 3.3. Download the file 3.3 DraggingTangents.gsp. This file shows the graph of the function \( f(x) = x^4 - 2x^3 - 5 \) and a tangent to the function at point A.

Suppose the graph represents the path of a car, A, as it drives along a road, and the tangent shows the direction the car is steering at any moment. Drag the ‘car’ from one end of the ‘road’ to the other. Describe what happens to the tangent line as the car moves.

Pick a fixed location for A.

a) What is the value of \( x \) at point A?

b) What is the slope of the tangent at point A?

c) Does the tangent lie above or below the curve at point A?

d) Describe the shape of \( f(x) \) around point A. Is the curve increasing or decreasing?

2. Drag point A to at least eight different locations on the curve. Repeat step 1 for each new location of point A.
3. Describe how the shape of the graph and the slope of the tangent are related.

4. Reflect Describe the shape of the graph of \( f(x) \) in each case.
   a) The tangent lies above the curve.
   b) The tangent lies below the curve.

**Method 2: Use Paper and Pencil**

1. On grid paper, graph the function \( g(x) = x^4 - 2x^3 - 5 \).
2. Choose at least eight different points on the function. Label the points A, B, C, and so on. Draw the tangent to the function at each point.
3. Answer these questions for each of the points.
   a) What is the value of \( x \) at the point?
   b) What is the slope of the tangent at this point?
   c) Does the tangent lie above or below the curve at this point?
   d) Describe the shape of \( g(x) \) around the point. Is the curve increasing or decreasing?
4. Describe how the shape of the graph and the slope of the tangent are related.
5. Reflect Describe the shape of the graph of \( g(x) \) in each case.
   a) The tangent lies above the curve.
   b) The tangent lies below the curve.

The graph of a function \( f(x) \) is **concave up** on the interval \( a < x < b \) if all the tangents on the interval are below the curve. The graph curves upward as if wrapping around a point above the curve.

The graph of a function \( f(x) \) is **concave down** on the interval \( a < x < b \) if all the tangents on the interval are above the curve. The graph curves downward as if wrapping around a point below the curve.

A point at which the graph changes from being concave up to concave down, or vice versa, is called a **point of inflection**.
How can the second derivative be used to classify critical points?

1. Consider the function \( f(x) = x^3 - 4x^2 + x + 6 \).
   a) Graph \( f(x) \).
   b) Determine \( f'(x) \). Graph \( f'(x) \) on the same set of axes as \( f(x) \).
   c) Over which intervals is \( f(x) \) concave up? concave down? Determine the local maxima and minima of \( f(x) \).
   d) Determine \( f''(x) \). Graph \( f''(x) \) on the same set of axes as \( f(x) \) and \( f'(x) \).
   e) Compare \( f'(x) \) and \( f''(x) \) to \( f(x) \) over each interval from part c). What do you notice?
   f) Determine the coordinates of the points of inflection for \( f(x) \). What are the values of \( f'(x) \) and \( f''(x) \) at these points?

2. Repeat step 1 for each function.
   i) \( k(x) = x^5 - x^4 - 4x^3 + 4x^2 \)
   ii) \( h(x) = -x^3 + 2x^2 + 5x - 6 \)
   iii) \( b(x) = x^4 - 5x^2 \)
   iv) \( g(x) = x^3 - 5x^2 + 2x + 8 \)

3. Reflect Describe how to use the first and second derivatives to determine the intervals over which a function is concave up or concave down. How does the second derivative relate to the concavity of a function?

The Second Derivative Test

If \( f'(a) = 0 \) and \( f''(a) > 0 \), \( f(x) \) is concave up. There is a local minimum at \( (a, f(a)) \).

If \( f'(a) = 0 \) and \( f''(a) < 0 \), \( f(x) \) is concave down. There is a local maximum at \( (a, f(a)) \).

If \( f''(x) = 0 \) and \( f''(x) \) changes sign at \( a \), there is a point of inflection at \( (a, f(a)) \).

**Connections**

Points of inflection occur only when \( f'''(a) \) is undefined, but neither of these conditions is sufficient to guarantee a point of inflection at \( (a, f(a)) \). A simple example is \( f(x) = x^4 \) at \( x = 0 \).
Example 1  Intervals of Concavity

For the function \( f(x) = x^4 - 6x^2 - 5 \), find the points of inflection and the intervals of concavity.

Solution

Find the first and second derivatives of the function.

\[
\begin{align*}
  f'(x) &= 4x^3 - 12x \\
  f''(x) &= 12x^2 - 12
\end{align*}
\]

Method 1: Use Algebra

At a point of inflection, the second derivative equals zero and changes sign from positive to negative or vice versa.

\[
\begin{align*}
  12x^2 - 12 &= 0 \\
  12(x^2 - 1) &= 0 \\
  (x + 1)(x - 1) &= 0 \\
  x &= 1 \text{ or } x = -1
\end{align*}
\]

These values divide the domain into three intervals: \( x < -1 \), \( -1 < x < 1 \), and \( x > 1 \).

Choose an \( x \)-value from each interval to test whether \( f''(x) \) is positive or negative. Determine the coordinates of the points of inflection by substituting \( x = 1 \) and \( x = -1 \) into \( f(x) = x^4 - 6x^2 - 5 \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Test Value} & x < -1 & x = -1 & -1 < x < 1 & x = 1 & x > 1 \\
\hline
f''(x) & f''(-2) = 36 & 0 & f''(0) = -12 & 0 & f''(2) = 36 \\
\text{Positive} & \text{Negative} & \text{Positive} & \text{Negative} & \text{Positive} \\
\hline
f(x) & \text{Concave up} & \text{Point of inflection \((-1, -10)\)} & \text{Concave down} & \text{Point of inflection \((1, -10)\)} & \text{Concave up} \\
\hline
\end{array}
\]

The concavity of the graph changes at \((-1, -10)\) and at \((1, -10)\), so these are the points of inflection. The function is concave up to the left of \( x = -1 \) and to the right of \( x = 1 \). The function is concave down between these \( x \)-values.
**Method 2: Graph \( f''(x) \)**

Recall that \( f(x) = x^4 - 6x^2 - 5, f'(x) = 4x^3 - 12x, \) and \( f''(x) = 12x^2 - 12. \)

Graph \( f''(x) = 12x^2 - 12. \)

From the graph:
- \( f''(x) > 0 \) when the graph of \( f''(x) \) lies above the \( x \)-axis, so \( f''(x) > 0 \) when \( x < -1 \) and when \( x > 1. \)
- \( f''(x) < 0 \) when the graph of \( f''(x) \) lies below the \( x \)-axis, so \( f''(x) < 0 \) when \( -1 < x < 1. \)

The graph of \( f''(x) \) intersects the \( x \)-axis at \( x = -1 \) and \( x = 1. \) At these points, the sign of \( f''(x) \) changes, so there are points of inflection on \( f(x) \) at \( x = -1 \) and \( x = 1. \)

Substituting \( x = -1 \) and \( x = 1 \) into \( y = f(x) \) to determine the coordinates of the points of inflection gives the points of inflection as \((-1, -10)\) and \((1, -10)\). The function is concave up to the left of \( x = -1 \) and to the right of \( x = 1. \) The function is concave down between these \( x \)-values.

---

**Example 2** **Classify Critical Points**

For each function, find the critical points. Then, classify them using the second derivative test.

a) \( f(x) = x^3 - 3x^2 + 2 \)

b) \( f(x) = x^4 \)

**Solution**

a) \( f(x) = x^3 - 3x^2 + 2 \)

Determine the critical numbers for \( f(x). \)

\[
\begin{align*}
    f'(x) &= 3x^2 - 6x \\
    3x^2 - 6x &= 0 \\
    3x(x - 2) &= 0 \\
    x &= 0 \text{ or } x = 2
\end{align*}
\]
The critical numbers are \( x = 0 \) and \( x = 2 \). Substitute the critical numbers into \( f(x) \) to find the critical points.

\[
\begin{align*}
   f(0) &= (0)^3 - 3(0)^2 + 2 = 2 \\
   f(2) &= (2)^3 - 3(2)^2 + 2 = -2
\end{align*}
\]

The critical points are \((0, 2)\) and \((2, -2)\).

Since \( f'(x) = x^3 - 3x^2 + 2 \), then \( f''(x) = 6x - 6 \).

\[
\begin{array}{|c|c|c|}
\hline
    & x = 0 & x = 2 \\
\hline
f''(x) & f''(0) = -6 & f''(2) = 6 \\
      & \text{Negative} & \text{Positive} \\
\hline
f(x) & \text{Concave down} & \text{Concave up} \\
\hline
\end{array}
\]

The second derivative is negative at \( x = 0 \), so the graph is concave down when \( x = 0 \), and there is a local maximum at the point \((0, 2)\).

The second derivative is positive at \( x = 2 \), so the graph is concave up when \( x = 2 \), and there is a local minimum at the point \((2, -2)\).

b) \( f(x) = x^4 \)

Determine the critical numbers for \( f(x) \).

\[
\begin{align*}
   f'(x) &= 4x^3 \\
   0 &= 4x^3 \\
   0 &= x
\end{align*}
\]

So, \( f'(x) = 0 \) when \( x = 0 \).

Substitute \( x = 0 \) into \( f''(x) = 12x^2 \).

\[
\begin{align*}
   f''(0) &= 12(0)^2 \\
   f''(0) &= 0
\end{align*}
\]

Since \( f''(0) = 0 \), it appears that this is a point of inflection. However, the second derivative, \( f''(x) = 12x^2 \), is always positive, so it does not change sign, and there is no change in concavity. This function is always concave up, because \( f''(x) \) is always greater than or equal to zero.
Example 3  Interpret the Derivatives to Sketch a Function

Sketch a graph of a function that satisfies each set of conditions.

a) \( f''(x) = -2 \) for all \( x \), \( f'(-3) = 0 \), \( f(-3) = 9 \)

b) \( f''(x) < 0 \) when \( x < -1 \), \( f''(x) > 0 \) when \( x > -1 \), \( f'(-3) = 0 \), \( f'(1) = 0 \)

Solution

a) \( f''(x) = -2 \) for all \( x \), so the function is concave down.

\( f'(-3) = 0 \), so there is a local maximum at \( x = -3 \).

The function passes through the point \((-3, 9)\).

b) \( f''(x) < 0 \) when \( x < -1 \), so the function is concave down to the left of \( x = -1 \).

\( f''(x) > 0 \) when \( x > -1 \), so the function is concave up to the right of \( x = -1 \).

\( f'(-3) = 0 \), so there is a local maximum at \( x = -3 \).

\( f'(1) = 0 \), so there is a local minimum at \( x = 1 \).

Note that this is only one of the possible graphs that satisfy the given conditions. If this graph were translated up by \( k \) units, \( k \in \mathbb{R} \), the new graph would also satisfy the conditions since no specific points were given.
KEY CONCEPTS

- The second derivative is the derivative of the first derivative. It is the rate of change of the slope of the tangent.
- Intervals of concavity can be found by using the second derivative test or by examining the graph of \( f''(x) \).
  - A function is concave up on an interval if the second derivative is positive on that interval. If \( f'(a) = 0 \) and \( f''(a) > 0 \), there is a local minimum at \((a, f(a))\).
  - A function is concave down on an interval if the second derivative is negative on that interval. If \( f'(a) = 0 \) and \( f''(a) < 0 \), there is a local maximum at \((a, f(a))\).
  - If \( f''(a) = 0 \) and \( f'''(x) \) changes sign at \( x = a \), there is a point of inflection at \((a, f(a))\).

Communicate Your Understanding

C1 Describe what concavity means in terms of the location of the tangent relative to the function.
C2 If a graph is concave up on an interval, what happens to the slope of the tangent as you move from left to right?
C3 When there is a local maximum or minimum on a function, the first derivative equals zero and changes sign on each side of the zero. Make a similar statement about the second derivative. Use a diagram to explain.
C4 Describe how to use the second derivative test to classify critical points.

A Practise

1. For each graph, identify the intervals over which the graph is concave up and the intervals over which it is concave down.

2. Given each graph of \( f''(x) \), state the intervals of concavity for the function \( f(x) \). Also indicate where any points of inflection will occur.
3. For each graph of \( f''(x) \) in question 2, sketch a possible graph of \( y = f(x) \).
4. Find the second derivative of each function.
   a) \( y = 6x^2 - 7x + 5 \)
   b) \( f(x) = x^3 + x \)
5. For each function in question 4, find the intervals of concavity and the coordinates of any points of inflection.
   c) \( g(x) = -2x^3 + 12x^2 - 9 \)
   d) \( y = x^6 - 5x^4 \)

B  Connect and Apply

6. Sketch a graph of a function that satisfies each set of conditions.
   a) \( f''(x) = 2 \) for all \( x \), \( f'(2) = 0 \), \( f(2) = -3 \)
   b) \( f''(x) < 0 \) when \( x < 0 \), \( f''(x) > 0 \) when \( x > 0 \), \( f'(0) = 0 \), \( f(0) = 0 \)
   c) \( f''(x) > 0 \) when \( x < -1 \), \( f''(x) < 0 \) when \( x > -1 \), \( f'(-1) = 1 \), \( f(-1) = 2 \)
   d) \( f''(x) < 0 \) when \( -2 < x < 2 \), \( f''(x) > 0 \) when \( |x| > 2 \), \( f(2) = 1 \), \( f(x) \) is an even function
   e) \( f''(x) > 0 \) when \( x < -5 \), \( f''(x) < 0 \) when \( x > -5 \), \( f'(-5) = 3 \), \( f(-5) = 2 \)
   f) \( f''(x) < 0 \) when \( -2 < x < 1 \), \( f''(x) > 0 \) when \( x < -2 \) and \( x > 1 \), \( f(-2) = -4 \), \( f(0) = 0 \)
7. For each function, find and classify all the critical points. Then, use the second derivative to check your results.
   a) \( y = x^2 + 10x - 11 \)
   b) \( g(x) = 3x^3 - 5x^3 - 5 \)
   c) \( f(x) = x^4 - 6x^3 + 10 \)
   d) \( h(x) = -4.9t^2 + 39.2t + 2 \)
8. The shape of a ski ramp is defined by the function \( h(x) = 0.01x^3 - 0.3x^2 + 60 \) on the interval \( 0 \leq x \leq 22 \).
   a) Find the intervals of concavity for the given interval.
   b) Find the steepest point on the ski ramp.
9. Is each statement always true, sometimes true, or never true? Explain.
   a) \( f'(x) = 0 \) at a local maximum or minimum on \( f(x) \).
   b) At a point of inflection, \( f''(x) = 0 \).
10. Chapter Problem  The equation representing the total area of Naveen’s gardens is
    \[ A(x) = \left( \frac{4 + \pi}{4\pi} \right) x^2 - 10x + 100, \] where \( x \) represents the length of the edging to be used for the quarter circle.
    a) What are the intervals of concavity for \( A(x) \)? How can you tell by looking at the equation?
    b) Does the graph of \( A(x) \) have a local maximum or a local minimum?
    c) Based on your answers to parts a) and b), what \( x \)-value provides the maximum area? Assume \( 0 \leq x \leq 20 \). Explain your reasoning.
11. The graph represents the position of a car, moving in a straight line, with respect to time. Describe what is happening at each of the key points shown on the graph, as well as what is happening in the intervals between those points.
12. The body temperature of female mammals varies over a fixed period. For humans, the period is about 28 days. The temperature $T$, in degrees Celsius, varies with time $t$, in days, and can be represented by the cubic function $T(t) = -0.0003t^3 + 0.012t^2 - 0.112t + 36$.

a) Determine the critical numbers of the function.

b) The female is most likely to conceive when the rate of change of temperature is a maximum. Determine the day of the cycle when this occurs.

c) What kind of point is the point described in part b)? Justify your answer.

13. The second derivative of a function, $f(x)$, is defined by $f''(x) = x^2(x - 2)$.

a) For what values of $x$ is $f''(x) = 0$?

b) Determine the intervals of concavity.

c) If $f(2) = 1$, sketch a possible graph of $f(x)$.

14. Use this graph of $f'(x)$. How many points of inflection are on the graph of $f(x)$? Explain your reasoning.

15. Prove that a polynomial function of degree four has either two points of inflection or no points of inflection.

16. A function is defined by $f(x) = ax^3 + bx^2 + cx + d$.

a) Find the values of $a$, $b$, $c$, and $d$ if $f(x)$ has a point of inflection at $(0, 2)$ and a local maximum at $(2, 6)$.

b) Explain how you know there must also be a local minimum.

17. Assume each function in question 6 is a polynomial function. What degree is each function? Is it possible to have more than one answer? Explain your reasoning.

18. Math Contest Which statement is always true for a function $f(x)$ with a local maximum at $x = a$?

A $f'(a) = 0$

B $f''(a) < 0$

C $f'(x_1)f'(x_2) < 0$ if $x_1 < a < x_2$ and both $f'(x_1)$ and $f'(x_2)$ exist.

D There exists an interval $I$ containing $a$, such that $f'(x) > 0$ for all $x < a$ in $I$ and $f'(x) < 0$ for all $x > a$ in $I$.

E There exists an interval $I$ containing $a$, such that $f(x) < f(a)$ for all $x$ in $I$.

19. Math Contest Which statement is always true for a function $f(x)$ with $f'(a) = 0 = f''(a)$, where $a$ is in the domain of $f(x)$?

A $f(x)$ has a local maximum at $x = a$.

B $f(x)$ has a local minimum at $x = a$.

C $f(x)$ has either a local maximum or a local minimum at $x = a$.

D $f(x)$ has a point of inflection at $x = a$.

E None of the above are true.
3.4

Simple Rational Functions

Rational functions can be used in a number of contexts. The function 
\[ v = \frac{100}{t} \]
relates the velocity, \( v \), in kilometres per hour, required to 
travel 100 km to time, \( t \), in hours. The function 
\[ T = \frac{k}{r^2} \]
relates temperature, \( T \), to distance, \( r \), from the sun; in the function, \( k \) is a constant. In this section, 
you will examine the features of derivatives as they relate to rational functions 
and practical situations.

**Investigate A** How does the graph of a rational function behave in the vicinity of its 
vertical asymptotes?

**Tools**
* graphing calculator

Recall that an asymptote is not part of a function, but a boundary that shows 
where the function is not defined. The line \( x = a \) is a vertical asymptote if 
\[ f(x) \to \pm \infty \text{ as } x \to a \text{ from the left and/or the right.} \]

1. Use a graphing calculator to graph \( f(x) = \frac{1}{x} \). Use the ZOOM or WINDOW 
commands to examine the graph in the vicinity of \( x = 0 \). Describe what you see. Sketch the 
graph.

2. Press \( \text{TRACE 0 ENTER} \). Record the \( y \)-value when \( x = 0 \).

3. Press \( \text{2ND [TBLSET]} \). Begin at -1 and set \( \Delta x \) to 0.1. Press \( \text{2ND [TABLE]} \). 
Describe what is happening to the \( y \)-values as \( x \) approaches zero. Include 
what happens on both sides of \( x = 0 \).

4. **Reflect** Explain why \( f(x) \) is not defined at \( x = 0 \). Explain why the \( y \)-value 
gets very large and positive as \( x \) approaches zero from the right, and large 
and negative as \( x \) approaches zero from the left.

5. Repeat steps 1 to 4 for \( g(x) = \frac{1}{x+1} \). How does it compare to \( f(x) = \frac{1}{x} \)?

6. Repeat steps 1 to 4 for \( h(x) = \frac{1}{x-3} + 2 \). How does it compare to \( f(x) = \frac{1}{x} \)?

7. **Reflect** Describe how you could graph \( h(x) \) or another similar function 
without graphing technology.

Many rational functions, such as \( y = \frac{1}{x-2} \), have 
**vertical asymptotes**. These usually occur at \( x \)-values 
for which the denominator is zero and the function is 
undefined. However, a more precise definition involves 
examining the limit of the function as these \( x \)-values are 
approached.
Investigate B How can you determine whether the graph approaches positive or negative infinity on either side of the vertical asymptotes?

1. Open The Geometer’s Sketchpad®.

Graph \( f(x) = \frac{1}{(x - 1)} \), \( g(x) = \frac{1}{(x - 1)^3} \), and \( h(x) = \frac{1}{(x - 1)^5} \) on the same set of axes. Use a different colour for each function.

2. Describe how the graphs in step 1 are similar and how they are different.

3. Graph \( k(x) = \frac{1}{(x - 1)^2} \), \( m(x) = \frac{1}{(x - 1)^4} \), and \( n(x) = \frac{1}{(x - 1)^6} \) on the same set of axes. Use a different colour for each function.

4. Compare the graphs and equations in step 3 to each other, and to the graphs and equations in step 1.

5. Reflect Explain how the graphs in step 3 are different from those in step 1.

6. Describe the effect of making each change to the functions in steps 1 and 3.
   a) Change the numerator to \(-1\).
   b) Change the numerator to \(x\).

7. Reflect Summarize what you have discovered about rational functions of the form \( f(x) = \frac{1}{(x - 1)^n} \).

One-sided limits occur as \( x \to a \) from either the left or the right.

- \( x \to 3^- \) reads “\( x \) approaches 3 from the left.” For example, 2.5, 2.9, 2.99, 2.999, ...
- \( x \to 3^+ \) reads “\( x \) approaches 3 from the right.” For example, 3.5, 3.1, 3.01, 3.001, ...
- \( x \to -2^- \) reads “\( x \) approaches \(-2\) from the left.” For example, \(-2.1\), \(-2.01\), \(-2.001\), ...
- \( x \to -2^+ \) reads “\( x \) approaches \(-2\) from the right.” For example, \(-1.9\), \(-1.99\), \(-1.999\), ...

Example 1 Vertical Asymptotes

Consider the function defined by \( f(x) = \frac{1}{(x + 2)(x - 3)} \).

a) Determine the vertical asymptotes.

b) Find the one-sided limits in the vicinity of the vertical asymptotes.

c) Sketch a graph of the function.
Solution

a) Vertical asymptotes occur at x-values for which the function is undefined. The function $f(x)$ is undefined when the denominator equals zero.

$$x + 2 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -2 \quad x = 3$$

The equations of the vertical asymptotes are $x = -2$ and $x = 3$.

b) Consider the vertical asymptote defined by $x = 3$. One way to determine the behaviour of the function as it approaches the limit is to consider what happens if $x$ is very close to the limit.

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{1}{(x + 2)(x - 3)}$$

$$= \frac{1}{(3 + 2)(\text{very small negative number})}$$

$$= \frac{1}{(5)(\text{very small negative number})}$$

$$= -\infty$$

As $x$ approaches 3 from the left, $f(x)$ approaches a very large negative number.

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{1}{(x + 2)(x - 3)}$$

$$= \frac{1}{(3 + 2)(\text{very small positive number})}$$

$$= \frac{1}{(5)(\text{very small positive number})}$$

$$= \infty$$

As $x$ approaches 3 from the right, $f(x)$ approaches a very large positive number.

The graph shows the behaviour of $f(x)$ near $x = 3$.

Now consider the vertical asymptote defined by $x = -2$. Another way to determine the behaviour of the function as it approaches the limit is to substitute values very close to the limit for $x$, and find the value of the function.

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{(x + 2)(x - 3)}$$

$$= \frac{1}{(-2.01 + 2)(-2.01 - 3)}$$

$$= \frac{1}{(-0.01)(-5.01)}$$

$$\approx 19.96$$

To approximate the limit as $x$ approaches $-2$ from the left, substitute a number slightly less than $-2$, such as $-2.01$. It is important to determine whether the factor that is causing the vertical asymptote is approaching a small positive or negative number when examining the one-sided limits.
As $x$ approaches $-2$ from the left, $f(x)$ approaches a large positive number.

\[
\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{1}{(x + 2)(x - 3)} = \frac{1}{(-1.99 + 2)(-1.99 - 3)} = \frac{1}{(0.01)(-4.99)} = -20.04
\]

$-1.99$ is close to, but greater than $-2$.

As $x$ approaches $-2$ from the right, $f(x)$ approaches a large negative number.

The graph shows the behaviour of $f(x)$ near $x = -2$. 
Example 2  Derivatives of Rational Functions

Consider the function defined by \( f(x) = \frac{1}{x^2 + 1} \).

a) Find the intervals over which the function is increasing and decreasing.

b) Find the locations of any points of inflection.

c) Explain why the graph never crosses the \( x \)-axis and why there are no vertical asymptotes.

d) Sketch a graph of the function.

Solution

a) A function is increasing if the first derivative is positive.

Express the function in the form \( f(x) = (x^2 + 1)^{-1} \), then find \( f'(x) \).

\[
\begin{align*}
   f'(x) &= -1(x^2 + 1)^{-2}(2x) \\
   &= \frac{-2x}{(x^2 + 1)^2}
\end{align*}
\]

\( (x^2 + 1)^{-2} \) becomes \( (x^2 + 1)^2 \) in the denominator.

Because the exponent is even, the denominator, \( (x^2 + 1)^2 \), is always positive. The numerator determines whether \( f'(x) \) is positive.

Find the values of \( x \) when \( f'(x) = 0 \).

Because \( f'(x) \) is a rational function, \( f'(x) = 0 \) when the numerator equals zero.

\( f'(x) = 0 \) when \( -2x = 0 \)

\( f'(x) = 0 \) when \( x = 0 \)

\( x = 0 \) divides the domain into two parts: \( x < 0 \) and \( x > 0 \). In the table, \( x = -1 \) and \( x = 1 \) are substituted into \( f'(x) \) for the two intervals.

<table>
<thead>
<tr>
<th>Test Value</th>
<th>( x &lt; 0 )</th>
<th>( x = 0 )</th>
<th>( x &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-1) )</td>
<td>( \frac{-2(-1)}{((-1)^2 + 1)^2} ) = ( \frac{2}{4} ) Positive</td>
<td>0</td>
<td>( \frac{-2(1)}{(1)^2 + 1)^2} ) = ( \frac{-2}{4} ) Negative</td>
</tr>
<tr>
<td>( f'(0) )</td>
<td>( \frac{-2(0)}{(0)^2 + 1)^2} ) = 0</td>
<td>0</td>
<td>( \frac{-2}{4} ) Negative</td>
</tr>
<tr>
<td>( f(0) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the table, \( f(x) \) is increasing when \( x < 0 \) and decreasing when \( x > 0 \).
b) A function has a point of inflection if the second derivative is zero or undefined and is changing sign at that point.

Express the first derivative of the function in the form

\[ f'(x) = (-2x)(x^2 + 1)^{-2} \]

then find \( f''(x) \).

\[
\begin{align*}
f''(x) &= (x^2 + 1)^{-2} \frac{d}{dx}(-2x) + (-2x) \frac{d}{dx}(x^2 + 1)^{-2} \\
&= -2(x^2 + 1)^{-2} \cdot 2x - 2(x^2 + 1)^{-3} \cdot 2x \\
&= -2(x^2 + 1)^{-2} - 8x^2(x^2 + 1)^{-3} \\
&= (x^2 + 1)^{-3}[-2(x^2 + 1) + 8x^2] \\
&= \frac{6x^2 - 2}{(x^2 + 1)^3}
\end{align*}
\]

The value of \( f''(x) \) is zero when the numerator is zero.

\[ 0 = 6x^2 - 2 \]
\[ x^2 = \frac{1}{3} \]
\[ x = \pm \frac{1}{\sqrt{3}} \]

Determine if \( f''(x) \) is changing sign when \( f''(x) = 0 \).

\[ f''(-1) = \frac{1}{2} \quad f''(0) = -2 \quad f''(1) = \frac{1}{2} \]

There are points of inflection at \( x = \pm \frac{1}{\sqrt{3}} \).

d) For the function \( f(x) = \frac{1}{x^2 + 1} \), the numerator is a positive constant, and the denominator is positive for all values of \( x \), because \( x^2 + 1 \) has a minimum value of 1.

Therefore, the value of \( f(x) \) is always positive. As the values of \( x \) become large (positive or negative), the denominator becomes large, and \( \frac{1}{x^2 + 1} \) becomes small and positive.

Since there are no values of \( x \) for which the function is undefined, there are no vertical asymptotes.

d) Because the \( y \)-value approaches 0 as \( x \to \pm \infty \), the graph must be concave up for large positive or negative values of \( x \).
Example 3  Concavity of Rational Functions

Find the intervals of concavity for \( f(x) = \frac{-1}{x + 2} \). Sketch the graph.

Solution

Rewrite \( f(x) = \frac{-1}{x + 2} \) as \( f(x) = -1(x + 2)^{-1} \).

\[
\begin{align*}
  f'(x) &= (-1)(-1)(x + 2)^{-2} \\
  f''(x) &= (x + 2)^{-2} \\
  f'''(x) &= (-2)(x + 2)^{-3} \\
  f''''(x) &= \frac{-2}{(x + 2)^3}
\end{align*}
\]

The numerator in \( f''(x) \) is a constant, so \( f''(x) \neq 0 \). There is a vertical asymptote at \( x = -2 \). The denominator changes sign at the vertical asymptote, so \( f''(x) \) also changes sign. This results in a change of concavity.

Test Value | \( x < -2 \) | \( x = -2 \) | \( x > -2 \)
---|---|---|---
\( f''(x) \) | Positive | Undefined | Negative
\( f''(x) \) | Concave up | Vertical asymptote | Concave down

KEY CONCEPTS

- Vertical asymptotes usually occur in rational functions at values of \( x \) that make the denominator equal to zero. The line \( x = a \) is a vertical asymptote if \( f(x) \to \pm \infty \) as \( x \to a \) from the left and/or the right.
- Vertical asymptotes must be considered when finding intervals of concavity or intervals of increase or decrease.
- Use patterns to determine how a function behaves in the vicinity of a vertical asymptote.
Communicate Your Understanding

C1 Changes in concavity can occur only at points of inflection. Is this statement true or false? Explain.

C2 Describe the domain of the function \( f(x) = \frac{1}{x - 1} \).

C3 Explain the conditions under which a rational function would have no vertical asymptotes.

A) Practise

1. For each function, find the equations of any vertical asymptotes that exist.
   a) \( f(x) = \frac{x}{x - 5} \)
   b) \( f(x) = \frac{x + 3}{x^2 - 4} \)
   c) \( k(x) = \frac{3}{x^2 + 5} \)
   d) \( y = \frac{x^2}{x^2 - 3x + 2} \)
   e) \( b(x) = \frac{x - 5}{x^2 + 2x - 4} \)
   f) \( y = 2x + \frac{1}{x} \)
   g) \( p(x) = \frac{x - 2}{x^4 + 8} \)
   h) \( g(x) = \frac{2x - 3}{x^2 - 6x + 9} \)

2. For each function in question 1 that has vertical asymptotes, find the one-sided limits approaching the vertical asymptotes.

3. Find the derivative of each function. Then, determine whether the function has any local extrema.
   a) \( y = \frac{1}{x^2} \)
   b) \( f(x) = \frac{2}{x + 3} \)
   c) \( g(x) = \frac{x}{x - 4} \)
   d) \( h(x) = \frac{3}{(x - 2)^2} \)
   e) \( y = \frac{x}{x^2 - 1} \)
   f) \( t(x) = \frac{2x}{3x^2 + 12x} \)

B) Connect and Apply

4. Consider the function \( f(x) = \frac{-2}{(x + 1)^2} \).
   a) Describe how \( f(x) \) compares to the function \( g(x) = \frac{1}{x^2} \).
   b) Find the intervals of increase and decrease for \( f(x) \).
   c) Find the intervals of concavity for \( f(x) \).

5. Consider the function \( h(x) = \frac{1}{x^2 - 4} \).
   a) Write the equations of the vertical asymptotes.
   b) Make a table showing the intervals over which the function is increasing and decreasing.
   c) How can you use the table from part b) to determine the behaviour of \( f(x) \) in the vicinity of the vertical asymptotes?
   d) Sketch a graph of the function.

6. After a chemical spill, the cost of cleaning up \( p \) percent of the contaminants is represented by the equation \( C(p) = \frac{75000}{100 - p} \).
   a) Find the cost of removing 50% of the contaminants.
   b) Find the limit as \( p \) approaches 100 from the left.
   c) Why is it not feasible to remove all of the contaminants?

7. A function has a vertical asymptote defined by \( x = 2 \). The function is concave down when \( x > 2 \). Find \( \lim_{x \to 2^+} f(x) \). Explain your reasoning.
8. A pollutant has been leaking steadily into a river. An environmental group undertook a clean-up of the river. The number of units of the pollutant in the river \( t \) years after the clean-up began is given by the equation

\[
N(t) = 2t + \frac{1}{10t + 1}.
\]

a) How many units of the pollutant were in the river when the clean-up began?
b) After how many years is the number of units a minimum?
c) What may have happened at this point?

\[\textbf{Achievement Check}\]

9. Consider the function \( f(x) = \frac{x}{x - 1} \).

a) State the equation of the vertical asymptote.
b) Make a table showing the intervals over which the function is increasing and decreasing.
c) How can you use your table from part b) to determine the behaviour of \( f(x) \) in the vicinity of the vertical asymptote?
d) Are there any turning points? Explain how this might help you graph \( f(x) \) for large values of \( x \).

\[\textbf{Extend and Challenge}\]

10. Use a graphing calculator to graph the function \( f(x) = \frac{x^2 - 4}{x - 2} \). Use the ZOOM or WINDOW commands to examine the graph near \( x = 2 \). Use the TABLE feature to examine the \( y \)-values at and near \( x = 2 \).

a) Why is \( x = 2 \) not a vertical asymptote?
b) Is the function defined at \( x = 2 \)? Explain.

11. Prove that a function of the form

\[ f(x) = \frac{ax}{bx + c}, \]

where \( a, b, \) and \( c \) are non-zero constants, will never have a turning point.

12. Write the equation of a function \( f(x) \) with vertical asymptotes defined by \( x = 2 \) and \( x = -1 \), and an \( x \)-intercept at 1.

13. \textbf{Math Contest} Which statements are true about the graph of the function

\[ y = \frac{(x + 1)^2}{2x^2 + 5x + 3} \]?

i) The \( x \)-intercept is \(-1\).
ii) There is a vertical asymptote at \( x = -1 \).
iii) There is a horizontal asymptote at \( y = \frac{1}{2} \).

\[\begin{align*}
\text{A} & \quad \text{i) only} \\
\text{B} & \quad \text{iii) only} \\
\text{C} & \quad \text{i) and iii) only} \\
\text{D} & \quad \text{ii) and iii) only} \\
\text{E} & \quad \text{i), ii), and iii)}
\end{align*}\]

14. \textbf{Math Contest} Consider these functions for positive values of \( n \). For which of these functions does the graph not have an asymptote?

\[\begin{align*}
\text{A} & \quad y = \frac{x^{2n} - 1}{x^{2n} + x^n} \\
\text{B} & \quad y = \frac{x^{2n} + 1}{x^n + 1} \\
\text{C} & \quad y = \frac{x^{2n} - 1}{x^n + 1} \\
\text{D} & \quad y = \frac{x^{2n} - x^n}{x^{2n} + x^n - 2} \\
\text{E} & \quad y = \frac{x^{2n+1} + x + 1}{x^{2n} + 1}
\end{align*}\]
Putting It All Together

Some investors buy and sell stocks as the price increases and decreases in the short term. Analysing patterns in stock prices over time helps investors determine the optimal time to buy or sell. Other investors prefer to make long-term investments and not worry about short-term fluctuations in price. In this section, you will apply calculus techniques to sketch functions.

Example 1 Analyse a Function

Consider the function \( f(x) = x^3 + 6x^2 + 9x \).

a) Determine whether the function is even, odd, or neither.

b) Determine the domain of the function.

c) Determine the intercepts.

d) Find and classify the critical points. Identify the intervals of increase and decrease, any extrema, the intervals of concavity, and the locations of any points of inflection.

Solution

a) \( f(-x) = (-x)^3 + 6(-x)^2 + 9(-x) \)
   \( = -x^3 + 6x^2 - 9x \)

   The function is neither even nor odd.

b) The function is defined for all values of \( x \), so the domain is \( x \in \mathbb{R} \).

c) The \( y \)-intercept is 0.

   The \( x \)-intercepts occur when \( f(x) = 0 \).

   \[ 0 = x^3 + 6x^2 + 9x \]
   \[ = x(x^2 + 6x + 9) \]
   \[ = x(x + 3)^2 \]

   The \( x \)-intercepts are 0 and \(-3\).

d) Determine the first and second derivatives, and find the \( x \)-values at which they equal zero.

   \[ f'(x) = 3x^2 + 12x + 9, \quad f''(x) = 6x + 12 \]

   \[ 0 = 3x^2 + 12x + 9 \]
   \[ 0 = 3(x^2 + 4x + 3) \]
   \[ 0 = 3(x + 3)(x + 1) \]

   \( x = -3 \) and \( x = -1 \)

   \[ 6x = -12 \]
   \[ x = -2 \]

   Since \( f''(-2) = 0 \), there may be a point of inflection at \( x = -2 \).
The critical numbers divide the domain into three intervals: \( x < -3 \), \( -3 < x < -1 \), and \( x > -1 \).

Test an \( x \)-value in each interval.

- **Test \( x = -4 \) in the interval \( x < -3 \).**
  
  \[
  f'( -4 ) = 3(-4)^2 + 12(-4) + 9 \quad f''( -4 ) = 6(-4) + 12 \\
  f'(-4) = 9 \quad f''(-4) = -12
  \]
  
  On the interval \( x < -3 \), \( f'(x) > 0 \), so \( f(x) \) is increasing and \( f''(x) < 0 \), so \( f(x) \) is concave down.

- **Test \( x = -2 \) in the interval \( -3 < x < -1 \).**
  
  \[
  f'(-2) = 3(-2)^2 + 12(-2) + 9 \\
  f'(-2) = -3
  \]
  
  Since \( f'(x) < 0 \), \( f(x) \) is decreasing on the interval \( -3 < x < -1 \).

- **Test \( x = 0 \) in the interval \( x > -1 \).**
  
  \[
  f'(0) = 3(0)^2 + 12(0) + 9 \quad f''(0) = 6(0) + 12 \\
  f'(0) = 9 \quad f''(0) = 12
  \]
  
  On the interval \( x > -1 \), \( f'(x) > 0 \), so \( f(x) \) is increasing and \( f''(x) > 0 \), so \( f(x) \) is concave up.

At \( x = -2 \), \( f''(x) = 0 \) and is changing sign from negative to positive. So there is a point of inflection at \( x = -2 \).

There are local extrema at \( x = -3 \) and \( x = -1 \).

Use the second derivative test to classify the local extrema as local maxima or local minima and to determine the concavity of the function.

\[
 f''(-3) = 6(-3) + 12 \
 f''(-3) = -6
\]

Since \( f''(x) < 0 \), \( f(x) \) is concave down and there is a local maximum at \( x = -3 \).

\[
 f''(-1) = 6(-1) + 12 \\
 f''(-1) = 6
\]

Since \( f''(x) > 0 \), \( f(x) \) is concave up and there is a local minimum at \( x = -1 \).
Summarize the information in a table.

<table>
<thead>
<tr>
<th>Test Value</th>
<th>$x &lt; -3$</th>
<th>$x = -3$</th>
<th>$-3 &lt; x &lt; -1$</th>
<th>$x = -1$</th>
<th>$x &gt; -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>Positive</td>
<td>0</td>
<td>Negative</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>Negative</td>
<td>Negative</td>
<td>0</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>Increasing Concave down</td>
<td>Local maximum</td>
<td>Decreasing Point of inflection</td>
<td>Local minimum</td>
<td>Increasing Concave up</td>
</tr>
</tbody>
</table>

Follow these steps to sketch the graph of a polynomial function $y = f(x)$:

**Step 1** Determine the domain of the function.

**Step 2** Determine the intercepts of the function.

**Step 3** Determine and classify the critical numbers of the function.

**Step 4** Determine the points of inflection.

**Step 5** Determine the intervals of increase and decrease and the intervals of concavity of the function.

**Step 6** Sketch the function.

---

**Example 2** Analyse and Sketch a Function

Analyse the key features of the function $f(x) = x^4 - 5x^3 + x^2 + 21x - 18$. Then sketch the function.

**Solution**

**Step 1** Determine the domain of the function.

The function is defined for all values of $x$, so the domain is $x \in \mathbb{R}$.

**Step 2** Determine the intercepts of the function.

The $y$-intercept is $-18$.

The $x$-intercepts occur when $f(x) = 0$. Use the factor theorem. Test factors of $-18$.

Try $x = 1$.

$f(1) = (1)^4 - 5(1)^3 + (1)^2 + 21(1) - 18$

$= 0$
So, \((x - 1)\) is a factor.

\[
\begin{align*}
    x^3 - 4x^2 - 3x + 18 \\
    x - 1) x^4 - 5x^3 + x^2 + 21x - 18 \\
    \underline{x^4 - x^3} \\
    -4x^3 + x^2 \\
    -4x^3 + 4x^2 \\
    \underline{-3x^2 + 21x} \\
    -3x^2 + 3x \\
    \underline{18x - 18} \\
    18x - 18 \\
    0
\end{align*}
\]

The other factor is \(x^3 - 4x^2 - 3x + 18\). Use the factor theorem again. Test factors of 18.

Try \(x = 3\).

\[
f(3) = (3)^3 - 4(3)^2 - 3(3) + 18 = 0
\]

So, \((x - 3)\) is a factor.

\[
\begin{align*}
    x^2 - x - 6 \\
    x - 3) x^3 - 4x^2 - 3x + 18 \\
    \underline{x^3 - 3x^2} \\
    -x^2 - 3x \\
    -x^2 + 3x \\
    \underline{-6x + 18} \\
    -6x + 18 \\
    0
\end{align*}
\]

Another factor is \(x^2 - x - 6\), which can be factored as \((x + 2)(x - 3)\).

\[f(x) = x^4 - 5x^3 + x^2 + 21x - 18\] can be written in factored form,

\[f(x) = (x + 2)(x - 1)(x - 3)^2.\]

The \(x\)-intercepts are \(-2, 1, \) and \(3\).

**Step 3** Determine and classify the critical numbers of the function.

\[f'(x) = 4x^3 - 15x^2 + 2x + 21\]

Use the factor theorem. Test factors of 21.

Try \(x = 3\).

\[
f(3) = 4(3)^3 - 15(3)^2 + 2(3) + 21 = 0
\]
So, \((x - 3)\) is a factor.

\[
\begin{align*}
4x^2 - 3x - 7 \\
x - 3 \left(4x^3 - 15x^2 + 2x + 21\right) \\
\frac{4x^3 - 12x^2}{4x^3 - 12x^2} \\
-3x^2 + 2x \\
\frac{-3x^2 + 9x}{-3x^2 + 9x} \\
-7x + 21 \\
0
\end{align*}
\]

Another factor is \(4x^2 - 3x - 7\), which can be factored as \((x + 1)(4x - 7)\).

0 = \((x + 1)(4x - 7)(x - 3)\)
The critical numbers are \(-1, 1.75, \) and \(3\).

Now use the second derivative test to classify the critical points.

\(f''(x) = 12x^2 - 30x + 2\)

- For \(x = -1\), \(f''(-1) = 12(-1)^2 - 30(-1) + 2 = 44\)
  \(f'(x) = 0\) and \(f''(-1) > 0\), so \(f(x)\) has a local minimum at \(x = -1\).
- For \(x = 1.75\), \(f''(1.75) = 12(1.75)^2 - 30(1.75) + 2 = -13.75\)
  \(f'(x) = 0\) and \(f''(1.75) < 0\), so \(f(x)\) has a local maximum at \(x = 1.75\).
- For \(x = 3\), \(f''(3) = 12(3)^2 - 30(3) + 2 = 20\)
  \(f'(x) = 0\) and \(f''(3) > 0\), so \(f(x)\) has a local minimum at \(x = 3\).

Substitute the critical numbers into \(f(x) = x^4 - 5x^3 + x^2 + 21x - 18\).

\[
\begin{align*}
f(-1) &= -32 \\
f(1.75) &= 4.4 \\
f(3) &= 0
\end{align*}
\]

There are local minima at \((-1, -32)\) and \((3, 0)\), and a local maximum at \((1.75, 4.4)\).

**Step 4 Determine the points of inflection.**

\(f''(x) = 12x^2 - 30x + 2\)

\[
0 = 12x^2 - 30x + 2 \\
= 6x^2 - 15x + 1
\]

\[
x = \frac{15 \pm \sqrt{(-15)^2 - 4(6)(1)}}{2(6)}
\]

\[
= \frac{15 \pm \sqrt{201}}{12}
\]

\[
x = 0.07 \quad \text{or} \quad x = 2.43
\]
Check that $f''(x)$ changes sign at $x = 0.07$.

$$f''(0) = 12(0)^2 - 30(0) + 2 = 2$$
$$f''(1) = 12(1)^2 - 30(1) + 2 = -16$$

At $x = 0.07$, $f''(x) = 0$ and is changing sign. There is a point of inflection at $x = 0.07$.

Check that $f''(x)$ changes sign at $x = 2.43$.

$$f''(2) = 12(2)^2 - 30(2) + 2 = -10$$
$$f''(3) = 12(3)^2 - 30(3) + 2 = 20$$

At $x = 2.43$, $f''(x) = 0$ and is changing sign. There is a point of inflection at $x = 2.43$.

**Step 5** Determine the intervals of increase and decrease and the intervals of concavity of the function.

The critical numbers divide the domain into four intervals: $x < -1$, $-1 < x < 1.75$, $1.75 < x < 3$, and $x > 3$. Determine the behaviour of the function in each interval.

- Test $x = -2$ in the interval $x < -1$.

$$f'(-2) = 4(-2)^3 - 15(-2)^2 + 2(-2) + 21 = -75$$
$$f''(-2) = -110$$

On the interval $x < -1$, $f'(x) < 0$, so $f(x)$ is decreasing and $f''(x) < 0$, so $f(x)$ is concave down.

- Test $x = 0$ in the interval $-1 < x < 1.75$.

$$f'(0) = 3(0)^3 + 12(0) + 9 = 9$$

On the interval $-1 < x < 1.75$, $x < -1$, $f'(x) > 0$, so $f(x)$ is increasing. From Step 3, $f''(x) > 0$ on the interval $-1 < x < 0.07$ and $f''(x) < 0$ on the interval $0.07 < x < 1.75$, so $f(x)$ changes from concave up to concave down at $x = 0.07$.

- Test $x = 2$ in the interval $1.75 < x < 3$.

$$f'(2) = 3(2)^3 + 12(2) + 9 = 45$$
$$f''(2) = -3$$

On the interval $1.75 < x < 3$, $f'(x) < 0$, so $f(x)$ is decreasing. From step 3, $f''(x) < 0$ on the interval $1.75 < x < 2.43$ and $f''(x) > 0$ on the interval $2.43 < x < 3$, so $f(x)$ changes from concave down to concave up at $x = 2.43$. 
• Test \( x = 5 \) in the interval \( x > 3 \).

\[
f'(5) = 4(5)^3 - 15(5)^2 + 2(5) + 21 \quad f''(5) = 12(-2)^2 - 30(-2) + 2
f'(5) = 156 \quad f''(5) = 152
\]

On the interval \( x > 3 \), \( f'(x) > 0 \), so \( f(x) \) is increasing and \( f''(x) > 0 \), so \( f(x) \) is concave up.

Summarize the information in a table.

<table>
<thead>
<tr>
<th>Test Value</th>
<th>( x &lt; -1 )</th>
<th>( x = -1 )</th>
<th>(-1 &lt; x &lt; 1.75 )</th>
<th>( x = 1.75 )</th>
<th>( 1.75 &lt; x &lt; 3 )</th>
<th>( x = 3 )</th>
<th>( x &gt; 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>Negative</td>
<td>0</td>
<td>Positive</td>
<td>0</td>
<td>Negative</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>Positive</td>
<td>Positive</td>
<td>Point of inflection at ( x = 0.07 )</td>
<td>Negative</td>
<td>Point of inflection at ( x = 2.43 )</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>Decreasing</td>
<td>Concave up</td>
<td>Increasing</td>
<td>Local maximum</td>
<td>Decreasing</td>
<td>Local minimum</td>
<td>Increasing</td>
</tr>
</tbody>
</table>

**Step 6 Sketch the function.**

**KEY CONCEPTS**

- The first and second derivatives of a function can be used to identify critical points, determine intervals of increase and decrease, and identify the concavity of the function.
- To sketch a function, determine the domain of the function; determine the critical points; identify any symmetry, intervals of increase and decrease, and the concavity of the function; and then sketch the curve.
Communicate Your Understanding

C1 Explain how symmetry can be used to help analyse functions.
C2 Describe some strategies you can use to approach a curve-sketching problem.

A) Practise

1. For each function, determine the coordinates of the local extrema. Classify each point as a local maximum or a local minimum.
   a) \( f(x) = x^3 - 6x \)
   b) \( g(x) = -x^4 + 2x^2 \)
   c) \( f(x) = -x^3 + 3x - 2 \)
   d) \( h(x) = 2x^2 + 4x + 5 \)

2. For each function, determine the coordinates of any points of inflection.
   a) \( f(x) = 2x^3 - 4x^2 \)
   b) \( f(x) = x^4 - 6x^2 \)
   c) \( f(x) = x^5 - 30x^3 \)
   d) \( f(x) = 3x^5 - 5x^4 - 40x^3 + 120x^2 \)

B) Connect and Apply

4. Sketch each function.
   a) \( f(x) = x^3 + 1 \)
   b) \( h(x) = x^5 + 20x^2 + 5 \)
   c) \( k(x) = \frac{1}{2}x^4 - 2x^3 \)
   d) \( b(x) = -(2x - 1)(x^2 - x - 2) \)

5. Consider the function \( f(x) = 2x^3 - 3x^2 - 72x + 7 \).
   a) What is the maximum number of local extrema this function can have? Explain.
   b) What is the maximum number of points of inflection this function can have? Explain.
   c) Find and classify the critical points. Identify the intervals of increase and decrease, and state the intervals of concavity.
   d) Sketch the function.
   e) Compare your answers to parts c) and d) with your answers to parts a) and b).

6. Repeat question 5 for each function.
   i) \( h(x) = 3x^2 - 27 \)
   ii) \( t(x) = x^3 - 2x^4 + 3 \)
   iii) \( g(x) = x^4 - 8x^2 + 16 \)
   iv) \( k(x) = -2x^4 + 16x^3 - 12 \)

7. Refer to your answers to questions 5 and 6.
   a) How is the maximum possible number of local extrema related to the degree of the function?
   b) Explain why a particular function may have fewer than the maximum possible number of local extrema.

8. Consider a polynomial function of degree 6. How many points of inflection will it have? Can it have zero points of inflection?

9. Analyse and sketch each function.
   a) \( k(x) = 3x^3 + 7x^2 + 3x - 1 \)
   b) \( t(x) = 2x^3 - 12x^2 + 18x - 1 \)
   c) \( f(x) = 2x^4 - 26x^2 + 72 \)
   d) \( h(x) = x^4 - 6x^3 + 9x^2 + 3x \)
   e) \( g(x) = 3x^4 + 2x^3 - 15x^2 + 12x - 2 \)
10. Prove or disprove. Every even function has at least one turning point.

11. Use Technology
   a) Graph each of the following functions using a graphing calculator or other graphing software.
   b) Use the graph to estimate the intervals of concavity and the locations of any points of inflection.
   c) Use the first and second derivatives to give better estimates for these values.
   i) \( f(x) = 2x^5 - 20x^3 + 15x \)
   ii) \( g(x) = x^5 - 8x^3 + 20x - 1 \)

12. Write the equation of a function that has a local minimum when \( x = 2 \), and a point of inflection at \( x = 0 \). Explain how these conditions helped you determine the function.

13. Joshua was sketching the graph of the functions \( f(x) \) and \( g(x) \). He lost part of his notes including the formulas for the functions. Given the partial information below, sketch each function.
   a) The domain of \( f \) is \( \mathbb{R} \).
      \[ \lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty \]
      The \( y \)-intercept is 2. There are \( x \)-intercepts at \(-1\), \(3\), and \(5\). There are local extrema at \( x = 2\), \(3\), and \(4\).
   b) The polynomial function \( g \) has domain \( \mathbb{R} \).
      \[ \lim_{x \to -\infty} g(x) = \infty \]
      \( g \) is an odd function.
      There are five \( x \)-intercepts, two of which are 4 and \(-2\). There are local maxima at \((3, 4)\) and \((-1, 5)\).

14. The function \( f(x) = -x + \frac{1}{x^2} \) can be considered a sum of two functions.
   a) Identify the two functions.
   b) Sketch the two functions lightly on the same set of axes.
   c) Use your results from part b) to predict what the graph of \( f(x) \) will look like.
   d) Verify your prediction using graphing technology.
   e) Find the first derivative of \( f(x) = -x + \frac{1}{x^2} \) and use it to find the turning point.
   f) Find the second derivative of \( f(x) = -x + \frac{1}{x^2} \) and use it to find the intervals of concavity.

15. A function has vertical asymptotes at \( x = -1 \) and \( x = 3 \), and approaches zero as \( x \) approaches \( \pm \infty \). It is concave down on the intervals \(-\infty < x < -1 \) and \(-1 < x < 1 \). It is concave up on the intervals \( 1 < x < 3 \) and \( 3 < x < \infty \).
   a) Sketch a possible graph of such a function.
   b) Is there more than one equation that could model such a graph? Justify your answer.
   c) Is it possible to have any local extrema between \(-1\) and \(3\)? Justify your answer.

16. Consider the function \( g(x) = \frac{1}{x^2 - 1} \).
   a) How does \( g(x) \) behave as \( x \to \pm \infty \)? Explain your reasoning.
   b) State the equations of the vertical asymptotes.
   c) Evaluate the one-sided limits at one of the asymptotes.
   d) What does symmetry tell you about the other asymptote?
   e) Find and classify the critical point(s).
17. Consider the function \( f(x) = \frac{x - 1}{x + 1} \).
   a) Find the \( x \)- and \( y \)-intercepts of its graph.
   b) State the equations of the asymptotes.
   c) Find the first derivative and use it to determine when the function is increasing and decreasing.
   d) Explain how your answer to part c) helps determine the behaviour of the function in the vicinity of the asymptotes.
   e) Sketch the graph.

18. For each function,
   a) How does the function behave as \( x \to \pm \infty \)? Explain your reasoning.
   b) Does the function have any symmetry? Explain.
   c) Use the derivative to find the critical point(s). Classify them using the second derivative test.
   d) Find the point(s) of inflection and make a table showing the intervals of concavity.
      i) \( g(x) = x^3 - 27x \)
      ii) \( y = x^4 - 8x^2 + 16 \)
      iii) \( k(x) = \frac{1}{1 - x^2} \)
      iv) \( f(x) = \frac{x}{x^2 + 1} \)
      v) \( h(x) = \frac{x - 4}{x^2} \)

19. For the function \( g(x) = \frac{1}{x^2 + 1} \), answer all of the questions without finding derivatives.
   a) Find all of the intercepts.
   b) Find the maximum value. Explain your reasoning.
   c) State the equation of the horizontal asymptote.
   d) Are there any local minima? Use your answers to parts a) through c) to explain.
   e) How many points of inflection are there? Explain your reasoning.
   f) Sketch the graph, then verify your work using a graphing calculator.

20. Consider the functions \( f(x) = \frac{x^2 + 2x + 3}{x + 1} \) and \( g(x) = x + 1 + \frac{2}{x + 1} \).
   a) Use a graphing calculator to verify that \( f(x) = g(x) \).
   b) Prove algebraically that \( f(x) = g(x) \). Try to find more than one method.
   c) The function \( g(x) \) can be considered the sum of what two functions? Describe how these functions can help predict the shape of \( g(x) \).

21. **Math Contest** How many local extreme points does the graph of the function \( y = (x + 2)^5(x^2 - 1)^4 \) have?
   A 2
   B 3
   C 4
   D 5
   E 6

22. **Math Contest** For which of these functions is the derivative discontinuous at a point in the function’s domain?
   A \( y = \frac{1}{x^2} \)
   B \( y = \frac{1}{x} \)
   C \( y = \frac{x^2}{x} \)
   D \( y = \sqrt{x} \)
   E \( y = \frac{1}{\sqrt{x}} \)
In order to remain competitive in a global market, businesses must continually strive to maximize revenue and productivity while minimizing costs. A factory wants to produce the number of units that will minimize the cost of production while maximizing profits. Packaging must be as inexpensive as possible, while allowing for efficient transportation and storage. A farmer needs to determine an appropriate number of seeds to buy and plant in order to maximize yield while minimizing planting costs. In this section, you will examine a variety of real-life problems in which it is necessary to find the best, or optimal, value.

### Investigate

**How can you find the maximum value of a quantity on a given interval?**

A 400-m track is to be constructed of two straightaways and two semicircular ends, as shown in the diagram. The straightaways can be no less than 100 m long.

What radius would produce the maximum area enclosed inside the track?

1. Write an expression for the area, \( A \), in terms of \( r \) and \( L \).
2. The perimeter of the track is 400 m. Write a function for the area, \( A \), in terms of \( r \) only.
3. Use a graphing calculator to graph the function from step 2. Find the maximum area and the radius at which this occurs. Find the length of the straightaways for this value of the radius.

**Optional:** Find the maximum area and the values of \( r \) and \( L \) by finding the derivative and proceeding as you did in Sections 3.2 and 3.3.

4. **Reflect** What type of track appears to give the maximum area? Does this make sense? Explain.
5. Does this answer fit the conditions in the question? Explain.
6. What radius gives the maximum area for the given conditions? How do you know?
7. **Reflect** When the interval contains no local maximum or minimum, describe how you would find the maximum or minimum value for the given interval.
Solving an optimization problem is similar to finding the absolute maximum or minimum values on a given interval. Refer back to the Investigate as you read the following suggested strategy.

When approaching optimization problems, follow these steps:

- Read the question carefully, identifying what the question is asking.
- Define the variables. A diagram is often a useful aid.
- Identify the quantity to be optimized.
- Write a word equation or formula, in terms of one or more variables, for the quantity to be optimized.
- Define the independent variable.
- Express all other variables in the word equation or formula in terms of the independent variable. See step 2 of the Investigate.
- Define a function in terms of the independent variable.
- Identify and state any restrictions on the independent variable.
- Differentiate the function.
- Solve \( f'(x) = 0 \) to identify critical points.
- Check the critical points and the endpoints you identified.
- Directly answer the question posed in the problem.

**Example 1**  Maximizing Area

A lifeguard has 200 m of rope and some buoys with which she intends to enclose a rectangular area at a lake for swimming. The beach will form one side of the rectangle, with the rope forming the other three sides. Find the dimensions that will produce the maximum enclosed area if

a) there are no restrictions on the dimensions

b) due to the depth of the water, the area cannot extend more than 40 m into the lake

**Solution**

a) Let \( A \) represent the area of the region to be enclosed, which is the quantity to be maximized.

Express \( A \) in terms of the other variables:

\[
A = L \times W
\]

From the length of the rope, \( 200 = 2L + W \).
Rearrange to isolate $W$.

$W = 200 - 2L$

Substitute $200 - 2L$ for $W$.

$A = L(200 - 2L)$

$= 200L - 2L^2$

It is useful to consider the domain for this function. Clearly, $L \geq 0$. Since $2L + W = 200$, the largest value for $L$ occurs when $W = 0$; i.e., $L = 100$. Thus the domain is $[0, 100]$.

Find the critical points.

$A'(x) = 200 - 4L$

$200 - 4L = 0$ \hspace{2cm} \text{Set the derivative equal to zero to find a potential local maximum.}$

$L = 50$

When $L = 50$, $W = 200 - 2L$

$= 200 - 2(50)$

$= 100$

The critical point is $(L, A)$ or $(50, 5000)$. Check that this critical point is a maximum.

**Method 1: Test the Endpoints of the Interval**

When $L = 0$, the area is zero. Similarly, when $L = 100$, then $W = 0$, so the area is zero.

**Method 2: Sketch a Graph**

Sketch the graph of $A = 200L - 2L^2$ on the interval $0 \leq L \leq 100$.

**Method 3: Use Derivatives**

Determine the second derivative.

$A'' = -4$

Since the second derivative is always negative, the graph is always concave down, including when $L = 50$. This means that there is a local maximum when $L = 50$.

The area is a maximum of $5000$ m$^2$ when the dimensions are $50$ m by $100$ m.

**b)** Due to the restriction, $L$ must not exceed $40$ m. Find the maximum area on the interval $0 < L < 40$.

From part a), the area is increasing on the interval $0 \leq L \leq 50$.

The maximum area is at the right end of the interval, when $L = 40$. 

---

3.6 Optimization Problems • MHR 197
Express \(SA\) in terms of other variables:

\[
\frac{SA}{H11005} = \frac{x^2}{H11001} + \frac{x^2}{H11001} + 4 \cdot xh
\]

\[
8000 = 2x^2 + 32000x^{-1}
\]

\[
SA' = 4x - 32000x^{-2}
\]

\[
SA' = \frac{4x^3 - 32000}{x^2}
\]

This can be confirmed by graphing.

When \(L = 40\),

\[
200 = 2L + W
\]

\[
200 = 2(40) + W
\]

\[
W = 120
\]

To find the area,

\[
A = L \times W = 40 \times 120 = 4800
\]

The maximum area of 4800 m\(^2\) occurs when the dimensions are 40 m by 120 m.

---

**Example 2**  **Packaging Problem**

A cardboard box with a square base is to have a volume of 8 L.

**a)** Find the dimensions that will minimize the amount of cardboard to be used.

**b)** The cardboard for the box costs 0.1¢/cm\(^2\), but the cardboard for the bottom is thicker, so it costs three times as much. Find the dimensions that will minimize the cost of the cardboard.

**Solution**

**a)** Let \(SA\) represent the surface area, the quantity to be minimized.

Express \(SA\) in terms of other variables:

\[
SA = (\text{bottom and top}) + 4(\text{sides})
\]

\[
SA = x^2 + x^2 + 4xh
\]

\[
8000 = 2x^2 + 32000x^{-1}
\]

\[
SA' = 4x - 32000x^{-2}
\]

\[
SA' = \frac{4x^3 - 32000}{x^2}
\]

8 L is equivalent to 8000 cm\(^3\).

From the volume equation, \(8000 = x^2h\).

Rearranging to isolate \(h\) gives \(h = \frac{8000}{x^2}\).

Substitute \(\frac{8000}{x^2}\) for \(h\).

Differentiate.

Write as a fraction.
Examine \( SA'(x) \) or \( SA''(x) \) in the vicinity of \( x = 20 \) to show that the function changes from decreasing to increasing and that it is concave up. Alternatively, graph the function. From the graph, the surface area is a minimum when \( x = 20 \).

Substitute \( x = 20 \) into \( h = \frac{8000}{x^2} \).

\[
b = \frac{8000}{20^2} = 20
\]

The dimensions of the box are 20 cm by 20 cm by 20 cm. In other words, it forms a cube.

b) The cost, \( C \), is to be minimized. Cost restrictions will change the solution from part a). Start with the surface area equation.

\[
SA = (\text{bottom and top}) + 4(\text{sides})
\]

\[
SA = x^2 + x^2 + \frac{32000}{x}
\]

\[
C(x) = 0.3x^2 + 0.1x^2 + 0.1\left(\frac{32000}{x}\right)
\]

\[
C = 0.4x^2 + \frac{32000}{x}
\]

\[
C'(x) = 0.8x - \frac{32000}{x^2}
\]

\[
C'(x) = \frac{0.8x^3 - 32000}{x^2}
\]

\[
0.8x^3 - 32000 = 0
\]

\[
x^3 = 4000
\]

\[
x = 15.9
\]
The minimum point is confirmed by a graph of $C(x)$.
Substitute $x = 15.9$ into $b = \frac{8000}{x^2}$.

$$b = \frac{8000}{15.9^2}$$
$$b \approx 31.7$$

The dimensions that minimize the cost of the box are approximately 15.9 cm by 15.9 cm by 31.7 cm. Note the 1:1:2 ratio.

### KEY CONCEPTS

- When solving optimization problems, follow these steps:
  - Identify what the question is asking.
  - Define the variables; draw a diagram if it helps.
  - Identify the quantity to be optimized and write an equation.
  - Define the independent variable. Express all other variables in terms of the independent variable.
  - Define a function in terms of the independent variable.
  - Identify and state any restrictions on the independent variable.
  - Differentiate the function.
  - Determine and classify the critical points.
  - Answer the question posed in the problem.

- The context of a problem often dictates the interval or domain to be considered.

- Answers should be verified to ensure they make sense, given the context of the question.

### Communicate Your Understanding

1. Finding where the derivative of a function equals zero will always produce the desired optimal value. Is this statement true or false? Explain your reasoning.

2. If $x$ represents the size of a price increase, what would be the meaning of $x = -5$?
A Practise

1. The height, \( h \), of a ball \( t \) seconds after being thrown into the air is given by the function \( h(t) = -4.9t^2 + 19.6t + 2 \). Find the maximum height of the ball.

2. Find two integers with a sum of 20 and whose product is a maximum.

3. At G&W Industries, it has been shown that the number of gizmos an employee can produce each day can be represented by the equation \( N(t) = -0.05t^2 + 3t + 5 \), where \( t \) is the number of years of experience the employee has, and \( 0 \leq t \leq 40 \). How many years of experience does it take to achieve maximum productivity?

B Connect and Apply

4. A rectangular pen is to be built with 1200 m of fencing. The pen is to be divided into three parts using two parallel partitions.
   a) Find the maximum possible area of the pen.
   b) Explain how the maximum area would change if each side of the pen had to be at least 200 m long.

5. Two pens with one common side are to be built with 60 m of fencing. One pen is to be square, the other rectangular, as shown. Find the dimensions that maximize the total area.

6. A showroom for a car dealership is to be built in the shape of a rectangle with brick on the back and sides, and glass on the front. The floor of the showroom is to have an area of 500 m².
   a) If a brick wall costs $1200/m while a glass wall costs $600/m, what dimensions would minimize the cost of the showroom?
   b) Should the cost of the roof be considered when deciding on the dimensions? Explain your reasoning. Assume the roof is flat.

7. A Norman window has the shape of a rectangle with a semicircular top, as shown.
   a) For a given perimeter, find the ratio of height to radius that will maximize the window's area.

8. A cylindrical can is to have a volume of 1 L.
   a) Find the height and radius of the can that will minimize the surface area.
   b) What is the ratio of the height to the diameter?
   c) Do pop cans have a similar ratio? If not, suggest why.

9. A soup can of volume 500 cm³ is to be constructed. The material for the top costs 0.4¢/cm² while the material for the bottom and sides costs 0.2¢/cm². Find the dimensions that will minimize the cost of producing the can.

10. A cylindrical drum with an open top is to be constructed using 1 m² of aluminum.
   a) Write an equation for the volume of the drum in terms of the radius.
   b) What radius gives the maximum volume?
   c) Use a graphing calculator to graph the equation from part a).
   d) What is the maximum volume if the radius can be a maximum of 0.2 m? Refer to the graph as you explain your reasoning.
11. A rectangular piece of paper with perimeter 100 cm is to be rolled to form a cylindrical tube. Find the dimensions of the paper that will produce a tube with maximum volume.

12. There are 50 apple trees in an orchard, and each tree produces an average of 200 apples each year. For each additional tree planted within the orchard, the average number of apples produced drops by 5. What is the optimal number of trees to plant in the orchard?

13. For an outdoor concert, a ticket price of $30 typically attracts 5000 people. For each $1 increase in the ticket price, 100 fewer people will attend. The revenue, $R$, is the product of the number of people attending and the price per ticket.

   a) Let $x$ represent the number of $1 price increases. Find an equation expressing the total revenue in terms of $x$.
   
   b) State any restrictions on $x$. Can $x$ be a negative number? Explain.
   
   c) Find the ticket price that maximizes revenue.
   
   d) Will your answer to part c) change if the concert area holds a maximum of 1200 people? Explain.

14. Find the area of the largest rectangle that can be inscribed between the $x$-axis and the graph defined by $y = 9 - x^2$.

15. The cost of fuel per kilometre for a truck travelling $v$ km/h is given by the equation $C(v) = \frac{v}{400} + \frac{12}{v}$.

   a) What speed will result in the lowest fuel cost per kilometre?
   
   b) Assume a driver is paid $20/h. What speed would give the lowest cost, including fuel and wages, for a 1000-km trip?

16. Find a positive number such that the sum of the square of the number and its reciprocal is a minimum.

17. Brenda drives an 18-wheeler. She plans to buy her own truck. Her research indicates that the expected running costs, $C$ in dollars, are given by $C(v) = 0.9 + 0.0005v^2$, where $v$ is the speed in kilometres per hour. Brenda’s first trip will be 1500 km, round trip. She plans to pay herself $30/h. Determine the speed that will minimize Brenda’s costs for the trip.

18. A piece of plexiglass is in the shape of a semicircle with radius 2 m. Determine the dimensions of the rectangle with the greatest area that can be cut from the piece of plexiglass.

19. **Chapter Problem.** Recall that 20 m of edging is to form the two sides of the square garden and the curved edge of the garden in the shape of a quarter circle. Let $x$ represent the length of edging used for the quarter circle.

   a) Find an expression for the length of edging used for each side of the square garden.
   
   b) Find an expression for the radius of the quarter circle.
   
   c) Use your answers to parts a) and b) to find an equation for the combined area of the two gardens.
   
   d) Show that the equation from part c) is equivalent to $A(x) = \frac{4 + \pi}{4\pi}x^2 - 10x + 100$.

20. A 60-cm by 40-cm piece of tin has squares cut out of each corner, then the sides are folded up to form an open-top box.

   a) Let $x$ represent the side length of the squares that are cut out. Draw a diagram showing all dimensions.
   
   b) Find an equation representing the volume of the box.
   
   c) State the domain of the relation. Explain your reasoning.
   
   d) Find the dimensions that will maximize the volume of the box.
C Extend and Challenge

21. Oil is shipped to a remote island in cylindrical containers made of steel. The height of each container equals the diameter. Once the containers are emptied on the island, the steel is sold. Shipping costs are $10/m³ of oil, and the steel is sold for $7/m².

a) Determine the radius of the container that maximizes the profit per container. Ignore any costs (other than shipping) or profits associated with the oil in the barrel.

b) Determine the maximum profit per container.

c) Check your answers to parts a) and b) by graphing.

22. A box with a square base and closed top is to be constructed out of cardboard.

a) Without finding derivatives, what shape do you expect would give the minimum surface area for a given volume? Explain.

b) Does the volume affect the shape of the box? Explain.

c) If the box has no top, will the shape change? Verify your answer by choosing a volume and solving the problem using calculus.

23. A series of rectangular fenced pens is to be built, each using 1000 m of fencing.

a) Find the dimensions that will maximize the total area in each situation.
   i) a rectangular pen with no restrictions
   ii) a rectangular pen, divided into two as shown
   iii) a rectangular pen, placed against a barn so it only requires three sides to be fenced.

b) Consider the three situations in part a). Do you see a pattern? Explain.

c) Explain how you could find the dimensions if the pens were similar to those in part a) ii), but divided into four equal parts.

24. a) For a rectangular prism of a given volume, what shape would have the least amount of surface area?

b) What three-dimensional shape would have the least amount of surface area for a given volume?

c) Think of the shapes of various packages you would see at the grocery store. Why do you suppose they are not necessarily the shapes you described in parts a) and b)? What other factors might be considered that would affect the shape of containers and packaging?

25. Math Contest A piece of wire of length \( L \) is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut so that the total area enclosed is a minimum?

A Use all of the wire for the circle.

B Use all of the wire for the square.

C Use \( \frac{L}{2} \) for the circle.

D Use \( \frac{4L}{\pi + 4} \) for the circle.

E Use \( \frac{\pi L}{\pi + 4} \) for the circle.

26. Math Contest Consider the function \( f(x) = \frac{x^n}{x-1} \), where \( n \) is a positive integer. Let \( G \) represent the greatest possible number of local extrema for \( f(x) \), and let \( L \) represent the least possible number of local extrema. Which statement is true?

A \( G = n \) and \( L = n - 1 \)

B \( G = n \) and \( L = 1 \)

C \( G = n - 1 \) and \( L = 0 \)

D \( G = 2 \) and \( L = 1 \)

E \( G = 2 \) and \( L = 0 \)
3.1 Increasing and Decreasing Functions

1. Find the intervals over which each function is increasing and decreasing.
   a) \( f(x) = 7 + 6x - x^2 \)
   b) \( y = x^3 - 48x + 5 \)
   c) \( g(x) = x^4 - 18x^2 \)
   d) \( f(x) = x^3 + 10x - 9 \)

2. The derivative of a function is \( f'(x) = x(x - 3)^2 \). Make a table showing the intervals over which the function is increasing and decreasing.

3.2 Maxima and Minima

3. Find the local extrema for each function and classify them as local maxima or local minima.
   a) \( y = 3x^2 + 24x - 8 \)
   b) \( f(x) = 16 - x^4 \)
   c) \( g(x) = x^3 + 9x^2 - 21x - 12 \)

4. The speed, in kilometres per hour, of a certain car \( t \) seconds after passing a police radar location is given by the function \( v(t) = 3t^2 - 24t + 88 \).
   a) Find the minimum speed of the car.
   b) The radar tracks the car on the interval \( 2 < t < 5 \). Find the maximum speed of the car on this interval.

5. Find and classify all critical points of the function \( f(x) = x^3 - 8x^2 + 5x + 2 \) on the interval \( 0 \leq x \leq 6 \).

3.3 Concavity and the Second Derivative Test

6. Which of these statements is most accurate regarding the number of points of inflection on a cubic function? Explain.
   A There is at least one point of inflection.
   B There is exactly one point of inflection.
   C There are no points of inflection.
   D It is impossible to tell.

7. A polynomial function of degree four (a quartic function) has either no points of inflection or two points of inflection. Is this statement true or false? Explain your reasoning.

8. For the function defined by \( f(x) = x^4 - 2x^3 - 12x^2 + 3 \), determine the points of inflection and the intervals of concavity.

9. The graph shows the derivative, \( f'(x) \), of a function \( f(x) \). Copy the graph into your notebook.

   a) Sketch a possible graph of \( f(x) \).
   b) Sketch a possible graph of the second derivative \( f''(x) \).

10. For the function \( f(x) = 2x^3 - x^4 \), determine the critical points and classify them using the second derivative test. Sketch the function.

3.4 Simple Rational Functions

11. For each function, state the equations of any vertical asymptotes.
   a) \( f(x) = \frac{x^2 - 4}{x} \)
   b) \( g(x) = \frac{2x - 3}{2x - 4} \)
   c) \( y = \frac{x^2 + 1}{x^2 - 3x - 10} \)
   d) \( h(x) = \frac{x - 1}{x^2 + 2x + 1} \)
12. Consider the function defined by \( f(x) = \frac{x + 4}{x^2} \).

   a) Without graphing, evaluate \( \lim_{x \to 0} f(x) \).
   
   b) How can you use your answer to part a) to evaluate \( \lim_{x \to 0} f(x) \)? Explain your reasoning.
   
   c) State the coordinates of the \( x \)-intercept.
   
   d) Find the coordinates of the turning point using the first derivative.
   
   e) Use a graphing calculator to verify your results.

3.5 Putting It All Together

13. Consider the function \( f(x) = x^3 - 3x \).

   a) Determine whether the function is even, odd, or neither.
   
   b) Determine the domain of the function.
   
   c) Determine the intercepts.
   
   d) Find and classify the critical points. Identify the intervals of increase and decrease, and state the intervals of concavity.

14. Analyse and sketch each function.

   a) \( f(x) = -x^2 + 2x \)
   
   b) \( k(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2 \)
   
   c) \( b(x) = 2x^3 - 3x^2 - 3x + 2 \)

3.6 Optimization Problems

15. The concentration, in mg/cm\(^3\), of a particular drug in a patient’s bloodstream is given by the formula \( C(t) = \frac{0.12t}{t^2 + 2t + 2} \), where \( t \) represents the number of hours after the drug is taken.

   a) Find the maximum concentration on the interval \( 0 \leq t \leq 4 \).
   
   b) Determine when the maximum concentration occurs.

16. An open-top box is to be constructed so that its base is twice as long as it is wide. Its volume is to be 2400 cm\(^3\). Find the dimensions that will minimize the amount of cardboard required.

   \[ \text{Problem Wrap-Up} \]

Recall the 400-m racetrack discussed in the Investigate in Section 3.6. It was to have two straightaways with semicircular ends. The Chapter Problem involves 20 m of garden edging to be cut into two pieces: one to form two sides of a square garden and one to be the edge of a garden in the shape of a quarter circle.

   a) How are these two problems similar?
   
   b) In the racetrack question, the maximum area, without restrictions, occurs when the length of the straightaways is zero. What shape results? Why does this make sense?
   
   c) The function \( A(x) = \left( \frac{4 + \pi}{4\pi} \right)x^2 - 10x + 100 \) represents the area of Naveen’s gardens, where \( x \) represents the length of the part used for the quarter circle.

   i) What does a value of \( x = 0 \) mean in this context? Describe the resulting shapes and the corresponding area.

   ii) What does a value of \( x = 20 \) mean in this context? Describe the resulting shapes and the corresponding area.

   d) The problem indicates that each piece of garden edging must be at least 5 m in length, meaning \( 5 \leq x \leq 15 \). Based on the similarity to the racetrack question, your experience throughout the chapter, and your answers to part c) above, explain which endpoint will provide the maximum area, given the restrictions on \( x \).
Chapter 3 PRACTICE TEST

For questions 1 to 5, choose the best answer.

1. On the interval \(0 \leq x \leq 3\), the function \(f(x) = x^2 - 8x + 16\)
   A is always increasing
   B is always decreasing
   C has a local minimum
   D is concave down

2. The graph of \(f''(x)\) is shown. Which of these statements is not true for the graph of \(f(x)\)?
   ![Graph of f''(x) with points A, B, and C]
   A It has one turning point.
   B It is concave down for all values of \(x\).
   C It is increasing for \(x < 2\).
   D It is decreasing for all values of \(x\).

3. For a certain function, \(f'(2) = 0\) and \(f'(x) > 0\) for \(-1 < x < 2\). Which statement is not true?
   A \((2, f(2))\) is a critical point
   B \((2, f(2))\) is a turning point
   C \((2, f(2))\) is a local minimum
   D \((2, f(2))\) is a local maximum

4. If \(f(x)\) is an odd function and \(f(a) = 5\), then
   A \(f(-a) = 5\)
   B \(f(-5) = a\)
   C \(f(-a) = -5\)
   D \(f(-a) = -a\)

5. For the function \(f(x) = \frac{-3}{(x-2)^2}\), which statement is not true?
   A The graph has no \(x\)-intercepts.
   B The graph is concave down for all \(x\) for which \(f(x)\) is defined.
   C \(f'(x) > 0\) when \(x < 2\) and \(f'(x) < 0\) when \(x > 2\)
   D \(\lim_{x \to 2^-} f(x) = -\infty\)

6. Copy and complete this statement.
   Given \(f(x) = x(x-1)^2\), the graph of \(f(x)\) has ____ critical points and ____ turning points.

7. The graph of \(f'(x)\) is shown. Identify the features on the graph of \(f(x)\) at each of points A, B, and C. Be as specific as possible.
   ![Graph of f'(x) with points A, B, and C]

8. Find the critical numbers for \(f(x) = x^3 - 5x^2 + 6x + 2\) on the interval \(0 \leq x \leq 4\).

9. Copy the graph of the function \(f(x)\) into your notebook. Sketch the first and second derivatives on the same set of axes.
   ![Graph of f(x) with points and axes]
10. Consider the function defined by \( f(x) = 3x^4 - 16x^3 + 18x^2 \).
   a) How will the function behave as \( x \to \pm \infty \)?
   b) Find the critical points and classify them using the second derivative test.
   c) Find the locations of the points of inflection.

11. The cost, in thousands of dollars, to produce \( x \) all-terrain vehicles (ATVs) per day is given by the function \( C(x) = 0.1x^2 + 1.2x + 3.6 \).
   a) Find a function \( U(x) \) that represents the cost per unit to produce the ATVs.
   b) How many ATVs should be produced per day in order to minimize the cost per unit?

12. Consider the function defined by \( y = x^2 + \frac{1}{x^2} \).
   a) Identify the vertical asymptote.
   b) Find and classify the critical points.
   c) Identify the intervals of concavity.
   d) Sketch the graph.

13. The graph shows the derivative, \( f'(x) \), of a function \( f(x) \).
   ![Graph of f'(x) with y = f'(x) and x-axis labeled from -2 to 4]
   a) Which is greater?
      i) \( f'(0) \) or \( f'(1) \)
      ii) \( f(-1) \) or \( f(3) \)
      iii) \( f(5) \) or \( f(10) \)
   b) Sketch a possible graph of \( f(x) \).

14. The function defined by \( g(x) = \frac{1}{(x - a)^2} \) has a vertical asymptote defined by \( x = a \). Without graphing, explain how you know how the graph will behave in the vicinity of \( x = a \).

15. A hotel chain typically charges $120 per room and rents an average of 40 rooms per night at this rate. They have found that for each $10 reduction in price, they rent an average of 10 more rooms.
   a) Find the rate they should be charging to maximize revenue.
   b) How does this change if the hotel only has 50 rooms?

16. a) For a given perimeter, what shape of rectangle encloses the most area?
   b) For a given perimeter, what type of triangle encloses the most area?
   c) Which shape would enclose more area: a pentagon or an octagon? Explain your reasoning.
   d) What two-dimensional shape would enclose the maximum area for a given perimeter?

17. In a certain region, the number of bushels of corn per acre, \( B \), is given by the function \( B(n) = -0.1n^2 + 10n \), where \( n \) represents the number of seeds, in thousands, planted per acre.
   a) What number of seeds per acre yields the maximum number of bushels of corn?
   b) If corn sells for $3/bushel and costs $2 for 1000 seeds, find the optimal number of seeds that should be planted per acre.

18. An isosceles triangle is to have a perimeter of 64 cm. Determine the side lengths of the triangle if the area is to be a maximum.
Chapter 1

1. The height of a Frisbee™, \( h \) m, above the ground \( t \) s after it is tossed into the air is modelled by \( h(t) = -0.003t^2 + 0.012t + 1. \)

   a) Determine the average rate of change of the height of the Frisbee™ between 10 s and 30 s. What does this value represent?

   b) Estimate the instantaneous rate of change of the height of the Frisbee™ at 20 s. Interpret the meaning of this value.

   c) Sketch a graph of the function and the tangent at \( t = 20 \).

2. The general term of an infinite sequence is \( t_n = \frac{3n}{n^2 + 1} \), where \( n \in \mathbb{N} \).

   a) Write the first five terms of the sequence.

   b) Does the sequence have a limit as \( n \to \infty \)? Justify your response.

3. What is true about the graph of \( y = f(x) \) if \( \lim_{x \to 5} f(x) = \lim_{x \to 5} f(x) = 2 \) and \( f(5) = 1 \)?

4. Evaluate each limit, if it exists.
   
   a) \( \lim_{x \to 3} (-2x^3 + x^2 - 4x + 7) \)
   
   b) \( \lim_{x \to 4} \frac{x^2 - 16}{x - 4} \)
   
   c) \( \lim_{x \to 0} \frac{3x - 5}{7 + 4x^3} \)

5. Consider this graph of a function.

   a) State the domain of the function.

   b) Evaluate \( \lim_{x \to 3^+} f(x) \) for this function.

   c) Is the graph continuous? If not, where is it discontinuous? Justify your answer.

Chapter 2

6. Sketch the derivative of each function.

   a) \( y = f(x) \)

   b)

7. Use the first principles definition to determine \( \frac{dy}{dx} \) for each function.

   a) \( y = \frac{2}{3}x^3 - 8x \)

   b) \( y = \sqrt{2x - 1} \)

   c) \( f(x) = 3x^2 - x + 1 \)

   d) \( h(t) = \frac{-2t}{t + 3} \)

Chapter 3

8. Differentiate each function.

   a) \( y = 6x^4 - 5x^3 + 3x \)

   b) \( y = (3 + t)(4t - 7) \)

   c) \( g(x) = (-2x^3 + 3)^4 \)

   d) \( s(t) = \frac{t}{\sqrt{5t - 1}} \)

9. a) Determine the equation of the tangent to the curve \( y = \frac{-4x}{(x + 3)^2} \) at \( x = -2 \).

   b) Determine the equation of the normal to the curve at \( x = -1 \).

10. a) Determine the coordinates of the points on \( f(x) = x^3 + 2x^2 \) where the tangent lines are perpendicular to \( y - x + 2 = 0 \).

    b) Determine the equations of the tangents to the graph of \( f(x) \) at the points from part a).

    c) Sketch the graph and the tangent lines.
11. Determine \( \frac{dy}{dx} \) at \( x = -2 \) if \( y = 5u - 2u^2 + 3u \) and \( u = \sqrt{2 - x} \). Use Leibniz notation.

12. a) Which graph represents each function? Justify your choices.
   i) \( y = s(t) \)
   ii) \( y = v(t) \)
   iii) \( y = a(t) \)

b) Determine the intervals over which the object was speeding up and slowing down.

c) How do your answers to part b) relate to the slope of the position function?

13. The voltage across a resistor is \( V = IR \), where \( I = 3.72 - 0.02t \) is the current through the resistor in amperes, \( R = 13.00 + 0.21t \) is the resistance in ohms, and \( t \) is time in seconds.

a) Write an equation for \( V \) in terms of \( t \).

b) Determine \( V'(t) \) and interpret its meaning for this situation.

c) Determine each rate of change after 3 s.
   i) voltage
   ii) current
   iii) resistance

14. The cost per day, \( C \), in dollars, of producing \( x \) gadgets is \( C(x) = -0.001x^2 + 1.5x + 500 \), \( 0 \leq x \leq 750 \). The demand function is \( p(x) = 4.5 - 0.1x \).

a) Determine the marginal cost at a production level of 300 gadgets.

b) Determine the actual cost of producing the 301st gadget.

c) Determine the marginal revenue and marginal profit for the sale of 300 gadgets.

15. The graph shows the derivative of a function.

a) Which is greater?
   i) \( f'(-2) \) or \( f'(2) \)
   ii) \( f(6) \) or \( f(12) \)

b) Sketch a possible graph of \( f(x) \).

16. Determine and classify all critical points of each function on the interval \(-4 < x < 4\).

a) \( f(x) = x^4 - 4x^3 \)

b) \( y = 2x^3 - 3x^2 - 11x + 6 \)

c) \( h(x) = x^4 - 5x^3 + x^2 + 21x - 18 \)

17. Analyse then sketch each function.

a) \( f(x) = x^3 - 9x^2 + 15x + 4 \)

b) \( f(x) = x^3 - 12x^2 + 36x + 5 \)

c) \( f(x) = \frac{2x}{x^2 - 5x + 4} \)

d) \( f(x) = 3x^3 + 7x^2 + 3x - 1 \)

18. A cylindrical can is to have a volume of 900 cm\(^3\). The metal costs $15.50 per m\(^2\). What dimensions produce a can with minimized cost? What is the cost of making the can?

19. A farmer has 6000 m of fencing and wishes to create a rectangular field subdivided into 4 congruent plots of land. Determine the dimensions of each plot of land if the area to be enclosed is a maximum.
**TASK**

**An Intense Source of Light**

Photographers often use multiple light sources to control shadows or to illuminate their subject in an artistic manner. The illumination from a light source is inversely proportional to the square of the intensity of the source. Two light sources, L1 and L2, are 10 m apart, and have intensity of 4 units and 1 unit, respectively. The illumination at any point P is the sum of the illuminations from the two sources.

Hint: When a quantity \( y \) is inversely proportional to the square of another quantity \( x \), then \( y = \frac{k}{x^2} \), where \( k \) is a constant.

\[
\text{L1} \quad \text{L2} \quad \text{P}
\]

10 m

a) Determine a function that represents the illumination at the point P relative to the distance from L1.

b) At what distance from L1 is the illumination on P the greatest?

c) At what distance from L1 is the illumination on P the least?

d) Investigate whether it would be more effective to increase the intensity of L2 or to move L2 closer to L1 in order to increase the illumination at the point found in part b). Support your findings with mathematical evidence. Consider both the illumination and the rate of change of the illumination as L2 changes intensity or position.