The Capelli eigenvalue problem for Lie superalgebras

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- Base field: $\mathbb{C}$.
- $J = J_{\bar{0}} \oplus J_{\bar{1}}$: Jordan superalgebra:
  - $a \circ b := (-1)^{|a||b|} b \circ a$.
  - $(-1)^{|a||c|}[L_{a \circ b}, L_c] + (-1)^{|b||a|}[L_{b \circ c}, L_a] + (-1)^{|c||b|}[L_{c \circ a}, L_b] = 0.$

$$L_x : J \to J \ , \ L_x(y) := xy.$$

TKK Lie superalgebra (Kantor's functor)

$$\mathfrak{g}_J := \mathfrak{g}_J(-1) \oplus \mathfrak{g}_J(0) \oplus \mathfrak{g}_J(1)$$

- $\mathfrak{g}_J(-1) := J$.
- $\mathfrak{g}_J(0) := \langle L_a, [L_a, L_b] \ : \ a, b \in J \rangle \subseteq \mathrm{End}_{\mathbb{C}}(J)$.
- $\mathfrak{g}_J(1) := \langle P, [L_a, P] \ : \ a \in J \rangle \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}^2(J), J)$, where

$$P : \mathcal{S}^2(J) \to J \ , \ P(x, y) := x \circ y.$$

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The TKK Construction

Kantor, Kac, Cantarini-Kac

Assume that J is unital. Then $\mathfrak{g}_J$ is simple if and only if J is simple.

Remark

It is better to work with $\mathfrak{gl}(m|n)$ rather than $\mathfrak{psl}(m|n)$, and with $\mathfrak{q}(n)$ rather than $\mathfrak{sq}(n)$.

Classification of unital simple Lie superalgebras

J	$\mathfrak{g}^\flat$
$\mathrm{gl}(m,n)_+$	$\mathfrak{gl}(2m\|2n)$
$\mathrm{osp}(n,2m)_+$	$\mathfrak{osp}(4n\|2m)$
$\mathrm{p}(n)_+$	$\mathfrak{p}(2n)$
$\mathrm{q}(n)_+$	$\mathfrak{q}(2n)$
$(m,2n)_+$	$\mathfrak{osp}(m+3\|2n)$
$\mathrm{D}_t,\ t \neq -1$	$D(2\|1,t)$
F	$F(3\|1)$
$\mathrm{JP}(0,n)$	$H(n+3)$

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The Quadruple $(\mathfrak{g}^\flat, \mathfrak{g}, \mathfrak{k}, V)$

- Short subalgebra: $e := 1_J \in \mathfrak{g}^\flat(-1)$, $h := -L_{1_J} \in \mathfrak{g}^\flat$, $f := P \in \mathfrak{g}^\flat(1)$.

$$[h, e] = -e, \quad , \quad [e, f] = h \quad , \quad [h, f] = f.$$

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$$\mathfrak{k} = \{x \in \mathfrak{g} \ : \ w.x = x\} \text{ where } W := \{1, w\} \text{ is the Weyl group of } \mathfrak{s}.$$

- $(\mathfrak{g}, \mathfrak{k})$ is a supersymmetric pair.
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$\mathrm{osp}(n,2m)_+$	$\mathfrak{osp}(4n\|2m)$	$\mathfrak{gl}(m\|2n)$	$\mathfrak{osp}(m\|2n)$	$\mathcal{S}^2(\mathbb{C}^{m\|2n})^*$
$\mathrm{p}(n)_+$	$\mathfrak{p}(2n)$	$\mathfrak{gl}(n\|n)$	$\mathfrak{p}(n)$	$\Pi(\Lambda^2\mathbb{C}^{n\|n})^*$
$\mathrm{q}(n)_+$	$\mathfrak{q}(2n)$	$\mathfrak{q}(n) \oplus \mathfrak{q}(n)$	$\mathfrak{q}(n)$	$((\mathbb{C}^{n\|n})^* \otimes \mathbb{C}^{n\|n})^{\Pi\otimes\Pi}$
$(m,2n)_+$	$\mathfrak{osp}(m+3\|2n)$	$\mathfrak{osp}(m+1\|2n) \oplus \mathbb{C}$	$\mathfrak{osp}(m\|2n)$	$\mathbb{C}^{m+1,2n}$
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F	$F(3\|1)$	$\mathfrak{osp}(2\|4) \oplus \mathbb{C}$	$\mathfrak{osp}(1\|2) \oplus \mathfrak{osp}(1\|2)$	$\mathbb{C}^{6\|4}$
$\mathrm{JP}(0,n)$	$H(n+3)$	$H(n+1) \oplus \mathbb{C}$	$H(n)$	$\mathbb{C}^{2^n\|2^n}$

Natural Questions

Is $\mathscr{P}(V)$ a completely reducible $\mathfrak{g}$-module? Is it also multiplicity-free? Are the $\mathfrak{g}$-modules that arise $\mathfrak{k}$-spherical?

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Partitions

- $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_{\ell(\boldsymbol{\lambda})})$ such that $\boldsymbol{\lambda}_i \in \mathbb{Z}^+$, $\boldsymbol{\lambda}_i \geq \boldsymbol{\lambda}_{i+1}$ for all i.
- $|\boldsymbol{\lambda}| = \sum_i \boldsymbol{\lambda}_i$
- $\ell(\boldsymbol{\lambda})$: Number of parts of $\boldsymbol{\lambda}$.
- $\mathcal{H}_{m,n,d} := \{\boldsymbol{\lambda} \ : \ |\boldsymbol{\lambda}| = d$ and $\boldsymbol{\lambda}_{m+1} \leq n\}$.
- $\mathcal{DP}_{n,d} := \{\boldsymbol{\lambda} \ : \ |\ \ell(\boldsymbol{\lambda}) \leq n, \ \boldsymbol{\lambda}| = d$ and $\boldsymbol{\lambda}_i > \boldsymbol{\lambda}_{i+1}$ for every $i\}$.

$$\mathsf{D} : \mathcal{DP}_{n,d} \to \mathcal{H}_{n,n,2d} \ , \ \boldsymbol{\lambda} \mapsto \mathsf{D}(\boldsymbol{\lambda})$$

$(4, 2, 1) \quad \mapsto$

4	4	4	4
4	2	2	
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Partitions and Representations

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Irreducible representations from partitions

- $\mathfrak{g} = \mathfrak{gl}(m|n)$.
  Fundamental roots:

$$\{\varepsilon_i - \varepsilon_{i+1} \,:\, 1 \leq i \leq m-1\} \cup \{\delta_i - \delta_{i+1} \,:\, 1 \leq i \leq n-1\}.$$

  $\boldsymbol{\mu} \in \mathcal{H}_{m,n,d} \rightsquigarrow V_{\boldsymbol{\mu}}$ with h.w.

$$\mu := \boldsymbol{\mu}_1\varepsilon_1 + \cdots + \boldsymbol{\mu}_m\varepsilon_m + \boldsymbol{\mu}_{m+1}\delta_1 + \cdots + \boldsymbol{\mu}_{m+n}\delta_n.$$

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Theorem (Sahi, S., Serganova + Cheng–Wang '00', 01, ...)

The $\mathfrak{g}$-module $\mathscr{P}(V)$ is completely reducible in the following cases. The irreducible components of $\mathscr{P}(V)$ are parametrized as shown.

J	$\mathfrak{g}$	$\mathscr{P}^d(V)$	
$\mathrm{gl}(m,n)_+$	$\mathfrak{gl}(m\vert n)\oplus\mathfrak{gl}(m\vert n)$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{H}_{m,n,d}} V_{\boldsymbol{\lambda}}\otimes V_{\boldsymbol{\lambda}}^*$	
$\mathrm{osp}(n,2m)_+$	$\mathfrak{gl}(m\vert 2n)$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{H}_{m,n,d}} V_{2\boldsymbol{\lambda}}$	
$\mathrm{p}(n)_+$	$\mathfrak{gl}(n\vert n)$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{DP}_{n,d}} V_{\mathrm{D}(\boldsymbol{\lambda})}$	New
$\mathrm{q}(n)_+$	$\mathfrak{q}(n)\oplus\mathfrak{q}(n)$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{DP}_{n,d}} \frac{1}{2^{\delta(\ell(\boldsymbol{\lambda}))}}U_{\boldsymbol{\lambda}}\otimes U_{\boldsymbol{\lambda}}^*$	
$(m,2n)_+$, $\frac{m}{2}\in\mathbb{N}$ or $\frac{m}{2}\geq n$	$\mathfrak{osp}(m+1\vert 2n)\oplus\mathbb{C}$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{H}_{2,0,d}} V_{\boldsymbol{\lambda}}$	New
D_t, $\frac{t}{t+1}\notin\mathbb{Q}\cap[0,1]$	$\mathfrak{gl}(1\vert 2)$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{H}_{1,1,d}} V_{\boldsymbol{\lambda}}$	New
F	$\mathfrak{osp}(2\vert 4)\oplus\mathbb{C}$	$\bigoplus_{\boldsymbol{\lambda}\in\mathcal{H}_{2,1,d}} V_{\boldsymbol{\lambda}}$	New

In addition, the irreducible components of $\mathscr{P}(V)$ are $\mathfrak{k}$-spherical.

- For $(m,2n)_+$, the highest weight corresponding to $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\in\mathcal{H}_{2,0,d}$ is: $(\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2)\varepsilon_1+(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)\zeta$.
- For D_t, the highest weight corresponding to $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1,\underbrace{1,\dots,1}_{(d-\boldsymbol{\lambda}_1)})\in\mathcal{H}_{1,1,d}$ is:

  $(2d\frac{t}{1+t}-4d+2\boldsymbol{\lambda}_1)\varepsilon_1+(-d\frac{t}{1+t}+3d-\boldsymbol{\lambda}_1)(\delta_1+\delta_2)$.
- For F, the highest weight corresponding to $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1,\underbrace{2,\dots,2}_{r},\underbrace{1,\dots,1}_{d-\boldsymbol{\lambda}_1-2r})\in\mathcal{H}_{1,2,d}$ is

  $(d+2\boldsymbol{\lambda}_1-4)\varepsilon_1+(d-2r-\boldsymbol{\lambda}_1)(\delta_1+\delta_2)+d\zeta$.

  The fundamental system is $\{-\varepsilon_1-\delta_1,\delta_1-\delta_2,2\delta_2\}$

The Capelli Basis

We remark that $\mathscr{P}(V)$ is a multiplicity-free $\mathfrak{g}$-module.

$$\mathscr{P}(V) \cong \bigoplus_{\boldsymbol{\lambda}\in\mathcal{E}_V} V_{\boldsymbol{\lambda}} \Rightarrow \mathscr{D}(V) \cong \mathcal{S}(V) \cong \bigoplus_{\boldsymbol{\lambda}\in\mathcal{E}_V} V_{\boldsymbol{\lambda}}^*.$$

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Interpolation Jack Polynomials

Theorem (Sahi '94, Okounkov–Olshanski '97)

Assume that $\theta \notin \mathbb{Q}^{\leq 0}$. Fix an integer $N \geq 1$. Then for every partition $\boldsymbol{\lambda}$ such that $\ell(\boldsymbol{\lambda}) \leq N$, there exists a unique (up to scaling) polynomial $P^*_{\boldsymbol{\lambda}}(z_1, \ldots, z_N; \theta)$ such that:

- $\deg(P^*_{\boldsymbol{\lambda}}) \leq |\boldsymbol{\lambda}|$.
- $P^*_{\boldsymbol{\lambda}}(z_1, \ldots, z_N; \theta)$ is symmetric in $z_i + \theta(1 - i)$.
- $P^*_{\boldsymbol{\lambda}}(\boldsymbol{\mu}, \theta) = 0$ for all $\boldsymbol{\mu}$ satisfying $\ell(\boldsymbol{\mu}) \leq N$ and $|\boldsymbol{\mu}| \leq |\boldsymbol{\lambda}|$ and $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$.
- $P^*_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}; \theta) \neq 0$.

The $P^*_{\boldsymbol{\lambda}}$'s form a basis for the algebra $\Lambda_{N,\theta}$ of shifted symmetric polynomials in N-variables.

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Sergeev–Veselov Polynomials

Remark

The polynomials $P^*_{\boldsymbol{\lambda}}(z;\theta)$ are compatible with the inverse system

$$\cdots \to \Lambda_{N+1,\theta} \to \Lambda_{N,\theta} \to \cdots \to \Lambda_{0,\theta}.$$

Therefore we can consider $P^*_{\boldsymbol{\lambda}}(z;\theta)$ as elements of $\Lambda_\theta := \varprojlim_N \Lambda_{N,\theta}$ in infinitely many variables $z_1, z_2, z_3, \ldots$.

The Algebra $\Lambda^\natural_{m,n,\theta}$ (Sergeev–Veselov, 2005)

- $\Lambda^\natural_{m,n,\theta}$: algebra of polynomials $f(x;y)$ in $m+n$ variables $x_1,\ldots,x_m,y_1,\ldots y_n$ such that
  - f is separately symmetric in the x_i's and the y_j's.
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- Bernoulli sums:
  $b^\natural_k(z;\theta) := \sum_{i\geq 1} B_k\left(z_i + \frac{1}{2} + \theta(\frac{1}{2}-i)\right) - B_k\left(\frac{1}{2} + \theta(\frac{1}{2}-i)\right).$
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Factorial Schur Q-functions

Recall that:

$$\mathcal{DP}_{n,d} := \{\boldsymbol{\lambda} \ : \ | \ \ell(\boldsymbol{\lambda}) \leq n, \ \boldsymbol{\lambda}| = d \text{ and } \boldsymbol{\lambda}_i > \boldsymbol{\lambda}_{i+1} \text{ for every } i\}.$$

Let Γ_N be the algebra of N-variable symmetric polynomials $f(z_1, \ldots, z_N)$ such that $f(t, -t, z_3, \ldots, z_N)$ is independent of t.

Theorem (Ivanov '01)

For every $\boldsymbol{\lambda} \in \mathcal{DP}_N := \bigcup_d \mathcal{DP}_{N,d}$, there exists a unique (up to scalar) polynomial $Q^*_{\lambda} \in \Gamma_N$ such that

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The Capelli Eigenvalue Problem

Let $\mathfrak{g}$, V be as before. Also, let $\mathcal{E}_V$ be the set of partitions that parametrize the irreducible summands of $\mathscr{P}(V)$.

$$\boldsymbol{\lambda} \in \mathcal{E}_V \leftrightsquigarrow \lambda \in \widehat{\mathcal{E}}_V \subseteq \mathfrak{h}^*$$

Set $\mathfrak{a}^* :=$ Zariski closure of $\widehat{\mathcal{E}}_V$ in $\mathfrak{h}^*$.

- $\mathfrak{a} := \ker(\mathfrak{a}^{*\perp})$ where "$\perp$" is w.r.t. Killing form. (Dual to $\mathfrak{a}^*$.)
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Example

$J := \mathrm{osp}(n|2m)_+$, $\mathfrak{g} = \mathfrak{gl}(m|2n)$, $V = \mathcal{S}^2(\mathbb{C}^{m|2n})^*$.

- $\widehat{\mathcal{E}}_V = \{\sum_{i=1}^m 2\boldsymbol{\lambda}_i\varepsilon_i + \sum_{i=1}^n \langle \boldsymbol{\lambda}_i' - m\rangle(\delta_{2i-1} + \delta_{2i})\}$.
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The Capelli Eigenvalue Problem

Let $\mathfrak{g}$, V be as before. Also, let $\mathcal{E}_V$ be the set of partitions that parametrize the irreducible summands of $\mathscr{P}(V)$.

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The Capelli Eigenvalue Problem

Theorem (Sahi, S. Serganova)

Let $\mathfrak{g}$ and V be as before, and let $\mathcal{E}_V$ be the set of partitions that parametrize the irreducible components $V_{\boldsymbol{\lambda}} \subset \mathscr{P}(V)$ and also the basis $\{D_{\boldsymbol{\mu}}\}$ of $\mathscr{P}\mathscr{D}(V)^{\mathfrak{g}}$.

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{E}_V$, the action of $D_{\boldsymbol{\mu}}$ on $V_{\boldsymbol{\lambda}}$ is by the scalar $F_{\boldsymbol{\mu}} \circ \eta(\lambda\big|_{\mathfrak{a}})$, where

- $F_{\boldsymbol{\mu}}(\cdot) = SP^*_{\boldsymbol{\mu}}(\cdot, \theta)$ or $F_{\boldsymbol{\mu}}(\cdot) = Q^*_{\boldsymbol{\mu}}(\cdot)$, according to the table below.
- $\eta : \mathfrak{a}^* \to \mathbb{C}^N$ is an affine linear transformation, where $N := N_{\mathfrak{g}}$.
- $\lambda \in \widehat{\mathcal{E}}_V$ corresponds to $\boldsymbol{\lambda}$.

J	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{gl}_{m\|n}, \varepsilon\varepsilon, \varepsilon\delta, \delta\delta$	θ
$\mathfrak{gl}(m,n)_+$	$\mathfrak{gl}(m\|n) \oplus \mathfrak{gl}(m\|n)$	$\mathfrak{gl}(m\|n)$	$m\|n,\ 2, 2, 2$	1
$\mathfrak{osp}(n,2m)_+$	$\mathfrak{gl}(m\|2n)$	$\mathfrak{osp}(m\|2n)$	$m\|n,\ 1, 2, 4$	$\frac{1}{2}$
$\mathrm{D}_t,\ t \neq -1$	$\mathfrak{gl}(1\|2)$	$\mathfrak{osp}(1\|2)$	$1\|1,\ -, 2, -$	$-\frac{1}{t}$
F	$\mathfrak{osp}(2\|4) \oplus \mathbb{C}$	$\mathfrak{osp}(1\|2) \oplus \mathfrak{osp}(1\|2)$	$1\|2,\ -, 2, 3$	$\frac{2}{3}$
J	$\mathfrak{g}$	$\mathfrak{k}$	Q_n	θ
$\mathfrak{p}(n)_+$	$\mathfrak{gl}(n\|n)$	$\mathfrak{p}(n)$	$n,\ 2\|2$	
$\mathfrak{q}(n)_+$	$\mathfrak{q}(n) \oplus \mathfrak{q}(n)$	$\mathfrak{q}(n)$	$n,\ 2\|2$	
$(m,2n)_+$	$\mathfrak{osp}(m+1\|2n) \oplus \mathbb{C}$	$\mathfrak{osp}(m\|2n)$	$2,\ m-1\|2n$	$\frac{m-1}{2} - n$

Furthermore, identifying $\mathfrak{a}^*$ with $\mathfrak{a}.v_\circ \subset V$ where $v_\circ \in V$ is a spherical vector, we have

$$\text{top homog.}(F_{\boldsymbol{\mu}} \circ \eta) = p_{\boldsymbol{\mu}}\big|_{\mathfrak{a}},$$

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