On Isometries of the Euclidean Plane

1 Isometries: the unsung heroes

Among the known mathematical objects, there are those of such immeasurable importance that we scarcely notice their existence. That a rigid object, for example, can be moved through space without changing its size and shape is so fundamental a property of our world that one rarely pauses to take note of it. (Those who commit this transgression are in good company: in the opening pages of his *Elements* [3, p. 4], Euclid assumes the fact without proof in demonstrating side-angle-side congruence of triangles. Fortunately this and other defects in Euclid's proofs have since been remedied [6, pp. 181-182][1].) This property deserves our attention, however, since it informs so many pursuits in mathematics, sciences, and engineering. The property is described mathematically by the isometries: precisely, an *isometry* of Euclidean space is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with the property that

$$d(f(p), f(q)) = d(p, q) \quad \forall p, q \in \mathbb{R}^n,$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance. These functions are ubiquitous in engineering disciplines such as robotics and computer vision, among many others. In physics, equations describing motion must be invariant under isometries. A function that is not an isometry cannot be applied to a rigid object as a motion in space (indeed, 'rigid motion' is a commonly used synomym for isometry).

One may ask, then, under what conditions a function f is an isometry. There are conditions one might expect to guarantee isometry but which do not. For example, a mapping in the plane that preserves area is not in general an isometry. Using elementary calculus or linear algebra one can show that the shear transformation $f(x, y) = (x + y, y)$ preserves area (Figure 1); that it does not preserve distance may be seen by considering the points $O = (0, 0), A = (0, 1)$: the distance $d(O, A) = 1$ while $d(f(O), f(A)) = \sqrt{2}$.

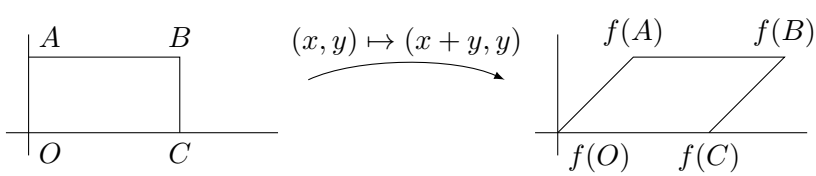

Figure 1:

We shall define some terminology and give a theorem that may be surprising. Let $s \in \mathbb{R}^+$. We say that f *preserves* s if any points $p, q \in \mathbb{R}^2$ such that $d(p, q) = s$ also satisfy $d(f(p), f(q)) = s$. We say that f *preserves* a given polygon if it maps its vertices to the vertices of a congruent polygon.

[1] This elucidating history of the human understanding of mathematics, its rigor, and its relation to the physical world may surprise readers who view mathematics as a body of unassailable truth.

Theorem 1 (Beckman-Quarles Theorem for $\mathbb{R}^2$) *If a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves some distance $s \in \mathbb{R}^+$, then it is an isometry.*

This fact is somewhat remarkable: the preservation of just one distance is a much weaker condition than isometry, but the two turn out to be equivalent. The result was first proven in 1953 by F.S. Beckman and D.A. Quarles, Jr., for IBM [1]. Ulrich Everling [4] proved an analog for the n-dimensional sphere in 1995, and similar theorems have been proven for other geometric spaces (see for example Benz [2] and references therein).

On the diamond anniversary of Beckman and Quarles's publication, we revisit the two-dimensional Euclidean case and adapt parts of Everling's proof to the plane. This new treatment simplifies the original proof and emphasizes the parallels between the planar and spherical cases. The idea of the proof is quite natural: starting with s, try to find more and more distances that are preserved under f. We find one other distance, then all integers, then arbitrarily small distances, and eventually we can extend the argument to all distances. The ingenuity of the proof lies in the sophisticated geometric constructions used to establish preservation of the distances.

2 The rhombus and the pentagon

Note that we may assume by a change of scale that $s = 1$. To begin the proof, first consider an equilateral triangle with vertices A, B, C and with unit side length; the image triangle $\triangle f(A)f(B)f(C)$ must also be a unit-side triangle. Now consider a rhombus $ABDC$ with unit side length, where A, B, C form a unit-side triangle and so do B, C, D (Figure 2, center). We would like to say that this unit-side rhombus maps to a congruent rhombus (Figure 2, left), but it may also map to a unit-side triangle with $f(A) = f(D)$ (Figure 2, right). These are the only two possibilities, since $\triangle ABC$ and $\triangle BCD$ are preserved.

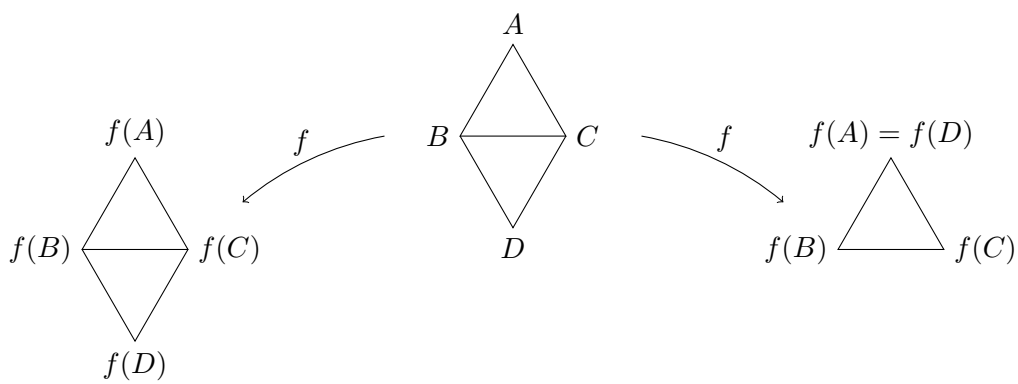

Figure 2: Can we show that the image shown at right cannot occur?

To establish that the rhombus is preserved it suffices to show that $d(f(A), f(D)) = d(A, D) = \sqrt{3}$. We use an argument presented by Everling [4] for the sphere but which also holds elegantly in the plane.

Lemma 1 (Pentagon Lemma) *The distance $\sqrt{3}$ is preserved under f.*

Proof. Consider the rhombus $ABDC$ as described above, together with points E, F, G such that $AEGF$ is a rhombus congruent to $ABDC$ and $d(D, G) = 1$ (Figure 3). The goal is to show that

$d(A, D) = \sqrt{3}$ is preserved. Note that $d(f(D), f(G)) = 1$. Since $d(A, D) = d(A, G) = \sqrt{3}$, we have from the above discussion that $d(f(A), f(D))$ and $d(f(A), f(G))$ are both in $\{0, \sqrt{3}\}$. Suppose that $d(f(A), f(D)) = 0$. Then $f(A) = f(D)$ and $d(f(A), f(G)) = d(f(D), f(G)) = 1 \notin \{0, \sqrt{3}\}$, a contradiction. Therefore $d(f(A), f(D)) = \sqrt{3}$. $\square$

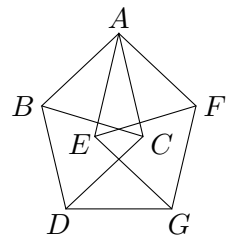

Figure 3: Two rhombi overlap to form a pentagon.

3 *Ex uno plures*: preservation of integer distances

We now arrive at a moment of ingenuity in this proof, and the reason for constructing the rhombus: for any integer $k \in \mathbb{Z}^+$ we can use unit-side rhombi to construct a shape containing a line segment of length k. We will show that this shape is preserved under f and therefore that any integer line segment is preserved. Beckman and Quarles begin with $k = 2$:

> That the distance 2 is preserved is seen by considering the regular hexagon with unit side. The consecutive vertices of such a figure must be transformed consecutively onto the vertices of a congruent figure. This becomes apparent when one considers the component overlapping rhombi. [1, p. 811]

Figure 4 shows the hexagon in question. Each component rhombus is preserved under f, so the hexagon must also be preserved. For any line segment AB of length two, then, we can construct a unit-side hexagon with AB as a diameter, proving that the distance two is preserved under f.

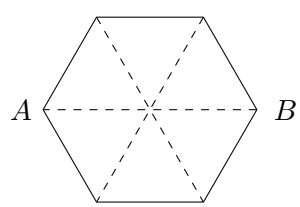

Figure 4:

That the distance three is preserved is shown by adding three new overlapping rhombi to the hexagon (Figure 5, left). The overall shape is again preserved, and so the distance $d(A, B) = 3$ is preserved. Thus continuing to add rhombi, we find that any integer distance $k > 0$ is preserved under f (Figure 5, right).

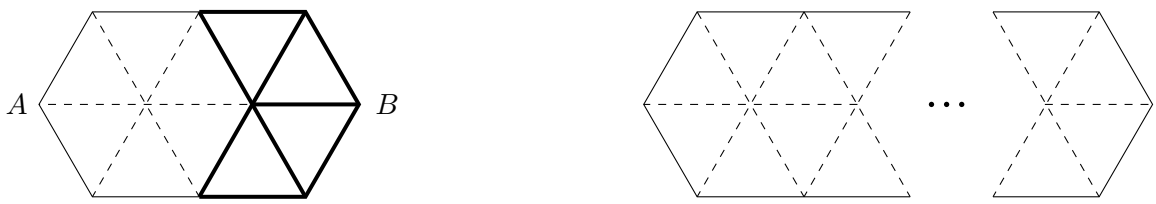

Figure 5:

4 The ferris wheel: preservation of small distances

In this section we establish that arbitrarily small preserved distances exist. We use the same method used for integer distances: build a structure of points that is preserved under f and show that it contains any desired (small) distance.

Let p, q_i be such that $d(p, q_i) = 2, d(q_i, q_{i+1}) = 1, q_{i+2} \neq q_i$ $(i = 0, 1, 2, \ldots)$ (Figure 6). To show that the structure is preserved it suffices to show that the distance $d(q_i, q_{i+2}) = \sqrt{15}/2$ is preserved, which is in essence a repetition of the pentagon lemma with a few changes of line segment lengths.

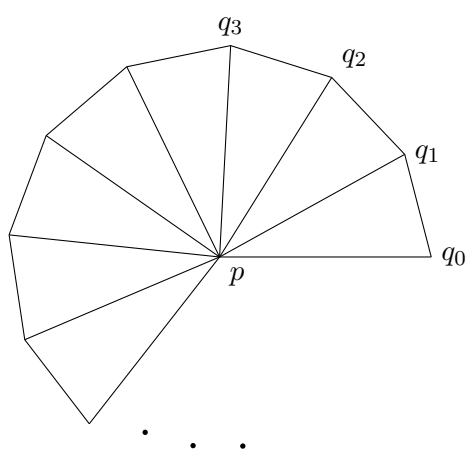

Figure 6: The "ferris wheel."

Lemma 2 (Pentagon Lemma Revisited) *The distance $\sqrt{15}/2$ is preserved by f.*

Proof. Consider the quadrilateral $ABDC$ with $d(B, A) = d(B, C) = d(B, D) = 2$, $d(A, C) = d(C, D) = 1$ (Figure 7, left). Our goal is to show that $d(A, D) = \sqrt{15}/2$ is preserved. Since integer distances are preserved, $\triangle CBA$ and $\triangle CBD$ are mapped to congruent triangles. Therefore either $f(A) = f(D)$ or the quadrilateral $ABDC$ is preserved, that is to say $d(f(A), f(D)) \in \{0, \sqrt{15}/2\}$. We will show that $d(f(A), f(D)) \neq 0$ and hence the distance $\sqrt{15}/2$ is preserved.

As in the proof of the pentagon lemma, we have established that $d(A, D)$ must map to zero or to our desired distance. Now mimicking that proof, we use another polygon congruent to $ABDC$ to create an isosceles triangle with one side length equal to one (a preserved distance) and the other two equal to the desired distance.

Consider the point G such that $d(G, A) = \sqrt{15}/2$, $d(G, D) = 1$, and $d(G, B) > d(D, B)$ (Figure 7, right). Generate a new polygon congruent to $ABDC$ by reflecting it in the altitude of $\triangle ADG$ passing

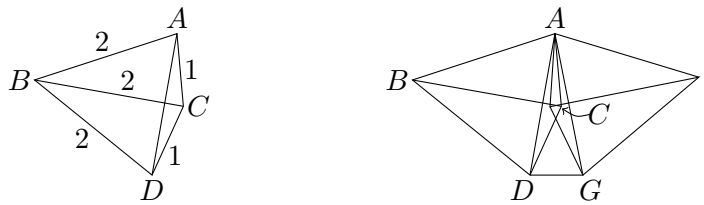

Figure 7: (Right figure not to scale.)

through A. By an argument similar to that above we have $d(f(A), f(G)) \in \{0, \sqrt{15}/2\}$. We also have $d(f(D), f(G)) = 1$.

Suppose that $d(f(A), f(D)) = 0$. Then $f(A) = f(D)$ and $d(f(A), f(G)) = d(f(D), f(G)) = 1 \notin \{0, \sqrt{15}/2\}$, a contradiction. Hence $d(f(A), f(D)) = \sqrt{15}/2$. $\square$

The ferris wheel can be used to generate arbitrarily small distances that are preserved under f. Let $\theta = \angle q_i p q_{i+1}$ and observe that $\theta = \arccos(7/8)$. If it can be established that $\theta \notin \pi\mathbb{Q}$, then there is no nonzero integer k for which the ferris wheel "comes back around" to its starting point, obtaining $q_k = q_0$; if there were such a k it would satisfy $k\theta = 2\pi$ and hence $\theta = 2\pi/k \in \pi\mathbb{Q}$. Furthermore every point q_i is distinct: by the same argument, for any q_i there is no nonzero $n \in \mathbb{N}$ such that $q_{i+n} = q_i$. Since there are infinitely many distinct q_i on the finite-circumference circle of the ferris wheel, we can find pairs of points that are arbitrarily close to one another[2]. That is, for any distance $0 < \varepsilon < 4$ there exist q_i and q_j such that $d(q_i, q_j) < \varepsilon$. All distances $d(q_i, q_j)$ are preserved by preservation of the ferris wheel, and hence arbitrarily small preserved distances can be found.

It remains to show that $\theta = \arccos(7/8) \notin \pi\mathbb{Q}$. Let $\phi = \frac{m}{n}\pi$ where $m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}$, and let $\cos\phi \in \mathbb{Q}$. Then

$$2\cos\phi = e^{i\phi} + e^{-i\phi} = e^{i\pi m/n} + e^{-i\pi m/n} \in \mathbb{Q}.$$

This last expression is the sum of two algebraic integers, which is itself an algebraic integer. (An algebraic integer is a zero of a polynomial with integer coefficients and a leading coefficient of one. Each of the above terms is a zero of $f(x) = x^{2n} - 1$. For the closure of the algebraic integers under addition, see for example [7, Section 10.3]). By the Rational Roots Theorem [7, Section 7.4] the only rational numbers that are also algebraic integers are integers. Therefore $2\cos\phi$ is an integer and $\cos\phi \in \{0, \pm\frac{1}{2}, \pm 1\}$. Since $\cos\theta = 7/8 \in \mathbb{Q} \setminus \{0, \pm\frac{1}{2}, \pm 1\}$, it must be that $\theta \notin \pi\mathbb{Q}$. This fact establishes that arbitrarily small preserved distances exist.

5 Getting to Q.E.D.: preservation of all real distances

The last intermediate result to be established is as follows. This argument was given by Everling for the sphere but applies without modification to the plane.

Lemma 3 (Accordion Lemma) *Let $s \in \mathbb{R}$ be a distance that is preserved under f. Then any distance less than s is also preserved under f.*

Proof. Let $0 < t < s$. There exist collinear points A, B, C such that $d(A, B) = t, d(A, C) = s$, and $d(B, C) = s - t$. By the result of the previous section, for small $\varepsilon' > 0$ there exists $0 < \varepsilon < \varepsilon'$ that is preserved under f.

[2]This argument is based on an observation likely first made by Leopold Kronecker.

We claim that for arbitrarily small preserved $\varepsilon > 0$ there exist $k, l \in \mathbb{N}$ and a sequence $p_0, p_1, \ldots, p_{k+l}$ such that

$$p_0 = A,\ p_k = B,\ p_{k+l} = C,\ d(p_i, p_{i+1}) = \varepsilon,\ k\varepsilon < t + 2\varepsilon, \text{ and } l\varepsilon < s - t + 2\varepsilon.$$

Clearly there exist $p_0, p_1, \ldots, p_{k+l}$ satisfying $p_0 = A$ and $d(p_i, p_{i+1}) = \varepsilon$. To satisfy $p_k = B$ is suffices that $k\varepsilon > t$, since any distance $d(p_0, p_k) < k\varepsilon$ can be obtained using k line segments of length ε (Figure 8 illustrates the case in which k is even). Similarly, to satisfy $p_{k+l} = C$ it suffices that $(k + l)\varepsilon > s$, for which it suffices that $k\varepsilon > t$ and $l\varepsilon > s - t$. Take k and l to be the integer floors of $t/\varepsilon + 1$ and $(s - t)/\varepsilon + 1$ respectively. Then $t < k\varepsilon < t + 2\varepsilon$ and $s - t < l\varepsilon < s - t + 2\varepsilon$ and the required conditions are satisfied.

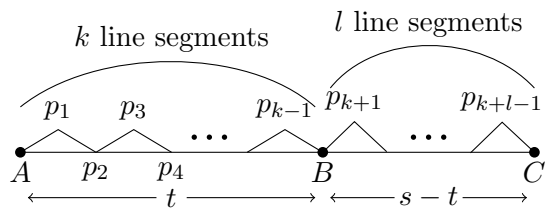

Figure 8: k small segments can be "squeezed" into an accordion shape to fit any line segment less than $k\varepsilon$.

Since each $d(p_i, p_{i+1})$ is preserved, we have $d(f(A), f(B)) \leq k\varepsilon < t + 2\varepsilon$. For the same reason $d(f(B), f(C)) \leq l\varepsilon < s - t + 2\varepsilon$, and by the triangle inequality $d(f(A), f(B)) \geq s - d(f(B), f(C)) > t - 2\varepsilon$. Hence $|d(f(A), f(B)) - t| < 2\varepsilon$, and since ε can be made arbitrarily small, $d(f(A), f(B)) = t$ and the distance t is preserved. $\square$

The final result now follows directly: all integer distances are preserved, and for any preserved distance s all distances less than s are preserved. Since every positive real number is less than some integer, all distances are preserved and f is an isometry.

The motivated reader can prove an analogous theorem for the sphere using many of the same arguments shown here, referring to Everling [4] for the details. One major difference between the two cases is that the sphere is of some fixed size, so we may not set s to unity by a change of scale. In fact the theorem statement itself is more limited, and for the unit sphere Everling claims only that a function preserving some distance $s \in (0, \pi/2)$ is an isometry. One should begin by proving the pentagon lemma for the sphere; it is nearly identical to that presented here, although the additional case where s is close to $\pi/2$ must be considered. Next establish preservation of arbitrarily small distances. The ferris wheel argument applies to one case of this proof, but there are two additional cases specific to the sphere. From there the accordion lemma can be used directly, followed by one final argument to 'grow' the preserved distances beyond s to any desired value.

References

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