Spherical Harmonics and the Capelli Eigenvalue problem for $\mathfrak{osp}(1|2n)$

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Abstract

In this thesis, we define a dual action of $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$ on the space of superpolynomials $\mathcal{P}(\mathbb{C}^{1|2n})$ and thereby study the spherical harmonics for $\mathfrak{osp}(1|2n)$. The harmonic polynomials are then used to give a decomposition of $\mathcal{P}(\mathbb{C}^{1|2n})$ into irreducible $\mathfrak{osp}(1|2n)$-modules. An action of $\mathfrak{gosp}(1|2n)$ consistent with the action of $\mathfrak{osp}(1|2n)$ on $\mathcal{P}(\mathbb{C}^{1|2n})$ decomposes $\mathcal{P}(\mathbb{C}^{1|2n})$ into a multiplicity-free decomposition and therefore defines Capelli operators. Lastly, we relate the surjectivity of the map $\mathcal{Z}(\mathfrak{g}) \to \mathcal{PD}(V)^{\mathfrak{g}}$ to the non-vanishing of certain determinants. These determinants are then given as polynomials in n along with a complete factorization with roots and their multiplicities.

The new results are Theorem 4.3.3 where we give explicit formulas for the joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vectors and Theorem 5.2.10 where we give the complete factorization of the aforementioned determinants.

Dedications

For camp, the people who can do anything, be anything, and achieve everything.

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Chapter 1

Introduction

The Capelli identity, discovered in 1887 by Alfredo Capelli [Cap87], is a mysterious-looking identity which highly influenced classical invariant theory. In particular, this identity gives equality between a differential operator, called the *Capelli element*, and a non-commutative determinant of polarization operators. In Hermann Weyl's book 'The Classical Groups' [Wey39] the Capelli identity played a key role. But it was first Howe and Umeda [HU91] who went on to reformulate the Capelli identity from the viewpoint of multiplicity-free actions and the enveloping algebra associated to a reductive Lie algebra.

With the new representation theoretic approach introduced by Howe and Umeda a basis D_λ parametrized by partitions of integers of the ring of invariant differential operators was defined by Kostant and Sahi [KS91]. This basis includes the Capelli element as a special case. Konstant and Sahi went on to consider the *Capelli eigenvalue problem*, which asks for the eigenvalues of D_λ. In a series of papers [KS91, KS93, Sah94] they investigated the eigenvalues and connected that area with Jordan theory and principal series modules. In the 1990's Sahi defined a related family of multi-variable polynomials uniquely characterized by certain symmetry and vanishing properties. For special choices of parameters, Sahi's polynomials yield the so called *interpolation Jack* *polynomials* which are also investigated by Okounkov and Olshanski [OO97, OO98].

Quite recently, Sahi, Salmasian and Serganova defined operators analogous to D_λ in the context of multiplicity-free actions of basic classical Lie superalgebras arising from Jordan superalgebras [SSS20]. They went on to prove that the eigenvalues of these operators are given as a special case of a family of polynomials defined by Sergeev and Veselov [SV05]. In particular, Sahi, Salmasian and Serganova covered most of the cases of the Capelli eigenvalue problem.

The work in [SSS20] studied the cases uniformly using a restricted root system structure associated to a symmetric pair. However, some examples coming from Jordan superalgebras lead to degenerate restricted root systems and therefore they need to be addressed by independent methods. In this thesis, we would like the work on a solution for the Capelli eigenvalue problem for $\mathfrak{osp}(1|2n)$. To begin, in Chapter 2 we give an outline of Lie superalgebras and the fundamental content needed for the subsequent chapters. In Chapter 3, we describe the highest weight theory for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ and define *Casimir operators*, particularly important central elements from the *universal enveloping algebra* of a Lie superalgebra.

Finally, in Chapter 4 we specialize to the family $\mathfrak{osp}(1|2n)$. Here we give an action of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{osp}(1|2n)$ on the space of superpolynomials $\mathcal{P}(\mathbb{C}^{1|2n})$. These actions define a dual pair and allow us to define the *spherical harmonics* of $\mathfrak{osp}(1|2n)$ and admit the necessary tools needed to, in Chapter 5, introduce the Lie superalgebra $\mathfrak{g} = \mathfrak{gosp}(1|2n)$ and define the Capelli operators for $\mathfrak{osp}(1|2n)$. The new results of this thesis are found in Chapter 4 where we give the explicit joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vector formulas and in Chapter 5 where we show the surjectivity of the map from $\mathcal{Z}(\mathfrak{g})$ to $\mathcal{PD}(V)^{\mathfrak{g}}$ is closely related to non-vanishing of certain determinants. We obtain a conjectural formula for these determinants as a polynomial in n, with a complete factorization with roots and their multiplicities.

Chapter 2

Lie Superalgebras

2.1 Basics

Throughout this thesis the base field will be $\mathbb{C}$. Many of the results of this thesis hold over other fields but we will not be considering this. We choose $\mathbb{C}$ as our base field for convenience and relevance towards our main results. Furthermore, throughout this thesis when we refer to $\mathbb{Z}_2$ we mean the additive group with 2 elements. i.e. $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

Definition 2.1.1 (Super vector spaces)**.** *A **super vector space** (or simply **superspace**) is a vector space, V, over $\mathbb{C}$ endowed with a $\mathbb{Z}_2$ grading. That is, $V = V_{\bar{0}} \oplus V_{\bar{1}}$.*

The elements of $V_{\bar{0}} \cup V_{\bar{1}}$ are said to be **homogeneous** and any $v \in V_i$, for $i = \bar{0}, \bar{1}$, is said to have **parity** i, denoted by $|v| = i$. Moreover, elements of $V_{\bar{0}}$ are said to be **even** and elements of $V_{\bar{1}}$ are said to be **odd**.

The **dimension** of a superspace, V, is $\dim V = \dim V_{\bar{0}} + \dim V_{\bar{1}}$. The **superdimension** of V is defined to be $\operatorname{sdim} V = \dim V_{\bar{0}} - \dim V_{\bar{1}}$.

Definition 2.1.2 (Homogeneous Linear Maps)**.** *A linear map $f\colon V \to W$, between superspaces, is said to be **homogeneous of parity** $|f| \in \mathbb{Z}_2$ if $f(V_i) \subseteq W_{i+|f|}$, for all $i \in \mathbb{Z}_2$. Furthermore, f is called **even** when $|f| = \bar{0}$ and **odd** when $|f| = \bar{1}$.*

When we talk about a subspace of a superspace, V, we usually mean a vector superspace $W = W_{\bar{0}} \oplus W_{\bar{1}} \subseteq V$ such that $W_i \subseteq V_i$, for each $i = \bar{0}, \bar{1}$.

Definition 2.1.3 (Tensor Product of Superspaces)**.** *Let V and W be superspaces over $\mathbb{C}$. Then $V \otimes W$ is a superspace over $\mathbb{C}$ with the $\mathbb{Z}_2$-grading structure defined by*

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}) \text{ and}$$
$$(V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}).$$

Remark 2.1.4. In the non-super setting there is the natural isomorphism between $V \otimes W$ and $W \otimes V$ defined on simple tensors by

$$V \otimes W \to W \otimes V$$
$$v \otimes w \mapsto w \otimes v.$$

These two spaces are naturally isomorphic in the super setting but not under the usual map defined above. Instead for superspaces V and W we choose the following isomorphism defined on simple tensors of homogeneous elements $v \in V$ and $w \in W$ by

$$V \otimes W \to W \otimes V$$
$$v \otimes w \to (-1)^{|w||v|} w \otimes v.$$

Definition 2.1.5 (Lie superalgebra)**.** *A **Lie superalgebra** is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with an even linear map $[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies the following conditions for all homogeneous $x, y, z \in \mathfrak{g}$:*

1. ***(Skew-supersymmetry)*** $[x, y] = -(-1)^{|x||y|}[y, x]$
2. ***(Super Jacobi identity)*** $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

*The map $[\cdot,\cdot]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is called the **Lie bracket** (or **Lie Superbracket**) of $\mathfrak{g}$.*

*A **homomorphism of Lie superalgebras** is an even linear map $f\colon \mathfrak{g} \to \mathfrak{g}'$, between Lie superalgebras, such that for all $x, y \in \mathfrak{g}$ we have*

$$f([x,y]) = [f(x), f(y)].$$

Example 2.1.6 (General Linear Lie Superalgebra)**.** Let V be a super vector space and denote the space of all linear maps $f\colon V \to V$ by $\mathrm{End}(V)$. Using the definition of even and odd linear maps from Definition 2.1.2, $\mathrm{End}(V)$ is a super vector space. Furthermore, define the map

$$[\cdot,\cdot]\colon \mathrm{End}(V) \otimes \mathrm{End}(V) \to \mathrm{End}(V)$$

for any $f, g \in \mathrm{End}(V)_{\bar{0}} \cup \mathrm{End}(V)_{\bar{1}}$ by

$$[f,g] = f \circ g - (-1)^{|f||g|} g \circ f,$$

then extend by linearity to all $f, g \in \mathrm{End}(V)$.

By direct computation it is straightforward to check that $[\cdot,\cdot]$ is an even linear map, skew-supersymmetric and satisfies the super Jacobi identity given in Definition 2.1.5. This Lie bracket is called the **supercommutator** and the resulting Lie superalgebra the **general linear Lie superalgebra**, denoted by $\mathfrak{gl}(V)$. When $V = \mathbb{C}^{m|n}$ we write $\mathfrak{gl}(m|n)$ instead of $\mathfrak{gl}(V)$.

Definition 2.1.7 (Bilinear Form)**.** *Let V be a superspace over $\mathbb{C}$.*

1. *We say a bilinear form $B(\cdot,\cdot)\colon V \times V \to \mathbb{C}$ is **even** (resp. **odd**) if $B(V_i, V_j) = \{0\}$ unless $i + j = \bar{0}$ (resp. $i + j = \bar{1}$).*

2. *Assume that the bilinear form $B(\cdot,\cdot)$ of part (1) is even.*

   *(a) $B(\cdot,\cdot)$ is called **supersymmetric** if $B(v,w) = (-1)^{|v||w|} B(w,v)$ for all*

$v, w \in V_{\bar{0}} \cup V_{\bar{1}}$.

*(b) $B(\cdot,\cdot)$ is called **skew-supersymmetric** if $B(v,w) = -(-1)^{|v||w|}B(w,v)$ for all $v, w \in V_{\bar{0}} \cup V_{\bar{1}}$.*

Definition 2.1.8 (Invariant Bilinear Form)**.** *If $\mathfrak{g}$ is a Lie superalgebra and B an even bilinear form on $\mathfrak{g}$, then B is called **invariant** if for all $x, y, z \in \mathfrak{g}$ we have*

$$B([x,y],z) = B(x,[y,z]).$$

Example 2.1.9 (Heisenberg-Clifford Lie superalgebra)**.** Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be any finite-dimensional super vector space equipped with a skew-supersymmetric non-degenerate bilinear form

$$\beta \colon V \times V \to \mathbb{C}.$$

Define the **Heisenberg-Clifford Lie superalgebra** by

$$\mathfrak{H}(V,\beta) = V \oplus \mathbb{C},$$

with the grading $\mathfrak{H}(V,\beta)_{\bar{0}} = V_{\bar{0}} \oplus \mathbb{C}$ and $\mathfrak{H}(V,\beta)_{\bar{1}} = V_{\bar{1}}$. Furthermore, the Lie superbracket on $\mathfrak{H}(V,\beta)$ is defined for all $x, y \in V$ and $a, b \in \mathbb{C}$ by

$$[x+a, y+b] = \beta(x,y).$$

Observe that the image of the above defined Lie superbracket is contained in $\mathbb{C}$. Thus it is an easy check to verify that this does define a Lie superbracket.

Definition 2.1.10 (Representation of a Lie superalgebra)**.** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra and $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vector space. If $\pi \colon \mathfrak{g} \to \mathrm{End}(V)$ is a Lie superalgebra homomorphism, then (V,π) is called a **representation of** $\mathfrak{g}$.*

We will often omit π if the action is clear and use the terminology **$\mathfrak{g}$-module** to mean a representation of $\mathfrak{g}$.

Example 2.1.11 (The adjoint representation)**.** Let $\mathfrak{g}$ be a Lie superalgebra. Define $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ for all $x, y \in \mathfrak{g}$ by $\mathrm{ad}_x(y) = [x, y]$. It follows from bilinearity of the Lie bracket and the Jacobi identity that ad is a Lie superalgebra homomorphism and thus $(\mathfrak{g}, \mathrm{ad})$ is a representation of $\mathfrak{g}$. We call this representation **the adjoint representation**.

Definition 2.1.12 (Irreducible Representation)**.** *Let $\mathfrak{g}$ be a Lie superalgebra and let (V, π) be a representation of $\mathfrak{g}$. We say (V, π) is* ***irreducible*** *if for any superspace $W \subseteq V$ such that $\pi(\mathfrak{g})W \subseteq W$, either $W = 0$ or $W = V$.*

Definition 2.1.13 (Basic Classical Lie superalgebra)**.** *A Lie superalgebra $\mathfrak{g}$ is called a* ***basic classical Lie superalgebra*** *if the following are satisfied:*

(a) the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible and

(b) there is a supersymmetric non-degenerate invariant bilinear form on $\mathfrak{g}$.

Structure Theory for Basic Classical Lie Superalgebras

For this section let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be any basic classical Lie superalgebra. Every simple Lie algebra is a basic classical Lie superalgebra. The classification of basic classical Lie superalgebras was first obtained by Victor Kac [Kac77]. Table 2.1 contains the list of simple basic classic Lie superalgebras with non-zero odd part.

Definition 2.1.14 (Cartan Subalgebra)**.** *For $\mathfrak{g}$ a basic classical Lie superalgebra, a* ***Cartan Subalgebra*** *$\mathfrak{h}$ of $\mathfrak{g}$ is defined to be a Cartan subalgebra of the even subalgebra $\mathfrak{g}_{\bar{0}}$.*

Remark 2.1.15. For the Lie superalgebras in Table 2.1 any Cartan subalgebra will be a maximal toral subalgebra. That is, it will satisfy

1. $[\mathfrak{h}, \mathfrak{h}] = 0$,
2. if there is some $x \in \mathfrak{g}_{\bar{0}}$ such that $[\mathfrak{h}, x] = 0$, then $x \in \mathfrak{h}$, and
3. for all $x \in \mathfrak{h}$, the map $\mathrm{ad}_x \colon \mathfrak{g} \to \mathfrak{g}$ is diagonalizable.

Lie Superalgebra	Constraints
$\mathfrak{sl}(m+1\|n+1)$	$m, n \geq 0,\ m \neq n$
$\mathfrak{sl}(m+1\|m+1)/\mathbb{C}I_{m+1}$	$m \geq 0$
$\mathfrak{osp}(m\|2n)$	$m, n \geq 1$
$D(2\|1; a)$	$a \neq 0, -1$
$G(3)$	
$F(4)$	

Table 2.1: Simple Basic Classical Lie Superalgebras with non-zero odd part

Definition 2.1.16 (Root System)**.** *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, the* ***root space*** *associated to α is defined by*

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x, \forall y \in \mathfrak{h}\}.$$

The ***root system*** *for $\mathfrak{g}$ corresponding to $\mathfrak{h}$ is defined to be*

$$\Phi = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \text{ and } \alpha \neq 0\}$$

and respectively define the ***even*** *and* ***odd roots*** *to be*

$$\Phi_{\bar{0}} = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0\} \text{ and } \Phi_{\bar{1}} = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0\}.$$

The next theorem can be found in [CW12, Theorem 1.15] and is also in [Kac77, Prop. 2.5.5].

Theorem 2.1.17 (Structure Theorem)**.** *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$. Then:*

1. *The Lie superalgebra* $\mathfrak{g}$ *decomposes with respect to* $\mathfrak{h}$ *as*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \text{ and } \mathfrak{g}_0 = \mathfrak{h}.$$

2. $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.

3. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, *for* $\alpha, \beta, \alpha + \beta \in \Phi$.

4. *There exists a non-degenerate even invariant supersymmetric bilinear form* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{g}$.

5. $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ *unless* $\alpha = -\beta \in \Phi$.

6. *The restriction of* $\langle \cdot, \cdot \rangle$ *on* $\mathfrak{h} \times \mathfrak{h}$ *is non-degenerate.*

7. *Fix some non-zero* $e_\alpha \in \mathfrak{g}_\alpha$. *Then* $[e_\alpha, e_{-\alpha}] = \langle e_\alpha, e_{-\alpha} \rangle H_\alpha$, *where* H_α *is such that* $\langle H_\alpha, x \rangle = \alpha(x)$ *for all* $x \in \mathfrak{h}$.

8. $\Phi = -\Phi$, $\Phi_{\bar{0}} = -\Phi_{\bar{0}}$, *and* $\Phi_{\bar{1}} = -\Phi_{\bar{1}}$.

9. *Let* $\alpha \in \Phi$. *Then,* $k\alpha \in \Phi$ *for* $k \neq \pm 1$ *if and only if* $\alpha \in \Phi_{\bar{1}}$ *and* $\langle \alpha, \alpha \rangle \neq 0$. *Furthermore, if* $k\alpha \in \Phi$ *then* $k = \pm 1, \pm 2$.

The space $\mathfrak{h}^*$ can be equipped with a non-degenerate symmetric bilinear form using the natural pairing induced by $\langle \cdot, \cdot \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$. We define the **orthogonal group** on $\mathfrak{h}^*$ as the group

$$\mathrm{O}(\mathfrak{h}^*) = \{x \in \mathfrak{gl}(\mathfrak{h}^*) \mid \langle x \cdot \alpha, x \cdot \beta \rangle = \langle \alpha, \beta \rangle \text{ for all } \alpha, \beta \in \mathfrak{h}^*\}.$$

Definition 2.1.18 (Weyl Group)**.** *Let* $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ *be as in Definition 2.1.16. For* $\alpha \in \Phi_{\bar{0}}$, *define the* ***reflection by*** α *to be the linear involution map* $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$ *defined for all* $\beta \in \mathfrak{h}^*$ *by*

$$s_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

The ***Weyl group*** *of $\mathfrak{g}$, denoted by $\mathcal{W}$, is the subgroup of $\mathrm{O}(\mathfrak{h}^*)$ generated by the reflections s_α, $\alpha \in \Phi_{\bar{0}}$. We write $(\lambda, \alpha) = \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$.*

Remark 2.1.19. Equivalently, we can define the Weyl group of $\mathfrak{g}$ as the Weyl group of $\mathfrak{g}_{\bar{0}}$.

The identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ induces an action of $\mathcal{W}$ on $\mathfrak{h}$ defined on the generators $s_\alpha \in \mathcal{W}$ for all $x \in \mathfrak{h}$ by

$$s_\alpha \cdot x = x - 2\frac{\langle x, H_\alpha\rangle}{\langle H_\alpha, H_\alpha\rangle} H_\alpha,$$

where α is identified with H_α.

The next proposition can be found in [CW12, Theorem 1.15].

Proposition 2.1.20. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathcal{W}$ be the associated Weyl group. Then $\Phi, \Phi_{\bar{0}}$, and $\Phi_{\bar{1}}$ are invariant under the action of $\mathcal{W}$. Furthermore, for all $x, y \in \mathfrak{h}$ and $w \in \mathcal{W}$*

$$\langle w \cdot x, w \cdot y\rangle = \langle x, y\rangle.$$

Let Φ be a root system for $\mathfrak{g}$ and let $\mathfrak{h}^*_{\mathbb{R}}$ be the vector space spanned by Φ over $\mathbb{R}$. A total ordering, $\leq$, on $\mathfrak{h}^*_{\mathbb{R}}$ is **compatible** with the real vector space structure if for all $v, v', w, w' \in \mathfrak{h}^*_{\mathbb{R}}$ and $c \in \mathbb{R}_{>0}$ the following are true

1. if $v \leq v'$ and $w \leq w'$, then $v + w \leq v' + w'$,

2. if $v \leq v'$ then $-v' \leq -v$, and

3. if $v \leq v'$ then $cv \leq cv'$.

Definition 2.1.21 (Positive and Negative System)**.** *Let $\leq$ be a total ordering of $\mathfrak{h}^*_{\mathbb{R}}$ that is compatible with the real vector space structure.*

- *A* ***positive system*** *of Φ, denoted Φ^+, is a subset of Φ consisting of all roots $\alpha \in \Phi$ such that $\alpha > 0$. We call the elements of Φ^+* ***positive roots****.*

- *A **negative system** of Φ if defined analogously and denoted by Φ^-. We call the element of Φ^- **negative roots**.*

Definition 2.1.22 (Fundamental System)**.** *If Φ^+ is a positive system of Φ, then a **fundamental system**, denoted by Π, is the set of all $\alpha \in \Phi^+$ such that α cannot be written as a non-trivial sum of two roots in Φ^+. Elements of Π are called **simple roots**.*

Following the notation of Definition 2.1.21 we also set $\Phi_i^\pm = \Phi^\pm \cap \Phi_i$ for $i \in \mathbb{Z}_2$. Furthermore, define

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha.$$

Thus $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

and we call such a decomposition a **triangular decomposition** of $\mathfrak{g}$.

Definition 2.1.23 (Borel Subalgebra)**.** *Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition of $\mathfrak{g}$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called a **Borel subalgebra** of $\mathfrak{g}$ (corresponding to Φ^+).*

The $\mathbb{Z}_2$-grading on $\mathfrak{b}$ is given by $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_i$, for $i \in \mathbb{Z}_2$.

Definition 2.1.24 (Highest Weight Vector)**.** *Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition for a basic classical Lie superalgebra and V a representation of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$ we say a non-zero vector $v_\lambda \in V$ is a **highest weight vector of weight** λ if for all $x \in \mathfrak{h}$ and $y \in \mathfrak{n}^+$,*

$$xv_\lambda = \lambda(x)v_\lambda \quad \textit{and} \quad yv_\lambda = 0.$$

*If $V = \operatorname{span}\{xv_\lambda \mid x \in \mathfrak{g}\}$ then we say that V is a **highest weight module of weight** λ.*

In Lie algebra theory there is a classical theorem of Cartan and Weyl which states that every irreducible finite-dimensional module of a semi-simple Lie algebra has a unique highest weight. It turns out that the same result holds in the case of basic classical Lie superalgebras. This was first proved by Kac in [Kac77, 5.2]. See [CW12, Prop. 1.35] for more details.

Theorem 2.1.25. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra with Borel subalgebra, $\mathfrak{b}$. Then any finite-dimensional irreducible $\mathfrak{g}$-module is a highest weight module.*

2.2 The General Linear Lie Superalgebra and its Structure

Recall the general linear Lie superalgebra from Example 2.1.6. Let $V = \mathbb{C}^{m|n}$ and choose ordered bases for $V_{\bar{0}}$ and $V_{\bar{1}}$. We parametrize this basis by $I(m|n) = \{1, \dots, m; \bar{1}, \dots, \bar{n}\}$ with total ordering

$$1 \prec \cdots \prec m \prec 0 \prec \bar{1} \prec \cdots \prec \bar{n},$$

where $\bar{i} := i + m$. With this parametrization every element in $\mathfrak{gl}(m|n)$ can be written in block matrix form. That is, elements of $\mathfrak{gl}(m|n)$ can be represented by matrices of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.2.1}$$

where $a \in \mathrm{Mat}_{m\times m}(\mathbb{C})$, $b, c^T \in \mathrm{Mat}_{m\times n}(\mathbb{C})$, and $d \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. In this matrix realization, $\mathfrak{gl}(V)_{\bar{0}}$ consists of matrices of the form (2.2.1) with $b = c^T = 0$ and $\mathfrak{gl}(V)_{\bar{1}}$ consists of such matrices with $a = 0$ and $d = 0$. Define the **supertrace** of $x \in \mathfrak{gl}(m|n)$

by

$$\mathrm{str}(x) = \mathrm{tr}(a) - \mathrm{tr}(d), \tag{2.2.2}$$

where x is written as in Equation (2.2.1) and tr is the standard trace.

The Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{gl}(m|n)$ is the set of all diagonal matrices of $\mathfrak{gl}(m|n)$. That is, $\mathfrak{h} = \mathrm{span}\{E_{i,i} \mid \ i \in I(m|n)\}$, where $E_{i,j}$ is the standard matrix element with zeros in all entries except 1 in the $(i,j)^{th}$ position. For all $i \in I(m|n)$, let $\varepsilon_i \in \mathfrak{h}^*$ denote the dual vector corresponding to $E_{i,i}$. The root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ for $\mathfrak{gl}(m|n)$ is given by

$$\Phi_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid i,j \in I(m|n),\ i \neq j,\ \text{and } i,j \prec 0 \text{ or } i,j \succ 0\} \text{ and}$$
$$\Phi_{\bar{1}} = \{\pm(\varepsilon_j - \varepsilon_i) \mid i,j \in I(m|n),\ j \prec 0 \prec i\}.$$

The space $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}}$ is a subspace of $\mathfrak{h}^*$ of codimension one. In the above matrix realization, the standard Borel subalgebra of $\mathfrak{gl}(m|n)$ consists of upper triangular matrices. This Borel subalgebra corresponds to the positive system

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i,j \in I(m|n),\ i \prec j\} \tag{2.2.3}$$

and the fundamental system

$$\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in I(m|n) \backslash \{\overline{n}\}\}. \tag{2.2.4}$$

2.3 The Orthosymplectic Lie Superalgebra and its Structure

The next Lie superalgebra, the orthosymplectic Lie superalgebra, is of particular interest in this thesis since the main results will be concerning a family of orthosymplectic Lie superalgebras. The orthogonal Lie algebra and the symplectic Lie algebra are closely related in that they are both defined by certain invariance conditions of an inner product. The orthosymplectic Lie superalgebra is a way to unify these Lie algebras.

Definition 2.3.1 (Orthosymplectic Lie Superalgebra)**.** *Let B be a non-degenerate even supersymmetric bilinear form on a superspace, V. The* ***orthosymplectic Lie superalgebra*** *is defined as $\mathfrak{osp}(V) = \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}$, where for $i \in \mathbb{Z}_2$*

$$\mathfrak{osp}(V)_i = \{x \in \mathfrak{gl}(V)_i \mid B(x(v), w) = -(-1)^{i|v|} B(v, x(w)),\ \textit{for all } v, w \in V\}.$$

The Lie bracket is induced from $\mathfrak{gl}(V)$.

We write $\mathfrak{osp}(m|2n)$ instead of $\mathfrak{osp}(V)$ when $V = \mathbb{C}^{m|2n}$.

Remark 2.3.2. The **symplectic-orthogonal Lie superalgebra**, denoted by $\mathfrak{spo}(V)$, can be analogously defined as the subalgebra of $\mathfrak{gl}(V)$ that preserves a non-degenerate skew-supersymmetric bilinear form on V. Furthermore, $\mathfrak{osp}(m|2n)$ and $\mathfrak{spo}(2n|m)$ are isomorphic via the parity map $\mathbb{C}^{m|2n} \to \mathbb{C}^{2n|m}$.

Similarly to $\mathfrak{gl}(m|n)$, we choose bases for $V_{\bar{0}}$ and $V_{\bar{1}}$, indexed by $I(m|2n) = \{1, \dots, m; \bar{1}, \dots, \overline{2n}\}$, where $\bar{i} = i + m$. Take a total ordering of $I(m|2n)$ analogous to $\mathfrak{gl}(m|n)$ and for each $i \in I(m|2n)$ define the **parity of** i to be

$$|i| = \begin{cases} \bar{0} & \text{if } i \prec 0 \\ \bar{1} & \text{if } i \succ 0. \end{cases}$$

Once a basis is chosen then any bilinear form on V has a matrix realization. That is, for any bilinear form $B(\cdot,\cdot)\colon \mathbb{C}^{m|2n}\times\mathbb{C}^{m|2n}\to\mathbb{C}$ there is $J\in \mathrm{Mat}_{m+2n}(\mathbb{C})$ such that for all $v,w\in\mathbb{C}^{m|2n}$

$$B(v,w)=w^TJv,$$

where w^T is the transpose of w and here we identify $\mathbb{C}^{m|n}$ with $\mathbb{C}^{m+n}$ in the standard fashion. When B is an even supersymmetric bilinear form, there is $J_1\in\mathrm{Mat}_m(\mathbb{C})$ and $J_2\in\mathrm{Mat}_{2n}(\mathbb{C})$ such that $J_1=J_1^T$, $J_2=-J_2^T$, and the matrix corresponding to B is $J=\mathrm{diag}(J_1,J_2)$. Thus $\mathfrak{osp}(m|2n)$ can be written as

$$\mathfrak{osp}(m|2n)_i=\{x\in\mathfrak{gl}(\mathbb{C}^{m|2n})_i \mid (xw)^TJv=-(-1)^{i|v|}w^TJxv,\ \text{for all } v,w\in\mathbb{C}^{m|2n}\}$$

Define the space

$$\mathrm{Skew}(\mathbb{C}^{m|2n})=\left\{X\in\mathfrak{gl}(m|2n)\mid X_{i,j}=-(-1)^{|i||j|}X_{j,i}\text{ for all } i,j\in I(m|2n)\right\}$$

and notice that we have a linear isomorphism

$$\begin{aligned}\Psi\colon\ \mathrm{Skew}(\mathbb{C}^{m|2n})&\to\mathfrak{osp}(m|2n)\\ X&\mapsto J^{-1}X.\end{aligned}\tag{2.3.1}$$

It is worthwhile to note that Equation (2.3.1) defines an isomorphism between these two vector spaces. This is easily checked by comparing dimensions of the domain and codomain. Furthermore, by the pullback along Ψ, we can equip $\mathrm{Skew}(\mathbb{C}^{m|2n})$ with a Lie superalgebra structure.

We now describe a matrix realization for $\mathfrak{osp}(V)$. For relevance towards our final results which are concerning $\mathfrak{osp}(1|2n)$, we work with $V=\mathbb{C}^{2m+1|2n}$ throughout the following. All of the following calculations can be done analogously for $\mathfrak{osp}(2m|2n)$, $\mathfrak{spo}(2n|2m+1)$, and $\mathfrak{spo}(2n|2m)$.

Define $\mathfrak{J}_{2m+1|2n}$ to be a $(2m+2n+1)\times(2m+2n+1)$ matrix in the $(1+m+m|n+n)$-block form by

$$\mathfrak{J}_{2m+1|2n} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & -I_n & 0 \end{bmatrix}.$$

The matrix $\mathfrak{J}_{2m|2n}$ is defined similarly as above but with the first row and column removed. Now define a bilinear form, $\langle \cdot, \cdot \rangle$, on $\mathbb{C}^{2m+1|2n}$ for any $v, w \in \mathbb{C}^{2m+1|2n}$ by

$$\langle v, w \rangle = w^T \, \mathfrak{J}_{2m+1|2n} \, v.$$

It is a straightforward calculation to see that $\langle \cdot, \cdot \rangle$ is a non-degenerate even supersymmetric bilinear form. For all $x \in \mathfrak{osp}(\mathbb{C}^{2m+1|2n})_{\bar{0}}$ write x in $(1+m+m|n+n)$-block form. That is,

$$x = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & d_4 & e_4 \\ 0 & 0 & 0 & d_5 & e_5 \end{bmatrix}$$

Moreover, for all $v, w \in \mathbb{C}^{2m+1|2n}$

$$w^T \, \mathfrak{J}_{2m+1|2n} \, xv = -(xw)^T \, \mathfrak{J}_{2m+1|2n} \, v.$$

Since this is for any $v, w \in \mathbb{C}^{2m+1|2n}$ we have $\mathfrak{J}_{2m+1|2n} \, x = -x^T \, \mathfrak{J}_{2m+1|2n}$. Therefore by direct computation

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & d_5 & e_5 \\ 0 & 0 & 0 & -d_4 & -e_4 \end{bmatrix} = - \begin{bmatrix} a_1^T & a_3^T & a_2^T & 0 & 0 \\ b_1^T & b_3^T & b_2^T & 0 & 0 \\ c_1^T & c_3^T & c_2^T & 0 & 0 \\ 0 & 0 & 0 & -d_5^T & d_4^T \\ 0 & 0 & 0 & -e_5^T & e_4^T \end{bmatrix}$$

and then by comparing the LHS with the RHS we conclude that

$$x = \begin{bmatrix} 0 & -a_3^T & -a_2^T & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & -b_2^T & 0 & 0 \\ 0 & 0 & 0 & d_4 & e_4 \\ 0 & 0 & 0 & d_5 & -d_4^T \end{bmatrix},$$

for $b_3 = -b_3^T$, $c_2 = -c_2^T$, $d_5 = d_5^T$, and $e_4 = e_4^T$. Next take any $x \in \mathfrak{osp}(2m+1|2n)_{\bar{1}}$ written in block form

$$x = \begin{bmatrix} 0 & 0 & 0 & d_1 & e_1 \\ 0 & 0 & 0 & d_2 & e_2 \\ 0 & 0 & 0 & d_3 & e_3 \\ a_4 & b_4 & c_4 & 0 & 0 \\ a_5 & b_5 & c_5 & 0 & 0 \end{bmatrix}.$$

Then for all homogeneous $v \in V$

$$\mathfrak{J}_{2m+1|2n}\, xv = -(-1)^{|x|} x^T\, \mathfrak{J}_{2m+1|2n} v.$$

In particular, for all $v \in V_{\bar{0}}$ then

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_5 & b_5 & c_5 & 0 & 0 \\ -a_4 & -b_4 & -c_4 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ d_1^T & d_3^T & d_2^T & 0 & 0 \\ e_1^T & e_3^T & e_2^T & 0 & 0 \end{bmatrix}.$$

Similarly for all $v \in V_{\bar{1}}$ we have

$$\begin{bmatrix} 0 & 0 & 0 & d_1 & e_1 \\ 0 & 0 & 0 & d_3 & e_3 \\ 0 & 0 & 0 & d_2 & e_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -a_5^T & a_4^T \\ 0 & 0 & 0 & -b_5^T & b_4^T \\ 0 & 0 & 0 & -c_5^T & c_4^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

That is,

$$x = \begin{bmatrix} 0 & 0 & 0 & -a_5^T & a_4^T \\ 0 & 0 & 0 & -c_5^T & c_4^T \\ 0 & 0 & 0 & -b_5^T & b_4^T \\ a_4 & b_4 & c_4 & 0 & 0 \\ a_5 & b_5 & c_5 & 0 & 0 \end{bmatrix}.$$

Given these calculations, Table 2.2 gives a basis for $\mathfrak{osp}(2m+1|2n)$.

Associated Blocks	Basis Elements	Index
a_2	$E_{i+1,1} - E_{1,i+m+1}$	$1 \preceq i \preceq m$
a_3	$E_{i+m+1,1} - E_{1,i+1}$	$1 \preceq i \preceq m$
b_2	$E_{i+1,j+1} - E_{j+m+1,i+m+1}$	$1 \preceq i, j \preceq m$
b_3	$E_{i+m+1,j+1} - E_{j+m+1,i+1}$	$1 \preceq i < j \preceq m$
c_2	$E_{i+1,j+m+1} - E_{j+1,i+m+1}$	$1 \preceq i < j \preceq m$
d_4	$E_{i,j} - E_{j+n,i+n}$	$\overline{1} \preceq i, j \preceq \overline{n}$
d_5	$E_{i+n,j} + E_{j+n,i}$	$\overline{1} \preceq i \preceq j \preceq \overline{n}$
e_4	$E_{i,j+n} + E_{j,i+n}$	$\overline{1} \preceq i \preceq j \preceq \overline{n}$
a_4	$E_{i,1} + E_{1,i+n}$	$\overline{1} \preceq i \preceq \overline{n}$
a_5	$E_{i+n,1} - E_{1,i}$	$\overline{1} \preceq i \preceq \overline{n}$
b_4	$E_{i,j+1} + E_{j+m+1,i+n}$	$\overline{1} \preceq i \preceq \overline{n},\ 1 \preceq j \preceq m$
b_5	$E_{i+n,j+1} - E_{j+m+1,i}$	$\overline{1} \preceq i \preceq \overline{n},\ 1 \preceq j \preceq m$
c_4	$E_{i,j+m+1} + E_{j+1,i+n}$	$\overline{1} \preceq i \preceq \overline{n},\ 1 \preceq j \preceq m$
c_5	$E_{i+n,j+m+1} - E_{j+1,i}$	$\overline{1} \preceq i \preceq \overline{n},\ 1 \preceq j \preceq m$

Table 2.2: Basis of $\mathfrak{osp}(2m+1|2n)$

The standard Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{osp}(2m+1|2n)$ has basis

$$\{E_{i+1,i+1} - E_{i+m+1,i+m+1} \mid 1 \preceq i \preceq m\} \cup \{E_{j,j} - E_{j+n,j+n} \mid \overline{1} \preceq j \preceq \overline{n}\} \qquad (2.3.2)$$

and dual basis

$$\{\varepsilon_i \mid 1 \preceq i \preceq m\} \cup \{\delta_j \mid 1 \preceq j \preceq n\}.$$

To save space in some of our calculations we will sometimes write ε_{j+m} to mean δ_j.

Define the bilinear form $\langle \cdot, \cdot \rangle \colon \mathfrak{osp}(2m+1|2n) \times \mathfrak{osp}(2m+1|2n) \to \mathbb{C}$ by

$$\langle x, y \rangle := \operatorname{str}(xy).$$

Then it is a simple calculation to verify that ε_i and δ_j can be identified with the maps

$$\varepsilon_i \mapsto \left(v \mapsto \frac{1}{2} \langle E_{i+1,i+1} - E_{i+m+1,i+m+1}, v \rangle \right)$$
$$\delta_j \mapsto \left(v \mapsto \frac{-1}{2} \langle E_{\overline{j},\overline{j}} - E_{\overline{j}+n,\overline{j}+n}, v \rangle \right).$$

Table 2.3 gives the roots that occur in $\mathfrak{osp}(2m+1|2n)$ and the associated block containing the root space associated to any given root. This table can be verified by a tedious but straightforward direct calculation.

In the case of $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{h}^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}^*$. Given this realization we take the fundamental system of $\mathfrak{osp}(\ell|2n)$ to be

$$\Pi = \begin{cases} \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{\overline{n}} \mid i \in I(m,n),\ i \prec \overline{n}\} & \text{for } \ell = 2m+1 \\ \{\varepsilon_i - \varepsilon_{i+1}, 2\varepsilon_{\overline{n}} \mid i \in I(m,n),\ i \prec \overline{n}\} & \text{for } \ell = 2m \end{cases} \tag{2.3.3}$$

and therefore the corresponding positive roots are

$$\Phi^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j, \varepsilon_s, 2\varepsilon_t \mid i,j,s,t \in I(m|n),\ i \prec j,\ 0 \prec t\} & \text{for } \ell = 2m+1 \\ \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_s \mid i,j,s \in I(m|n),\ i \prec j,\ 0 \prec s\} & \text{for } \ell = 2m. \end{cases} \tag{2.3.4}$$

Associated Block	Roots	Index
a_2	ε_i	$1 \preceq i \preceq m$
a_3	$-\varepsilon_i$	$1 \preceq i \preceq m$
b_2	$\varepsilon_i - \varepsilon_j$	$1 \preceq i, j \preceq m,\ i \neq j$
b_3	$-\varepsilon_i - \varepsilon_j$	$1 \preceq i \prec j \preceq m$
c_2	$\varepsilon_i + \varepsilon_j$	$1 \preceq i \prec j \preceq m$
d_4	$\delta_i - \delta_j$	$1 \preceq i, j \preceq n,\ i \neq j$
d_5	$-\delta_i - \delta_j$	$1 \preceq i \preceq j \preceq n$
e_4	$\delta_i + \delta_j$	$1 \preceq i \preceq j \preceq n$
a_4	δ_i	$1 \preceq i \preceq n$
a_5	$-\delta_i$	$1 \preceq i \preceq n$
b_4	$\delta_i - \varepsilon_j$	$1 \preceq i \preceq n,\ 1 \preceq j \preceq m$
b_5	$-\delta_i - \varepsilon_j$	$1 \preceq i \preceq n,\ 1 \preceq j \preceq m$
c_4	$\delta_i + \varepsilon_j$	$1 \preceq i \preceq n,\ 1 \preceq j \preceq m$
c_5	$\varepsilon_j - \delta_i$	$1 \preceq i \preceq n,\ 1 \preceq j \preceq m$

Table 2.3: Roots of $\mathfrak{osp}(2m+1|2n)$

2.4 Enveloping Algebra of a Lie Superalgebra

Throughout this section let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over $\mathbb{C}$. We introduce the associative superalgebra called the universal enveloping algebra of $\mathfrak{g}$, denoted by $\mathfrak{U}(\mathfrak{g})$. One important property of $\mathfrak{U}(\mathfrak{g})$ is that the category of all representations of $\mathfrak{g}$ is isomorphic to the category of left modules over $\mathfrak{U}(\mathfrak{g})$. That is, the study of representations of a Lie superalgebra can be translated to the study of superalgebra modules over the related universal enveloping algebra.

Definition 2.4.1 (Superalgebra)**.** *An **associative superalgebra**, $\mathcal{A}$, is a super vector space equipped with an even associative multiplication $\cdot\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.*

When the multiplication is clear we omit the $\cdot$ and use juxtaposition instead. A Lie superalgebra structure can be defined on any superalgebra $\mathcal{A}$ by defining the Lie superbracket for all $a, b \in \mathcal{A}_{\bar{0}} \cup \mathcal{A}_{\bar{1}}$ by

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

Definition 2.4.2 (Universal Enveloping Algebra)**.** *A* ***universal enveloping algebra*** *of $\mathfrak{g}$ is an associative superalgebra $\mathfrak{U}(\mathfrak{g})$ together with a homomorphism of Lie superlagebras $\iota\colon \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ such that the following universal property is satisfied: for any associative superalgebra A and homomorphism of Lie superalgebras $\psi\colon \mathfrak{g} \to A$, there exists a unique homomorphism of associative superalgebras $\Psi\colon \mathfrak{U}(\mathfrak{g}) \to A$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \iota\ } & \mathfrak{U}(\mathfrak{g}) \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\Psi} \\ & & A \end{array}$$

The construction of the universal enveloping algebra of $\mathfrak{g}$ and its uniqueness follows from a standard (but non-trivial) argument, see [Hum78, Section 17.2] or [Car05, Section 9.1] for the details in the standard setting or [Cao18, Chapter 3] for the details in the super setting. Define the **tensor algebra** of $\mathfrak{g}$ as the associative superalgebra

$$T(\mathfrak{g}) := \bigoplus_{j=0}^{\infty} T^i(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots ,$$

where $T^i(\mathfrak{g}) := \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{i \text{ times}}$ and the $\mathbb{Z}_2$-grading is given by Definition 2.1.3. The multiplication in $T(\mathfrak{g})$ is defined on pure tensors by

$$(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) := x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m,$$

where $x_1, \ldots, x_n, y_1, \ldots, y_m \in \mathfrak{g}$ and $m, n \in \mathbb{N}$. The universal enveloping algebra of $\mathfrak{g}$ is then defined to be quotient algebra

$$\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g})/J,$$

where J is the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$. The map $\iota \colon \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is the composition of the inclusion map $\mathfrak{g} \to T(\mathfrak{g})$ and the quotient map $T(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$. It is easy to verify that ι satisfies the required universal property.

Since $\mathfrak{U}(\mathfrak{g})$ is an associative superalgebra we define a Lie superbracket on $\mathfrak{U}(\mathfrak{g})$ homogeneous elements $x, y \in \mathfrak{U}(\mathfrak{g})_{\bar{0}} \cup \mathfrak{U}(\mathfrak{g})_{\bar{1}}$ by

$$[x, y] = xy - (-1)^{|x||y|} yx,$$

then extend by linearity to all of $\mathfrak{U}(\mathfrak{g})$.

Definition 2.4.3 (Centre of a Superalgebra)**.** *The* ***centre*** *of an associative superalgebra, $\mathcal{A}$, is the space*

$$\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} \mid \forall b \in \mathcal{A},\ [a, b] = 0\}.$$

For $\mathfrak{U}(\mathfrak{g})$ we will write $\mathcal{Z}(\mathfrak{U}(\mathfrak{g}))$ or $\mathcal{Z}(\mathfrak{g})$ as the centre of $\mathfrak{U}(\mathfrak{g})$. Furthermore, $\mathcal{Z}(\mathfrak{g})$ decomposes as $\mathcal{Z}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})_{\bar{0}} \oplus \mathcal{Z}(\mathfrak{g})_{\bar{1}}$, where $\mathcal{Z}(\mathfrak{g})_i = \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{U}(\mathfrak{g})_i$ for $i \in \mathbb{Z}_2$.

Theorem 2.4.4 (Poincaré-Birkhoff-Witt Theorem)**.** *Let $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots, y_q\}$ be the bases of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$, respectively. The set*

$$\left\{x_1^{r_1} x_2^{r_2} \cdots x_p^{r_p} y_1^{s_1} y_2^{s_2} \cdots y_q^{s_q} \mid r_1, \ldots, r_p \in \mathbb{Z}_{\geq 0}, s_1, s_2, \ldots, s_q \in \{0, 1\}\right\}$$

is a basis for $\mathfrak{U}(\mathfrak{g})$.

Proof: This proof is done analogously to the original PBW Theorem in the non-super setting. See [Hum78, Section 17.4] or [Car05, Section 9.2]. For explicit details in the super setting see [Cao18, Chapter 3]. ∎

2.5 The Harish-Chandra Projection Map

Let $\mathfrak{g}$ be a basic classical Lie superalgebra with Cartan subalgebra, $\mathfrak{h}$, and Weyl group, $\mathcal{W}$. In this section we introduce supersymmetric polynomials and relate them to $\mathcal{Z}(\mathfrak{g})$ via the Harish-Chandra homomorphism. The results of this section can be found in [CW12, Section 2.2].

For this section, fix $m, n \in \mathbb{Z}_{\geq 0}$ and let $\mathbb{C}[x, y]$ be the algebra of complex polynomials in indeterminates $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$.

Definition 2.5.1 (Supersymmetric polynomials)**.** *We say a polynomial $f \in \mathbb{C}[x, y]$ is **supersymmetric**, if:*

1. *f is symmetric in $x_1, \ldots, x_m$,*
2. *f is symmetric in $y_1, \ldots, y_n$, and*
3. *the polynomial obtained from setting $x_m = y_n = t$ is independent of t.*

The $\mathbb{C}$-algebra of all supersymmetric polynomials is denoted by $\mathbb{C}[x, y]_{\text{sup}}$.

Example 2.5.2. Let $k \in \mathbb{Z}_{\geq 0}$ then we have the family of supersymmetric polynomials $\sigma^k_{m,n}$ defined by

$$\sigma^k_{m,n} := \sum_{i=1}^{m} x_i^k - \sum_{j=1}^{n} y_j^k.$$

Another supersymmetric polynomial is

$$\tau_{m,n} := \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i - y_j).$$

Next we will describe a particular subset of $S(\mathfrak{h}^*)$ and Theorem 2.5.7 will give justification to the importance of this subset. First note that the action of $\mathcal{W}$ on $\mathfrak{h}^*$ gives rise to a natural action of $\mathcal{W}$ on $S(\mathfrak{h}^*)$. That is, the action is induced from the diagonal action of $\mathcal{W}$ on $T(\mathfrak{h}^*)$. We denote $S(\mathfrak{h}^*)^{\mathcal{W}}$ as the space of symmetric tensors invariant under this $\mathcal{W}$-action. Second, we fix the basis $\{\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n\}$ for $\mathfrak{h}^*$ and identify $S(\mathfrak{h}^*)$ with $\mathbb{C}[x,y]$ by the algebra isomorphism defined via

$$\begin{gathered}\xi\colon S(\mathfrak{h}^*) \to \mathbb{C}[x,y]\\ \varepsilon_i \mapsto x_i\\ \delta_j \mapsto y_j.\end{gathered}$$

Thus we can translate the definition of supersymmetric polynomials to $S(\mathfrak{h}^*)$ by

$$S(\mathfrak{h}^*)_{\text{sup}} = \xi^{-1}(\mathbb{C}[x,y]_{\text{sup}})$$

and define the set

$$S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}} = S(\mathfrak{h}^*)^{\mathcal{W}} \cap S(\mathfrak{h}^*)_{\text{sup}}.$$

We will calculate $S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}}$ for the cases $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2m|2n)$. The calculations supporting the following examples can be found in [CW12, Sec. 2.2.4].

Example 2.5.3 (Invariant Symmetric Tensors of $\mathfrak{gl}(m|n)$)**.** Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ with the standard Cartan subalgebra, $\mathfrak{h} = \text{span}\{E_{i,i} |\ i \in I(m|n)\}$. The Weyl group of $\mathfrak{g}$ is $\mathcal{W} \cong \mathfrak{S}_m \times \mathfrak{S}_n$. Choose $\{\varepsilon_i \mid i \in I(m|n)\}$ as our basis for $\mathfrak{h}^*$. Thus the action of $\mathcal{W}$ on $S(\mathfrak{h}^*)$ gives us

$$S(\mathfrak{h}^*)^{\mathcal{W}} = \left\{f \in S(\mathfrak{h}^*) \mid f \text{ is symmetric in } \{\varepsilon_i \mid 1 \preceq i \preceq m\} \text{ and } \{\varepsilon_j \mid \overline{1} \preceq j \preceq \overline{n}\}\right\}.$$

In particular,

$$S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}} = \{f \in S(\mathfrak{h}^*) \mid f \text{ is supersymmetric in } \varepsilon_i \text{ and } \varepsilon_j \text{ for } 1 \preceq i \preceq m, \overline{1} \preceq j \preceq \overline{n}\}.$$

Example 2.5.4 (Invariant Symmetric Tensors of $\mathfrak{osp}(2m+1|2n)$)**.** The Weyl group, $\mathcal{W}$, for $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ is $\mathcal{W} \cong (\mathbb{Z}_2^m \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2^n \rtimes \mathfrak{S}_n)$ and the Cartan subalgebra

$$\mathfrak{h} = \operatorname{span}\{E_{i+1,i+1} - E_{i+m+1,i+m+1} \mid 1 \preceq i \preceq m\} \cup \left\{E_{j,j} - E_{j+n,j+n} \mid \overline{1} \preceq j \preceq \overline{n}\right\}.$$

Let $\{\varepsilon_i \mid 1 \preceq i \preceq m\}$ and $\{\delta_j \mid \overline{1} \preceq j \preceq \overline{n}\}$ be the basis for $\mathfrak{h}^*$ for $\mathfrak{osp}(2m+1|2n)$. Therefore, $S(\mathfrak{h}^*)^{\mathcal{W}}$ is the algebra of polynomials symmetric in $\{\varepsilon_i^2 \mid 1 \preceq i \preceq m\}$ and $\{\delta_j^2 \mid \overline{1} \preceq j \preceq \overline{n}\}$. Furthermore,

$$S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}} = \{f \in S(\mathfrak{h}^*) \mid f \text{ is supersymmetric in } \varepsilon_i^2, \delta_j^2, 1 \preceq i \preceq m, \overline{1} \preceq j \preceq \overline{n}\}.$$

Example 2.5.5 (Invariant Symmetric Tensors of $\mathfrak{osp}(2m|2n)$)**.** Consider $\mathfrak{osp}(2m|2n)$ as a subalgebra of $\mathfrak{osp}(2m+1|2n)$. We write W_{+1} to be the Weyl group of $\mathfrak{osp}(2m+1|2n)$. Then the Weyl group, $\mathcal{W}$, of $\mathfrak{osp}(2m|2n)$ is an index 2 subgroup of

$$\mathcal{W}_{+1} \cong (\mathbb{Z}_2^m \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2^n \rtimes \mathfrak{S}_n),$$

consisting of elements which contain an even number of -1 in $\mathbb{Z}_2^m$. Here we see $\mathbb{Z}_2 = \{1, -1\}$ as a group under multiplication. To calculate $S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}}$, first notice that since $\mathcal{W} \subseteq \mathcal{W}_{+1}$ we can conclude that $S(\mathfrak{h}^*)^{W_{+1}}_{\text{sup}} \subseteq S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}}$, where $S(\mathfrak{h}^*)^{W_{+1}}$ and $S(\mathfrak{h}^*)^{W_{+1}}_{\text{sup}}$ are given explicitly in Example 2.5.4. Define the element

$$Q = \prod_{\substack{1 \preceq i \preceq m \\ 1 \preceq j \preceq n}} (\varepsilon_i^2 - \delta_j^2).$$

There is a detailed discussion in [CW12, Sec. 2.2] showing that for any $g \in S(\mathfrak{h}^*)^{\mathcal{W}_{+1}}$

$$\varepsilon_1 \cdots \varepsilon_m Q g \in S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}}.$$

In particular, it is proved in [CW12, Sec. 2.2] that

$$S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}} = S(\mathfrak{h}^*)^{\mathcal{W}_{+1}}_{\text{sup}} \oplus \mathbb{C}[\varepsilon, \delta]^{\mathcal{W}_{+1}} \varepsilon_1 \cdots \varepsilon_m Q.$$

Definition 2.5.6 (Weyl Vector)**.** *Let Φ^+ be a positive system in the root system Φ for $\mathfrak{g}$. Let ρ_i be the sum of positive roots of parity i and we define the **Weyl vector** corresponding to Φ^+ to be*

$$\rho = \frac{1}{2}(\rho_0 - \rho_1). \tag{2.5.1}$$

Since $\mathfrak{h}$ is naturally isomorphic to $\mathfrak{h}^{**}$ then we extend this isomorphism to a natural algebra isomorphism between $S(\mathfrak{h})$ and $\mathcal{P}(\mathfrak{h}^*)$, here $\mathcal{P}(\mathfrak{h}^*)$ denotes the space of polynomials in $\mathfrak{h}^*$. We take this isomorphism as identification. Define the automorphism τ on $\mathcal{P}(\mathfrak{h}^*)$ for all $f \in \mathcal{P}(\mathfrak{h}^*)$ by

$$\tau(f)(\lambda) := f(\lambda - \rho),$$

for all $\lambda \in \mathfrak{h}^*$. The isomorphism between $S(\mathfrak{h})$ and $\mathcal{P}(\mathfrak{h}^*)$ allows us to consider τ as an isomorphism of $S(\mathfrak{h})$.

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of $\mathfrak{g}$ corresponding to Φ^+. We then pick a basis of $\mathfrak{g}$ consisting of bases of $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$. Using Theorem 2.4.4 we can then write $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^+)$ and set

$$\phi\colon \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h}) \tag{2.5.2}$$

to be the associated projection to the first component of this direct sum. Furthermore, note that when we restrict ϕ to $\phi'\colon \mathcal{Z}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{h})$ the result is an algebra homomorphism, see [CW12, Sec. 2.2.2].

Since $\mathfrak{h}$ is commutative we have that $\mathfrak{U}(\mathfrak{h}) \cong S(\mathfrak{h})$ and take this as identification.

Thus we define the algebra homomorphism

$$\mathfrak{hc} := \tau\phi' \colon \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h}). \tag{2.5.3}$$

Let $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the non-degenerate invariant bilinear form given by part 6 of Theorem 2.1.17. This form induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ and subsequently a $\mathcal{W}$-equivariant algebra isomorphism $\theta \colon S(\mathfrak{h}) \to S(\mathfrak{h}^*)$. Thus we define the **Harish-Chandra Homomorphism** to be

$$\mathfrak{hc}_* = \theta \mathfrak{hc} \colon \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h}^*), \tag{2.5.4}$$

which is an algebra homomorphism since each map in the composition is an algebra homomorphism.

Theorem 2.5.7. *Let $\mathfrak{g}$ be either $\mathfrak{gl}(m|n)$ or $\mathfrak{osp}(m|2n)$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{W}$ the corresponding Weyl group. Then*

$$\mathfrak{hc}_*(\mathcal{Z}(\mathfrak{g})) = S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}}.$$

For the proof and further references of Theorem 2.5.7 see [CW12, Sec. 2.2.5].

Example 2.5.8. Consider Theorem 2.5.7 in the case $\mathfrak{g} = \mathfrak{osp}(1|2n)$. From [CW12, Sec. 2.2] it follows that the set $\{\sigma^{2k}_{0,n} \mid k \in \mathbb{Z}_+\}$ generates the algebra $S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}}$, where

$$\sigma^{2k}_{0,n} := \sum_{i=1}^{n} \delta_i^{2k}.$$

Since there are no ε_i that occur we have that

$$S(\mathfrak{h}^*)^{\mathcal{W}}_{\text{sup}} = \{\text{supersymmetric polynomials in } \delta_j^2 \text{ for } 1 \preceq j \preceq n\} = \mathbb{C}[\delta_1^2, \dots, \delta_n^2]^{\mathfrak{S}_n}.$$

Thus by the fundamental theorem of symmetric polynomials [Mac15, Sec. 2.4] we have that

$$S(\mathfrak{h}^*)^W_{\text{sup}} = \langle e_k(\delta_1^2, \dots, \delta_n^2) \mid k = 1, \dots, n \rangle,$$

where for $k = 1, \dots, n$

$$e_k(\delta_1^2, \dots, \delta_n^2) = \sum_{1 \preceq j_1 \prec j_2 \prec \cdots \prec j_k \preceq n} \delta_{j_1}^2 \cdots \delta_{j_k}^2.$$

We call e_k the k^{th} **elementary symmetric polynomials** in variables $\delta_1^2, \dots, \delta_n^2$. In particular, this tells us that $\mathcal{Z}(\mathfrak{g})$ is a polynomial algebra generated by the pre-image of $e_k(\delta_1^2, \dots, \delta_n^2)$, for $1 \leq k \leq n$, by the map (2.5.4).

Chapter 3

Highest Weight Theory and Casimir Operators

In this chapter we would like to work towards specializing to the Lie superalgebra $\mathfrak{osp}(1|2n)$. Keeping that in mind, we present the classification of irreducible finite-dimensional highest-weight modules for the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(2m+1|2n)$. We will finish the chapter by defining Casimir operators of a Lie superalgebra and prove some properties of these Casimir operators. In later chapters, the Casimir operator will prove to be fundamental in relating $\mathcal{Z}(\mathfrak{g})$ to $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{g}}$.

3.1 Irreducible Finite Dimensional Highest Weight Modules of $\mathfrak{gl}(m|n)$

Recall that $\mathfrak{gl}(m|n)$ has Cartan subalgebra $\mathfrak{h} = \text{span}\{E_{i,i} \mid i \in I(m|n)\}$ and positive system from Equation (2.2.3). Hence, there is a triangular decomposition $\mathfrak{gl}(m|n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the subalgebra of strictly upper (resp. strictly lower) triangular matrices. Furthermore, $\mathfrak{h}^* = \text{span}\{\varepsilon_i \mid i \in I(m|n)\}$, where ε_i is dual to $E_{i,i}$ for each $i \in I(m|n)$, and we will adopt the notation $\delta_i = \varepsilon_{i+m}$.

The next theorem can be found in [CW12, Prop. 2.2] and [Kac77, Thm. 8].

Theorem 3.1.1. *Let V be a finite-dimensional $\mathfrak{gl}(m|n)$-module. Then V is irreducible if and only if V is a highest weight module. Moreover, the highest weight V is of the form*

$$\lambda = \sum_{i=1}^{m} \mu_i \varepsilon_i + \sum_{j=1}^{n} \nu_j \delta_j,$$

where $\mu_i - \mu_{i+1}, \nu_j - \nu_{j+1} \in \mathbb{Z}_+$, for all possible i, j.

Definition 3.1.2 (Integer Partition)**.** *An **integer partition** is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers such that*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_\ell.$$

*We say that λ is a **partition of** $k = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ and denote this by $\lambda \vdash k$. Furthermore, the **length** of λ is defined to be $l(\lambda) = \ell$.*

Corollary 3.1.3. *Let μ and ν be integer partitions such that $l(\mu) \leq m$ and $l(\nu) \leq n$. Then there exists a $\mathfrak{gl}(m|n)$-highest weight module with weight*

$$\lambda = \sum_{i=1}^{m} \mu_i \varepsilon_i + \sum_{j=1}^{n} \nu_j \delta_j.$$

The highest weight modules of Corollary 3.1.3 are exactly the modules that can be integrated to irreducible representations of the supergroup $GL(m|n)$.

Example 3.1.4 (Standard Representation of $\mathfrak{gl}(m|n)$)**.** Let $V = \mathbb{C}^{m|n}$ and let $\mathfrak{g} = \mathfrak{gl}(m|n)$ act by left matrix multiplication. Furthermore, let $\{e_i \mid i \in I(m|n)\}$ be the standard basis of V. Observe that for all $x \in \mathfrak{h}$ and $y \in \mathfrak{n}^+$

$$xe_i = \varepsilon_1(x) e_1 \quad \text{and} \quad ye_1 = 0.$$

Thus e_1 is a highest weight vector of V with highest weight ε_1. Furthermore, for any standard basis element $e_j \in V$, notice that $x = E_{j,1} \in \mathfrak{g}$ is such that $xe_1 = e_j$. That is, $V = \operatorname{span}\{xe_1 \mid g \in \mathfrak{g}\}$ and thus V is an irreducible module of highest weight ε_1.

Example 3.1.5 (Adjoint Representation)**.** Let $V = \mathfrak{gl}(m|n)$ be the adjoint representation of $\mathfrak{gl}(m|n)$; see Example 2.1.11. Define the subspace

$$\mathfrak{sl}(m|n) = \{x \in \mathfrak{gl}(m|n) \mid \operatorname{str}(x) = 0\},$$

where $\operatorname{str}\colon \mathfrak{gl}(m|n) \to \mathbb{C}$ is the supertrace defined in Equation (2.2.2). It follows from the linearity and elementary properties of the trace map that $\mathfrak{sl}(m|n)$ is a $\mathfrak{gl}(m|n)$-submodule.

If $m \neq n$, then $I_{m+n} \notin \mathfrak{sl}(m|n)$ and as $\mathfrak{gl}(m|n)$-modules we have the decomposition

$$\mathfrak{gl}(m|n) = \mathbb{C}I_{m+n} \oplus \mathfrak{sl}(m|n).$$

Indeed, the action of $\mathfrak{gl}(m|n)$ on I_{m+n} is trivial. Hence $\mathbb{C}I_{m+n}$ is a highest weight module of weight 0. Furthermore, notice that for all $x \in \mathfrak{h}$ and $y \in \mathfrak{n}^+$

$$\operatorname{ad}_x(E_{1,m+n}) = (\varepsilon_1 - \varepsilon_{m+n})(x)E_{1,m+n} \quad \text{and} \quad \operatorname{ad}_y(E_{1,m+n}) = 0.$$

That is, $E_{1,m+n}$ is a highest weight vector of $\mathfrak{sl}(m|n)$. Furthermore, one can show that $\mathfrak{sl}(m|n) = \operatorname{span}\{\operatorname{ad}_x E_{1,m+n} \mid x \in \mathfrak{gl}(m|n)\}$. Thus under the adjoint action $\mathfrak{gl}(m|n)$ is the direct sum of two highest weight modules of weights 0 and $\varepsilon_1 - \varepsilon_{m+n}$, respectively.

When $m = n$, we will show that $\mathfrak{gl}(m|m)$ cannot be decomposed into the direct sum of subrepresentations. We will explain the argument since we could not find a straightforward reference. In particular, we claim that the only non-zero proper subrepresentations of V are $\mathfrak{sl}(m|m)$ and $\mathbb{C}I_{m+m}$. To this end, a useful and easily verified fact is that for all $x, y \in \mathfrak{gl}(m|m)$ we have $\operatorname{str}([x, y]) = 0$.

Thus if there is a subrepresentation $W \subseteq V$ such that $V = \mathfrak{sl}(m|m) \oplus W$, then $[\mathfrak{gl}(m|m), W] \subseteq \mathfrak{sl}(m|m)$. By assumption $\mathfrak{sl}(m|m) \cap W = 0$ and thus $[\mathfrak{gl}(m|m), W] = 0$. This implies that $W \subseteq \mathcal{Z}(\mathfrak{gl}(m|m)) = \mathbb{C}I_{m+m} \subseteq \mathfrak{sl}(m|m)$, which is a contradiction to $\mathfrak{sl}(m|m) \cap W = 0$.

Now suppose that there is a non-zero subrepresentation $0 \subsetneq W \subsetneq V$ such that $W \neq \mathbb{C}I_{m+m}$. Let $W' := \mathfrak{sl}(m|m) \cap W$. Since we know, from the last paragraph, that $V \neq W \oplus \mathfrak{sl}(m|n)$ and $\operatorname{codim}(\mathfrak{sl}(m|n)) = 1$ then $W' \neq 0$. But W' is a subrepresentation of $\mathfrak{sl}(m|m)$ and subsequently an ideal of $\mathfrak{sl}(m|m)$. It is well known that the only ideals of $\mathfrak{sl}(m|m)$ are $0, \mathbb{C}I_{m+m}$, and $\mathfrak{sl}(m|m)$.

If $W' = \mathfrak{sl}(m|m)$ then $\dim V - 1 = \dim W' \leq \dim W < \dim V$. Therefore, $W = \mathfrak{sl}(m|m)$.

If $W' = \mathbb{C}I_{m+m}$, then since $W \neq \mathbb{C}I_{m+m}$ we can write $W = \operatorname{span}\{I_{m+m}, g_{\bar{0}} + g_{\bar{1}}\}$, with $g_{\bar{i}} \in V_{\bar{i}}$ and $g_{\bar{0}} + g_{\bar{1}} \notin \mathfrak{sl}(m|m)$. Write

$$g_{\bar{0}} + g_{\bar{1}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and since W is a subrepresentation then there are some $a_1, b_1 \in \mathbb{C}$ such that

$$\left[\begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, g_{\bar{0}} + g_{\bar{1}}\right] = \begin{bmatrix} 0 & 2b \\ -2c & 0 \end{bmatrix} = a_1 I_{m+m} + b_1(g_{\bar{0}} + g_{\bar{1}}).$$

Since $g_{\bar{0}} + g_{\bar{1}} \notin \mathfrak{sl}(m|m)$ then $g_{\bar{0}} \notin \mathfrak{sl}(m|m)$. That is, $a_1 = b_1 = 0$, because otherwise $a_1 I_{m+m} + b_1 g_{\bar{0}} \neq 0$, and therefore $b = c = 0$. Thus W is purely even. Keeping this in mind then

$$\left[\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, g_{\bar{0}}\right] = \begin{bmatrix} 0 & a-d \\ d-a & 0 \end{bmatrix} \in W_{\bar{1}}.$$

But then $a - d = 0$ and therefore $g_{\bar{0}} \in \mathfrak{sl}(m|m)$. This gives us a contradiction and therefore $W' \neq \mathbb{C}I_{m+m}$.

That is, if $0 \subsetneq W \subsetneq V$ is a subrepresentation, then $W = \mathbb{C}I_{m+m}$ or $W = \mathfrak{sl}(m|m)$.

3.2 Irreducible Finite Dimensional Highest Weight Modules of $\mathfrak{osp}(2m+1|2n)$

In this section we will explore several examples of $\mathfrak{osp}(2m+1|2n)$-modules, give a complete classification of irreducible finite-dimensional $\mathfrak{osp}(2m+1|2n)$-modules, and show that any finite-dimensional $\mathfrak{osp}(1|2n)$-modules is completely reducible. The reason for focusing on $\mathfrak{osp}(2m+1|2n)$-modules is that the final results of this thesis are related to the representation theory of $\mathfrak{osp}(1|2n)$. For the analogous results for $\mathfrak{osp}(2m|2n)$ see [CW12, Sections 2.1.2 & 2.1.5].

Recall that $\mathfrak{osp}(2m+1|2n)$ has Cartan subalgebra

$$\mathfrak{h} = \mathrm{span}\{E_{i+1,i+1} - E_{i+m+1,i+m+1} \mid 1 \preceq i \preceq m\} \cup \{E_{j,j} - E_{j+n,j+n} \mid \overline{1} \preceq j \preceq \overline{n}\}$$

and we write ε_i and δ_j for the dual vectors to $E_{i+1,i+1} - E_{i+m+1,i+m+1}$ and $E_{j,j} - E_{j+n,j+n}$, respectively. Furthermore, $\mathfrak{osp}(2m+1|2n)$ has fundamental system, Π, given by Equation (2.3.3) and positive system, Φ^+, given by Equation (2.3.4). Now let

$$\mathfrak{osp}(2m+1|2n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

be the corresponding triangular decomposition. In particular, $\mathfrak{n}^+$ is the subalgebra containing all $1 + 2m + 2n$ block matrices of the form

$$\begin{bmatrix} 0 & 0 & -a_2^t & 0 & a_4^t \\ a_2 & b_2 & c_2 & -c_5^t & c_4^t \\ 0 & 0 & -b_2^t & 0 & 0 \\ a_4 & 0 & c_4 & d_4 & e_4 \\ 0 & 0 & c_5 & 0 & -d_4^t \end{bmatrix},$$

where b_2 and d_4 are strictly upper triangular, $c_2 = -c_2^t$, and $e_4 = e_4^t$.

We will give a classification of all finite dimensional irreducible $\mathfrak{osp}(2m+1|2n)$-modules. Before doing so, we consider several examples.

Example 3.2.1 (Standard Representation of $\mathfrak{osp}(2m+1|2n)$)**.** For $m \in \mathbb{Z}_{>0}$, let $V = \mathbb{C}^{2m+1|2n}$ and let $\mathfrak{osp}(2m+1|2n)$ act on V by left matrix multiplication. Take the standard basis of V to be $\{e_i \mid i \in I(2m+1|2n)\}$ and observe that for all $x \in \mathfrak{h}$ and $y \in \mathfrak{n}^+$

$$xe_2 = \varepsilon_1(x)e_2 \quad \text{and} \quad ye_2 = 0.$$

Furthermore, notice that $(E_{1,2} + E_{2+m,1})e_2 = e_1$ and when we consider the basis of $\mathfrak{osp}(2m+1|2n)$ acting on e_1 it is straightforward to see that $V = \text{span}\{xe_2 \mid x \in \mathfrak{osp}(2m+1|2n)\}$. That is, V is an irreducible module of highest weight ε_1.

Example 3.2.2. Let $\mathfrak{osp}(2m+1|2n)$ act on $V = \mathbb{C}^{2m+1|2n}$ as in Example 3.2.1 and define $S^2(V) = T^2(V)/I$, where $I = \text{span}\{v \otimes w - (-1)^{|v||w|}w \otimes v \mid v, w \in V_{\bar{0}} \cup V_{\bar{1}}\}$. The action of $\mathfrak{osp}(2m+1|2n)$ on $V \otimes V$ is given by the standard construction of tensor modules. That is, the action is given by

$$x(v \otimes w) := xv \otimes w + (-1)^{|x||v|}v \otimes xw$$

for all homogeneous $x \in \mathfrak{osp}(2m+1|2n)$, $v, w \in V$, and then extended linearly to non-homogeneous elements. It is straightforward to check that I is a subrepresentation of $V \otimes V$. Therefore the action on $V \otimes V$ induces an action of $\mathfrak{osp}(2m+1|2n)$ on $S^2(V)$. It follows from Example 3.2.1 that for all $x \in \mathfrak{h}$ and $y \in \mathfrak{n}^+$

$$x(e_2 \otimes e_2 + I) = 2\varepsilon_1(x)e_2 \otimes e_2 + I \quad \text{and} \quad y(e_2 \otimes e_2 + I) = 0.$$

Thus $V_{2\varepsilon_1} := \operatorname{span}\{x(e_2 \otimes e_2) + I \mid x \in \mathfrak{osp}(2m+1|2n)\} \subseteq S^2(V)$. The trivial representation V_0 is also a subrepresentation of V. In particular, one can check that the element

$$J = e_1 \otimes e_1 + 2\sum_{i=1}^{m} e_{i+1} \otimes e_{i+m+1} + 2\sum_{j=1}^{n} e_{j+2m+1} \otimes e_{i+n+2m+1} + I \in S^2(\mathbb{C}^{1|2n})$$

is such that $xJ = 0$ for all $x \in \mathfrak{osp}(2m+1|2n)$. It will not be shown here but $S^2(V)$ decomposes as $S^2(V) = V_{2\varepsilon_1} \oplus V_0$, where $V_0 = \mathbb{C}J$.

The last example we explore will be particularly relevant in this thesis. In fact, in Chapter 4 the following example will be used to give a complete decomposition of $S^\ell(V)$ as $\mathfrak{osp}(1|2n)$-modules.

Example 3.2.3. Let $V = \mathbb{C}^{1|2n}$ be the standard representation of $\mathfrak{osp}(1|2n)$ as described in Example 3.2.1. Then the module V has highest weight δ_1 with highest weight vector e_2. Let $\mathfrak{osp}(1|2n)$ act on $S(V) = T(V)/I$, where I is defined as in Example 3.2.2 and the action of $\mathfrak{osp}(2m+1|2n)$ on $S(V)$ is defined as in Example 3.2.2 and extended to higher rank tensors.

For any $1 \le \ell \le n$ consider the element $e_2 \otimes \cdots \otimes e_{\ell+1} + I$. It is a straightforward calculation to show that for any $x \in \mathfrak{h}$ and $y \in \mathfrak{n}^+$

$$x(e_2 \otimes \cdots \otimes e_{\ell+1} + I) = (\delta_1 + \cdots + \delta_\ell)(x)(e_2 \otimes \cdots \otimes e_{\ell+1}) + I$$

and

$$y(e_2 \otimes \cdots \otimes e_{\ell+1} + I) = 0.$$

Thus $e_2 \otimes \cdots \otimes e_{\ell+1} + I$ is a $\mathfrak{osp}(1|2n)$-highest weight vector. Furthermore, in the following subsection we will see that the module generated by this highest weight vector is an irreducible submodule of $S^\ell(V)$.

Irreducible finite-dimensional $\mathfrak{osp}(2m+1|2n)$-modules

Now that we have considered some examples of $\mathfrak{osp}(2m+1|2n)$-modules let us focus on the results needed to give a classification of irreducible finite-dimensional $\mathfrak{osp}(2m+1|2n)$-modules. Throughout this section we fix the standard Cartan subalgebra given by Equation (2.3.2) and positive system Φ^+ given by Equation (2.3.4). In this subsection we will be presenting results without proof. We do this for brevity and we refer the reader to [CW12, Section 2.1.4] where the details of the proofs are given.

Definition 3.2.4 (Dominant and Integral Weights)**.** *Let $\lambda \in \mathfrak{h}^*$.*

*We say λ is **dominant with respect to** $\Phi_{\bar{0}}^+$ (or $\Phi_{\bar{0}}^+$-dominant) if $\lambda(H_\alpha) \geq 0$, for all $\alpha \in {\Phi_{\bar{0}}}^+$.*

*Similarly, λ is **integral** if $2\frac{\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$, for all roots $\alpha \in \Phi$.*

When it is clear which set $\Phi_{\bar{0}}^+$ we are working with we will say that λ is dominant, without reference to $\Phi_{\bar{0}}^+$.

Lemma 3.2.5. *Let V_λ be an irreducible $\mathfrak{osp}(2m+1|2n)$-module of highest weight λ. If V_λ is finite dimensional, then λ is a $\Phi_{\bar{0}}^+$-dominant integral.*

See [CW12, Lemma 2.7] for details of the proof of Lemma 3.2.5.

Lemma 3.2.6. *A weight $\lambda = \sum_{i=1}^m \lambda_i \varepsilon_i + \sum_{j=1}^n \nu_j \delta_j$ is $\Phi_{\bar{0}}^+$-dominant integral if and only if*

1. *$\lambda_i \geq \lambda_{i+1}$ with either (a) all $\lambda_i \in \mathbb{Z}_{\geq 0}$, or (b) all $\lambda_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$.*

2. *ν is a partition.*

Lemma 3.2.6 is a consequence of the dominance integral condition for the Lie algebras $\mathfrak{o}(2m+1)$ and $\mathfrak{sp}(2n)$; see [GW09, Sec. 3.1.4].

When $\lambda_i \in \mathbb{Z}_+$, for all i, we say that λ is an **integer weight**. If $\lambda_i \in \frac{1}{2} + \mathbb{Z}_+$ for all i, then we say λ is a **half-integer weight**.

Definition 3.2.7 (Hook Partition)**.** *Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition. Then μ is called an $(n|m)$-**hook partition**, if $\mu_{n+1} \leq m$.*

Given a $(n|m)$-hook partition, μ, set $\nu = (\nu_1, \dots, \nu_m)$, where ν_j is the number of parts μ_ℓ such that $\mu_\ell \geq j$. Set

$$\mu^\natural = \sum_{i=1}^{m} \nu_i \varepsilon_i + \sum_{j=1}^{n} \max\{\mu_j - m, 0\}\delta_j.$$

Proposition 3.2.8. *Let $\lambda = \sum_{i=1}^m \lambda_i \varepsilon_i + \sum_{j=1}^n \nu_j \delta_j \in \mathfrak{h}^*$ be a $\Phi_{\bar{0}}^+$-dominant integral weight and let V_λ be the corresponding irreducible highest weight module. Then V_λ is finite dimensional if and only if either*

- *λ is a half-integer weight and $\lambda_m \geq n$, or*
- *λ is of the form $\mu^\natural$, with μ a $(n|m)$-hook partition.*

We will prove Proposition 3.2.8 for the family $\mathfrak{osp}(1|2n)$. For a complete proof of Proposition 3.2.8 see [CW12, Thm. 2.11] or [Mus12, Sec. 4].

Proof: Notice that the half-integer condition becomes vacuous. Therefore we are showing that V_λ is finite dimensional if and only if $\lambda = \sum_{j=1}^n \nu_j \delta_j$, for $\nu = (\nu_1, \dots, \nu_n)$ a partition.

Lemma 3.2.5 and Lemma 3.2.6 imply that if V_λ is finite dimensional, then $\lambda = \sum_{j=1}^n \nu_j \delta_j$, for $\nu = (\nu_1, \dots, \nu_n)$ a partition.

Conversely, suppose that $\lambda = \sum_{j=1}^{n} \nu_j \delta_j$, for $\nu = (\nu_1, \ldots, \nu_n)$ a partition. Let $\mu = (\mu_1, \mu_2, \ldots)$ be the conjugate partition of ν. We have seen in Example 3.2.3 that $S^{\mu_\ell}(\mathbb{C}^{1|2n})$ contains a $\mathfrak{osp}(1|2n)$-submodule of highest weight $\delta_1 + \cdots + \delta_{\mu_\ell}$ and highest weight vector v_{μ_ℓ}. Consider the inner tensor product module

$$S^{\mu_1}(\mathbb{C}^{1|2n}) \otimes \cdots \otimes S^{\mu_\ell}(\mathbb{C}^{1|2n}).$$

Clearly, this space is finite dimensional and it is easy to verify that $v_{\mu_1} \otimes \cdots \otimes v_{\mu_\ell}$ is a highest weight vector of weight λ. Thus we can conclude that V_λ is finite dimensional. ■

Complete Reducibility of $\mathfrak{osp}(1|2n)$

Lastly, in the remainder of this section we focus on the complete reducibility of $\mathfrak{osp}(1|2n)$. We will present general results for $\mathfrak{osp}(2m+1|2n)$ and then specialize to $\mathfrak{osp}(1|2n)$. In this section, let $\mathfrak{osp}(2m+1|2n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition associated to Φ^+ given by Equation (2.3.4). Write

$$\mathfrak{U}(\mathfrak{osp}(2m+1|2n)) = \mathfrak{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathfrak{U}(\mathfrak{osp}(2m+1|2n)) + \mathfrak{U}(\mathfrak{osp}(2m+1|2n))\mathfrak{n}^+)$$

and recall the projection map from Equation (2.5.2)

$$\phi\colon \mathfrak{U}(\mathfrak{osp}(2m+1|2n)) \to \mathfrak{U}(\mathfrak{h})$$

and its restriction to the algebra homomorphism

$$\phi'\colon \mathcal{Z}(\mathfrak{osp}(2m+1|2n)) \to \mathfrak{U}(\mathfrak{h}).$$

Lemma 3.2.9. *Let V_λ be a finite-dimensional $\mathfrak{osp}(2m+1|2n)$-module of highest weight*

$\lambda \in \mathfrak{h}^*$. *Then every* $z \in \mathcal{Z}(\mathfrak{osp}(2m+1|2n))$ *acts on* V_λ *by* $\lambda(\phi'(z))$.

Proof: Pick a basis of $\mathfrak{osp}(2m+1|2n)$ consisting of only weight vectors and apply Theorem 2.4.4 to write any $z \in \mathcal{Z}(\mathfrak{osp}(2m+1|2n))$ as

$$z = h_z + \sum_\ell n_\ell^- h_\ell n_\ell^+,$$

where $h_z, h_\ell \in \mathfrak{U}(\mathfrak{h})$, $n_\ell^\pm \in \mathfrak{U}(\mathfrak{n}^\pm)$ are monomials in weight vectors. Notice that for all $x \in \mathfrak{h}$

$$\begin{aligned}[x, z] &= [x, h_z] + \sum_\ell [x, n_\ell^- h_\ell n_\ell^+] \\ &= 0 + \sum_\ell [x, n_\ell^-] h_\ell n_\ell^+ + n_\ell^- h_\ell [x, n_\ell^+] \\ &= \sum_\ell (\alpha_\ell^- + \alpha_\ell^+) n_\ell^- h_\ell n_\ell^+,\end{aligned}$$

for $\alpha_\ell^\pm$ some constant dependent on the monomials $n_\ell^\pm$. But z is central and therefore $[x, z] = 0$. That is, $\alpha_\ell^- + \alpha_\ell^+ = 0$, for each ℓ, and therefore

$$\sum_\ell n_\ell^- h_\ell n_\ell^+ \in \mathfrak{n}^- \mathfrak{U}(\mathfrak{osp}(2m+1|2n)) \cap \mathfrak{U}(\mathfrak{osp}(2m+1|2n)) n^+.$$

Let v_λ be a highest weight vector for V_λ. Then for any $x \in \mathfrak{osp}(2m+1|2n)$ and $z \in \mathcal{Z}(\mathfrak{osp}(2m+1|2n))$ we have the following

$$\begin{aligned}z(xv_\lambda) &= x(h_z + \sum_\ell n_\ell^- h_\ell n_\ell^+) v_\lambda \\ &= \lambda(h_z) x v_\lambda + \sum_\ell n_\ell^- h_\ell n_\ell^+ v_\lambda \\ &= \lambda(h_z) x v_\lambda = \lambda(\phi'(z)) x v_\lambda.\end{aligned}$$

Thus we have that any $z \in \mathcal{Z}(\mathfrak{osp}(2m+1|2n))$ acts on V_λ by $\lambda(\phi'(z))$. ■

The next definition can be posed for any basic classical Lie superalgebra, $\mathfrak{g}$. Since this section focuses on $\mathfrak{osp}(2m+1|2n)$ we will assume that $\mathfrak{g}$ is $\mathfrak{osp}(2m+1|2n)$ to keep our presentation seamless.

Definition 3.2.10 (Infinitesimal Character)**.** *For any finite dimensional $\mathfrak{osp}(2m+1|2n)$-module, V_λ, of highest weight $\lambda \in \mathfrak{h}^*$, the **infinitesimal character** of V_λ is the map*

$$\begin{aligned}\chi_\lambda \colon \mathcal{Z}(\mathfrak{g}) &\to \mathbb{C}\\ z &\mapsto \lambda(\phi'(z)).\end{aligned}$$

Recall a root $\alpha \in \mathfrak{h}^*$ is called **isotropic** if $\langle \alpha, \alpha \rangle = 0$.

Example 3.2.11. For $\mathfrak{osp}(2m+1|2n)$, we check for isotropic roots:

$$\begin{aligned}\langle \varepsilon_i, \varepsilon_i \rangle &= -\langle \delta_j, \delta_j \rangle = \frac{1}{2},\\ \langle \varepsilon_i \pm \varepsilon_k, \varepsilon_i \pm \varepsilon_k \rangle &= -\langle \delta_j \pm \delta_\ell, \delta_j \pm \delta_\ell \rangle = 1,\\ \langle \varepsilon_i \pm \delta_j, \varepsilon_i \pm \delta_j \rangle &= 0.\end{aligned}$$

Thus $\{\pm(\varepsilon_i \pm \delta_j) \mid 1 \preceq i \preceq m,\ 1 \preceq j \preceq n\}$ is the set of all isotropic roots for $\mathfrak{osp}(2m+1|2n)$.

Recall that the Weyl vector, $\rho \in \mathfrak{h}^*$, defined in Equation (2.5.1) is

$$\rho = \frac{1}{2}(\rho_0 - \rho_1),$$

where ρ_i is the sum of all roots in Φ_i^+.

Definition 3.2.12 (Linked Weights)**.** *For $\lambda, \mu \in \mathfrak{h}^*$ we say that $\lambda \sim \mu$ if there exist $w \in \mathcal{W}$, $c_1, \ldots, c_\ell \in \mathbb{C}$, and mutually orthogonal isotropic odd roots $\alpha_1, \ldots, \alpha_\ell$ such* *that*

$$\mu + \rho = w\left(\lambda + \rho - \sum_{i=1}^{\ell} c_i\alpha_i\right) \quad \text{and} \quad \langle \lambda + \rho, \alpha_i \rangle = 0,\ i = 1, \ldots, \ell.$$

*When $\lambda \sim \mu$ we say that λ and μ are **linked**.*

The following theorem can be found with proof in [CW12, Theorem 2.27].

Theorem 3.2.13. *For any $\lambda, \mu \in \mathfrak{h}^*$, λ is linked to μ if and only if $\chi_\lambda = \chi_\mu$.*

Definition 3.2.14 (Typical Module)**.** *A finite-dimensional irreducible $\mathfrak{osp}(2m+1|2n)$-module V_λ of highest weight λ is called **typical** if $\langle \lambda + \rho, \alpha \rangle \neq 0$, for every isotropic root α.*

Lemma 3.2.15. *Let V_λ be a finite-dimensional irreducible $\mathfrak{osp}(2m+1|2n)$-module of highest weight λ. If λ is typical, then V_λ is the unique (up to isomorphism) irreducible finite dimensional $\mathfrak{osp}(2m+1|2n)$-module with infinitesimal character χ_λ.*

Proof: By Theorem 3.2.13, it suffices to show that if $\mu \in \mathfrak{h}^*$ is a highest weight linked to λ and associated to a finite dimensional irreducible representation of weight μ, then $\mu = \lambda$. Since λ is typical there are no isotropic α such that $\langle \lambda + \rho, \alpha \rangle = 0$, where ρ is defined in Equation (2.5.1). Therefore, there is some $w \in \mathcal{W}$ such that

$$\mu + \rho = w(\lambda + \rho).$$

Since $\mathcal{W}$ is generated by even reflections we have that $w\rho_1 = \rho_1$, for all $w \in \mathcal{W}$. That is,

$$\mu + \frac{1}{2}\rho_0 = w\left(\lambda + \frac{1}{2}\rho_0\right).$$

Thus we are in the standard Lie algebra situation. For standard Lie algebras there is the fact: If x, y are elements of the closure of the fundamental Weyl chamber corresponding to a positive system Π (of a Lie algebra) and there is some $w \in \mathcal{W}$ such that $y = wx$, then $x = y$. For an argument on this fact see [Hum78, Lemma B pg. 52].

But $\mu + \frac{1}{2}\rho_0$ and $\lambda + \frac{1}{2}\rho_0$ are in the fundamental Weyl chamber of $\mathfrak{osp}(2m+1|2n)_{\bar{0}}$ if for all $\varepsilon_i - \varepsilon_{i+1}, \delta_j - \delta_{j+1} \in \Pi$

$$\left\langle \mu + \frac{1}{2}\rho_0, \varepsilon_i - \varepsilon_{i+1} \right\rangle, \left\langle \lambda + \frac{1}{2}\rho_0, \varepsilon_i - \varepsilon_{i+1} \right\rangle \geq 0,$$
$$-\left\langle \mu + \frac{1}{2}\rho_0, \delta_j - \delta_{j+1} \right\rangle, -\left\langle \lambda + \frac{1}{2}\rho_0, \delta_j - \delta_{j+1} \right\rangle \geq 0.$$

Here the negative sign is so that $\langle \cdot, \cdot \rangle$ forms a Euclidean space on $\mathfrak{h}^*$. This is straightforward to check by a direct computation. Therefore $\mu + \rho_0$ and $\lambda + \rho_0$ are elements of the fundamental domain and it follows that $\mu = \lambda$. ■

Lemma 3.2.16. *Let V be a finite dimensional $\mathfrak{osp}(2m+1|2n)$-module with composition series*

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_r = V$$

and let L_λ be an irreducible $\mathfrak{osp}(2m+1|2n)$-module. If each subquotient is isomorphic to L_λ, then V is a direct sum of modules isomorphic to L_λ.

Proof: We prove this by induction on r. Clearly, for $r = 1$ then $V \cong L_\lambda$. Now suppose that the statement is true for $r - 1$ and V has composition series

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_r = V.$$

Thus M_r/M_1 has composition series of length $r - 1$ and our induction hypothesis implies that $M_r/M_1 = N_1 \oplus \cdots \oplus N_{r-1}$,where $N_i \cong L_\lambda$ for all $1 \leq i \leq r - 1$. Let

$n_i + M_1$ be the highest weight vector of N_i.

Note that $\mathfrak{g}_{\bar{0}} \cong \mathfrak{o}(2m+1) \oplus \mathfrak{sp}(2n)$ and therefore $\mathfrak{g}_{\bar{0}}$ is semi-simple. It then follows that as a $\mathfrak{g}_{\bar{0}}$-module V can be written as the direct sum of weight spaces. In particular, the weight spaces of V correspond with those of L_λ, otherwise if a weight occurs in V but not in L_λ then this weight must occur in one of the subquotients. Thus let I be the set of weights of L_λ and write each $n_i = n_{i,\lambda} + \sum_{\mu<\lambda} n_{i,\mu}$, where $n_{i,\nu}$ is a weight vector of weight $\nu \in I$. Notice that for all $h \in \mathfrak{h}$

$$hn_i = \lambda(h)n_{i,\lambda} + \sum_{\mu<\lambda} \mu(h)n_{i,\mu}.$$

Now pick $h \in \mathfrak{h}$ such that the numbers $\lambda(h)$ and $\mu(h)$, for $\mu \in I \backslash \{\lambda\}$, are all distinct. We can then consider the elements

$$\begin{aligned} n_i &= n_{i,\lambda} + \sum_{\mu<\lambda} n_{i,\mu} \\ hn_i &= \lambda(h)n_{i,\lambda} + \sum_{\mu<\lambda} \mu(h)n_{i,\mu} \\ h^2 n_i &= \lambda(h)^2 n_{i,\lambda} + \sum_{\mu<\lambda} \mu(h)^2 n_{i,\mu} \\ \vdots\ &= \ \vdots \qquad\qquad \vdots \\ h^{|I|-1} n_i &= \lambda(h)^{|I|-1} n_{i,\lambda} + \sum_{\mu<\lambda} \mu(h)^{|I|-1} n_{i,\mu}. \end{aligned}$$

We can rewrite this system of linear equations as the $Ax = b$,where we define the matrix $A = [\beta(h)^j]_{\beta\in I, 0\le j\le |I|-1}$, $x = [n_{i,\beta}]_{\beta\in I}$, and $b = [h^j n_i]_{0\le j\le |I|-1}$. Notice that A is a Vandermonde matrix and is therefore invertible. In particular this implies that each $n_{i\mu}$ can be written as a linear combination of the $\{h^j n_i \mid 0 \le j \le |I| - 1\}$. Now take

$n_{i,\mu}$ and write

$$n_{i,\mu} = \sum_{j=0}^{|I|-1} a_{j,\mu} h^j n_i.$$

Therefore we have that

$$\mu(h) n_{i,\mu} = \sum_{j=0}^{|I|-1} a_{j,\mu} h^{j+1} n_i$$

and therefore

$$\begin{aligned} \mu(h) n_{i,\mu} + M_1 &= \lambda(h) \left(\sum_{j=0}^{|I|-1} a_{j,\mu} \lambda(h)^j n_i + M_1 \right) \\ &= \lambda(h) \left(n_{i,\mu} \right) + M_1. \end{aligned}$$

But $\lambda(h) \neq \mu(h)$ and therefore $n_{i,\mu} + M_1 = 0$. That is, $n_{i,\mu} \in M_1$ for each $\mu < \lambda$. Thus we can disregard the $n_{i,\mu}$ and assume the n_i is of weight λ. That is, each n_i is a highest weight vector of weight λ. Furthermore, the set of n_i is a linearly independent set since $n_i + M_1$ generate distinct modules N_i. Thus we can write

$$\begin{aligned} V &= M_1 \oplus \left(\bigoplus_{i=1}^{r-1} \mathfrak{U}(\mathfrak{g}) n_i \right) \\ &\cong L_\lambda^{\oplus(r)}. \end{aligned}$$

Therefore, we have that if each subquotient of V is isomorphic to L_λ then V is a direct sum of modules isomorphic to L_λ. ■

Theorem 3.2.17. *Let V be a finite-dimensional representation of $\mathfrak{gosp}(2m+1|2n)$. Then all typical representations of $\mathfrak{g}$ appearing as subquotients of V split off as direct* *summands of* V.

Proof: Since V is finite dimensional we can write

$$V = \bigoplus_{\lambda \in I} V_\lambda,$$

where $I \subseteq \mathfrak{h}^*$ is a finite set and

$$V_\lambda = \left\{ v \in V \mid \text{ for all } z \in \mathcal{Z}(\mathfrak{g}),\ (z - \chi_\lambda(z))^N v = 0,\ \text{for some } N = N(z) \in \mathbb{N} \right\}.$$

This is a slight generalization of the primary decomposition from linear algebra and is a consequence of commutativity of $\mathcal{Z}(\mathfrak{g})$ and the fact that V is finite-dimensional. Notice that each V_λ is $\mathfrak{g}$-invariant and finite dimensional. Take typical $\lambda \in I$ and a composition series

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_r = V_\lambda.$$

The space M_i/M_{i-1}, for $1 \leq i \leq r$, is irreducible and therefore by Proposition 3.2.8 is a highest weight space of weight μ, for some μ. If $\lambda \neq \mu$ then take $z \in \mathcal{Z}(\mathfrak{g})$ such that $\chi_\lambda(z) \neq \chi_\mu(z)$. Let $v_\mu + M_{i-1}$ be a highest weight vector of M_i/M_{i-1}. Then there is some $N \in \mathbb{N}$ such that $(z - \chi_\lambda(z))^N v_\lambda = 0$ and therefore

$$\begin{aligned} 0 + M_i &= (z - \chi_\lambda(z))^N (v_\mu + M_i) \\ &= (\chi_\mu(z) - \chi_\lambda(z))^N v_\mu + M_i. \end{aligned}$$

But this would imply that $(\chi_\mu(z) - \chi_\lambda(z))^N v_\lambda \in M_i$ and therefore $v_\mu \in M_i$. This is a contradiction and therefore $\lambda = \mu$. That is, all the subquotients of V_λ are isomorphic to L_λ. We then apply Lemma 3.2.16 to decompose

$$V_\lambda = V_{\lambda,1} \oplus \cdots \oplus V_{\lambda,s_\lambda},$$

for some $s_\lambda \in \mathbb{N}$. ■

Corollary 3.2.18. *All finite-dimensional $\mathfrak{osp}(1|2n)$-modules are completely reducible.*

Proof: Example 3.2.11 gives us all isotropic roots of $\mathfrak{osp}(2m+1|2n)$. For $m = 0$ there are no isotropic roots and therefore all irreducible $\mathfrak{osp}(1|2n)$-modules are typical, thus we apply Theorem 3.2.17 to any finite-dimensional $\mathfrak{osp}(1|2n)$-module. ■

3.3 Casimir Operators

In this section we define the Casimir operator of a Lie superalgebra, prove several properties of this operator, and finally compute the Casimir operator for $\mathfrak{osp}(m|2n)$. Throughout this section we let $\mathfrak{g}$ be an n-dimensional Lie superalgebra equipped with even invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, homogeneous basis $\{X_i\}_{i=1}^n$, and the corresponding dual basis $\{X^i\}_{i=1}^n$.

Definition 3.3.1 (Casimir Operator)**.** *The **Casimir operator** of $\mathfrak{g}$ is the element of $\mathfrak{U}(\mathfrak{g})$ given by*

$$C := \sum_{i=1}^{n} (-1)^{|X_i|} X_i X^i.$$

Although not explicitly clear in the definition, it is worthwhile to note that the Casimir operator of $\mathfrak{g}$ is dependent on the invariant non-degenerate bilinear form. Furthermore, we will show that C is independent of the choice of a homogeneous basis.

Proposition 3.3.2. *Let $\mathfrak{g}$ be an n-dimensional Lie superalgebra equipped with invariant non-degenerate bilinear form, $\langle \cdot, \cdot \rangle$. Then the Casimir element, C, is independent of the basis $\{X_i\}_{i=1}^n$ that is chosen.*

Proof: Let $\{Y_i\}_{i=1}^n$ be a basis of homogeneous elements of $\mathfrak{g}$. There is an even invertible matrix $[a_{i,j}]$ given by the relations

$$Y_i = \sum_{k=1}^{n} a_{k,i} X_k.$$

Let $[b_{i,j}]$ be the even inverse matrix to $[a_{i,j}]$ and for $0 \leq i \leq n$ define

$$Y^i := \sum_{k=1}^{n} b_{i,k} X^k.$$

Notice that

$$\begin{aligned} \langle Y_i, Y^j \rangle &= \sum_{k,\ell=1}^{n} a_{k,i} b_{j,\ell} \langle X_k, X^\ell \rangle \\ &= \sum_{k=1}^{n} a_{k,i} b_{j,k} = \delta_{i,j}. \end{aligned}$$

Thus $\{Y^i\}_{i=1}^n$ is dual to $\{Y_i\}_{i=1}^n$. Since both $[a_{i,j}]$ and $[b_{i,j}]$ are block diagonal matrices we have the following calculations

$$\begin{aligned} \sum_{i=1}^{n} (-1)^{|Y_i|} Y_i Y^i &= \sum_{i=1}^{n} (-1)^{|Y_i|} \left(\sum_{k=1}^{n} a_{k,i} X_k \right) \left(\sum_{\ell=1}^{n} b_{i,\ell} X^\ell \right) \\ &= \sum_{i,k,\ell} (-1)^{|Y_i|} a_{k,i} b_{i,\ell} X_k X^\ell \\ &= \sum_{k,\ell} \left(\sum_{i=1}^{n} (-1)^{|X_k|} a_{k,i} b_{i,\ell} \right) X_k X^\ell \\ &= \sum_{k,\ell} (-1)^{|X_k|} \delta_{k,\ell} X_k X^\ell \\ &= \sum_{k=1}^{n} (-1)^{|X_k|} X_k X^k \\ &= C. \end{aligned}$$

This shows that the Casimir element is independent of choice of basis. ■

Proposition 3.3.3. *Let $\mathfrak{g}$ be as in Proposition 3.3.2. The Casimir operator, C, of $\mathfrak{g}$ is in the center of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$.*

Proof: Take any $y \in \mathfrak{g}_{\overline{0}} \cup \mathfrak{g}_{\overline{1}}$ and for all $0 \leq j \leq n$ write

$$[y, X_j] = \sum_{k=1}^{n} a_{k,j} X_k$$
$$[y, X^j] = \sum_{k=1}^{n} b_{k,j} X^k,$$

where $a_{k,j}, b_{k,j} \in \mathbb{C}$ and $a_{k,j} = b_{k,j} = 0$ when $|X_k| \neq |y| + |X_j|$. Furthermore, for $0 \leq j, k \leq n$,

$$\begin{aligned} a_{k,j} &= \langle [y, X_j], X^k \rangle \\ &= -(-1)^{|y||X_j|} \langle [X_j, y], X^k \rangle \\ &= -(-1)^{|y||X_j|} \langle X_j, [y, X^k] \rangle \\ &= -(-1)^{|y||X_j|} b_{j,k}. \end{aligned}$$

Moreover, for $0 \leq j \leq n$

$$\begin{aligned} X_j X^j y - y X_j X^j &= X_j (X^j y - (-1)^{|X^j||y|} y X^j) - (y X_j - (-1)^{|X_j||y|} X_j y) X^j \\ &= X_j [X^j, y] - [y, X_j] X^j. \end{aligned}$$

Thus we have

$$Cy - yC = \sum_{j=1}^{n} (-1)^{|X_j|} (X_j [X^j, y] - [y, X_j] X^j)$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{n} -(-1)^{|y||X^j|+|X_j|} b_{k,j} X_j X^k - (-1)^{|X_j|} a_{k,j} X_k X^j \\
&= \sum_{j,k=1}^{n} (-1)^{|X_k|} a_{j,k} X_j X^k - (-1)^{|X_j|} a_{k,j} X_k X^j \\
&= \sum_{j,k=1}^{n} (-1)^{|X_j|} a_{k,j} X_k X^j - \sum_{j,k=1}^{n} (-1)^{|X_j|} a_{k,j} X_k X^j \\
&= 0.
\end{aligned}
$$

Since this is for any $y \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$, which generate $\mathfrak{U}(\mathfrak{g})$ as an algebra, it follows that $C \in \mathcal{Z}(\mathfrak{g})$. ■

Proposition 3.3.4. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra with non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. The Casimir operator, C, acts on an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ by the scalar*

$$\langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle,$$

where ρ is Weyl vector as defined in Equation (2.5.1).

Proof: Let v_λ be a highest weight vector of V_λ, an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ.

Using Theorem 2.1.17, $\mathfrak{g}$ can be decomposed into the direct sum of 1-dimensional weight spaces of distinct weights. That is, if $\{H_i\}_{i \in I}$ is a basis for $\mathfrak{h}$, then $\{H_i\}_{i \in I} \cup \{e_\alpha\}_{\alpha \in \Phi}$ is a basis for $\mathfrak{g}$. Furthermore, since $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ unless $\alpha = -\beta$ then by scaling each e_α appropriately we have that $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha,-\beta}$, for $\alpha \in \Phi^+$. It follows that $\langle e_\alpha, (-1)^{|\beta|} e_\beta \rangle = \delta_{\alpha,-\beta}$, for $\alpha \in \Phi^-$. Theorem 2.1.17 tells us that $[e_\alpha, e_{-\alpha}] =$

$\langle e_\alpha, e_{-\alpha}\rangle H_\alpha$ then

$$\begin{aligned} C &= \sum_{i\in I} \frac{H_i^2}{\langle H_i, H_i\rangle} + \sum_{\alpha\in\Phi^+} (-1)^{|\alpha|} e_\alpha e_{-\alpha} + \sum_{\alpha\in\Phi^+} e_{-\alpha} e_\alpha \\ &= \sum_{i\in I} \frac{H_i^2}{\langle H_i, H_i\rangle} + \sum_{\alpha\in\Phi^+} 2 e_{-\alpha} e_\alpha + \sum_{\alpha\in\Phi^+} (-1)^{|\alpha|} H_\alpha. \end{aligned}$$

Now apply C to v_λ to get the following

$$\begin{aligned} Cv_\lambda &= \sum_{i\in I} \frac{H_i^2}{\langle H_i, H_i\rangle} v_\lambda + \sum_{\alpha\in\Phi^+} 2 e_{-\alpha} e_\alpha v_\lambda + \sum_{\alpha\in\Phi^+} (-1)^{|\alpha|} H_\alpha v_\lambda \\ &= \sum_{i\in I} \frac{\lambda(H_i)^2}{\langle H_i, H_i\rangle} v_\lambda + \sum_{\alpha\in\Phi_0^+} (-1)^{|\alpha|} \lambda(H_\alpha) v_\lambda \\ &= (\langle \lambda, \lambda\rangle + \langle \lambda, \rho_0 - \rho_1\rangle) v_\lambda. \end{aligned}$$

Since v_λ generates V_λ and C is central in $\mathcal{Z}(\mathfrak{g})$ it follows that C acts on V_λ by $\langle \lambda+\rho, \lambda+\rho\rangle - \langle \rho, \rho\rangle$. ■

Recall from Section 2.3 that $\mathfrak{g} = \mathfrak{osp}(m|2n)$ is defined using $J = \mathrm{diag}(J_1, J_2) \in \mathrm{Mat}_{m|2n}(\mathbb{C})$ with $J_1 = J_1^T$, $J_2 = -J_2^T$, and J_1 and J_2 both invertible. Furthermore, recall the vector space isomorphism between $\mathfrak{osp}(m|2n)$ and $\mathrm{Skew}(m|2n)$ given in Equation (2.3.1) by

$$\begin{aligned} \Psi\colon\ \mathrm{Skew}(\mathbb{C}^{m|2n}) &\to \mathfrak{osp}(m|2n) \\ X &\to J^{-1}X. \end{aligned}$$

Proposition 3.3.5 (Casimir Operator for $\mathfrak{osp}(m|2n)$)**.** *The map*

$$\langle X, Y\rangle := \mathrm{str}(J^{-1}XJ^{-1}Y),$$

defined for all $X, Y \in \mathrm{Skew}(\mathbb{C}^{m|2n})$ *is an invariant non-degenerate even supersymmetric bilinear form on* $\mathrm{Skew}(\mathbb{C}^{m|2n})$ *and therefore on* $\mathfrak{osp}(m|2n)$*. Furthermore, with respect to this bilinear form, the Casimir element for* $\mathfrak{osp}(m|2n)$ *is given by the image under* Ψ *of the element*

$$C = -\frac{1}{2}\sum_{1\preceq i\prec j\preceq m} L_{i,j}L^*_{i,j} + \frac{1}{2}\sum_{\overline{1}\preceq i\prec j\preceq \overline{2n}} L_{i,j}L^*_{i,j} + \frac{1}{4}\sum_{\overline{1}\preceq i\preceq \overline{2n}} L_{i,i}L^*_{i,i} + \frac{1}{2}\sum_{\substack{1\preceq i\preceq m\\ \overline{1}\preceq j\preceq \overline{2n}}} L_{i,j}L^*_{i,j},$$

where $L_{i,j} := E_{i,j} - (-1)^{|i||j|}E_{j,i}$ *and* $L^*_{i,j} := JL_{i,j}J^T$ *for* $i, j \in I(m|2n)$*.*

Proof: It is clear that $\langle\cdot,\cdot\rangle$ is a nondegenerate bilinear form. Furthermore, invariance can be seen by taking any $X, Y, Z \in \mathfrak{g}$ and the computation

$$\begin{aligned}
\langle [X,Y], Z\rangle &= \mathrm{str}([X,Y]Z)\\
&= \mathrm{str}(XYZ) - (-1)^{|X||Y|}\,\mathrm{str}(YXZ)\\
&= \mathrm{str}(XYZ) - (-1)^{|X||Y|+|Y|(|X|+|Z|)}\,\mathrm{str}(XZY)\\
&= \mathrm{str}(XYZ - (-1)^{|Y||Z|}XZY)\\
&= \mathrm{str}(X[Y,Z]) = \langle X, [Y,Z]\rangle.
\end{aligned}$$

For $i, j \in I(m|2n)$ set $L_{i,j} := E_{i,j} - (-1)^{|i||j|}E_{j,i}$ and $L^*_{i,j} := JL_{i,j}J^T$. Using the facts that $L_{i,j} = -(-1)^{|i||j|}L_{j,i}$ and $L^*_{i,j} = \sum_{a,b} J_{a,i}J_{b,j}L_{a,b}$, we have the following:

- For $1 \preceq i \prec j \preceq m$ and $1 \preceq k \prec \ell \preceq m$

$$\begin{aligned}
\langle L_{i,j}, L^*_{k,\ell}\rangle &= \mathrm{str}(J^{-1}L_{i,j}J^{-1}L^*_{i,j}) = \mathrm{tr}(J_1^{-1}(E_{i,j}-E_{j,i})(E_{k,\ell}-E_{\ell,k})J_1^T)\\
&= \mathrm{tr}((E_{i,j}-E_{j,i})(E_{k,\ell}-E_{\ell,k})J_1^TJ_1^{-1})\\
&= \mathrm{tr}((E_{i,j}-E_{j,i})(E_{k,\ell}-E_{\ell,k}))\\
&= \begin{cases} 0 & \text{if } (i,j)\neq(k,\ell) \text{ and}\\ -2 & \text{if } (i,j)=(k,\ell).\end{cases}
\end{aligned}$$

- For $\overline{1} \preceq i \preceq j \preceq \overline{2n}$ and $\overline{1} \preceq k \preceq \ell \preceq \overline{2n}$

$$\langle L_{i,j}, L^*_{k,\ell}\rangle = \begin{cases} 0 & \text{if } (i,j) \neq (k,\ell), \\ 2 & \text{if } (i,j) = (k,\ell) \text{ and } i \neq j, \text{ and} \\ 4 & \text{if } (i,j) = (k,\ell) \text{ and } i = j. \end{cases}$$

- For $1 \preceq i \preceq m$, $\overline{1} \preceq j \preceq \overline{2n}$ and $k, \ell \in I(m|2n)$

$$\langle L_{i,j}, L^*_{k,\ell}\rangle = \begin{cases} 0 & \text{if } (i,j) \neq (k,\ell), \text{ and} \\ -2 & \text{if } (i,j) = (k,\ell). \end{cases}$$

In particular, this says that the basis consisting of the $L_{i,j}$, which generates Skew$(m|2n)$, has dual basis given in Table 3.1:

Index of $L_{i,j}$	Associated dual element
$1 \preceq i, j \preceq m$	$\frac{-1}{2} L^*_{i,j}$
$\overline{1} \preceq i \prec j \preceq \overline{2n}$	$\frac{1}{2} L^*_{i,j}$
$\overline{1} \preceq i \preceq \overline{2n}$	$\frac{1}{4} L^*_{i,i}$
$1 \preceq i \preceq m$ and $\overline{1} \preceq j \preceq \overline{2n}$	$\frac{-1}{2} L^*_{i,j}$

Table 3.1: Dual Basis of Skew$(m|2n)$

That is, the Casimir for $\mathfrak{osp}(m|2n)$ is given by the image of

$$C = -\frac{1}{2} \sum_{1 \preceq i \prec j \preceq m} L_{i,j} L^*_{i,j} + \frac{1}{2} \sum_{\overline{1} \preceq i \prec j \preceq \overline{2n}} L_{i,j} L^*_{i,j} + \frac{1}{4} \sum_{\overline{1} \preceq i \preceq \overline{2n}} L_{i,i} L^*_{i,i} + \frac{1}{2} \sum_{\substack{1 \preceq i \preceq m \\ \overline{1} \preceq j \preceq \overline{2n}}} L_{i,j} L^*_{i,j},$$

by the map induced from Equation (2.3.1). ■

Chapter 4

Spherical Harmonics for $\mathfrak{osp}(1|2n)$

4.1 $\mathfrak{sl}_2(\mathbb{C})$-action on $\mathcal{P}(\mathbb{C}^{1|2n})$

Recall $\mathfrak{sl}_2(\mathbb{C}) = \text{span}\{H, E, F\}$, where

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F.$$

For $\lambda \in \mathbb{C}$ define a *lowest weight module* of weight λ to be the space V_λ with basis $\{v_j \mid j \in \mathbb{N} \cup \{0\}\}$, where for all $j \in \mathbb{N}$ we have

$$\begin{aligned} Hv_j &= (\lambda + 2j)v_j, \\ Ev_j &= v_{j+1}, \\ Fv_j &= -j(\lambda + j - 1)v_{j-1}, \text{ and} \\ Fv_0 &= 0. \end{aligned} \tag{4.1.1}$$

We call v_0 the *lowest weight vector* of V_λ.

Proposition 4.1.1. *If $\lambda \notin \mathbb{Z}_{<0}$, then V_λ is irreducible. If $\lambda \in \mathbb{Z}_{<0}$, then*

$$V'_\lambda = \text{span}\{E^j v_{1-\lambda} \mid j \in \mathbb{Z}_{\geq 0}\}$$

is an $\mathfrak{sl}_2(\mathbb{C})$-submodule and V_λ/V'_λ is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $1-\lambda$.

Proof: This can be proved by observing the action of $\mathfrak{sl}_2(\mathbb{C})$ on the basis of V_λ in both cases. See [How92, Prop. 1.2.6]. ■

For $\lambda \in \mathbb{C}$ define W_λ to be the unique irreducible quotient of V_λ. That is,

$$W_\lambda \cong \begin{cases} V_\lambda & \text{if } \lambda \notin \mathbb{Z}_- \\ V_\lambda/V'_\lambda & \text{otherwise.} \end{cases}$$

Lemma 4.1.2. *Let $\lambda_1, \dots, \lambda_\ell \in \mathbb{C}$. If the λ_i can be ordered such that $\sum_{i=1}^{s} \lambda_i \notin \mathbb{Z}_{<0}$, for all $s = 2, \dots, \ell$, then*

$$W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_\ell} \cong \bigoplus_{j=0}^{\infty} \binom{\ell+j-2}{\ell-2} W_{\lambda+2j},$$

where this scalar multiplication indicates multiplicity and $\lambda = \sum_{i=1}^{\ell} \lambda_i$.

Proof: It is enough to prove this for $\ell = 2$ and then apply induction. For this case, see [How92, Cor. 2.1.3]. ■

Definition 4.1.3 (Super Polynomials)**.** *Denote by $\mathcal{P}(\mathbb{C}^{1|2n})$ the superalgebra generated by the indeterminates $\{y\} \cup \{x_i \mid 1 \le i \le 2n\}$, satisfying the relations*

$$yx_i = x_i y \quad \text{and} \quad x_i x_j = -x_j x_i, \quad \text{for all } 1 \le i, j \le 2n.$$

We call the elements of $\mathcal{P}(C^{1|2n})$ ***superpolynomials****. For $k \in \mathbb{Z}_{\ge 0}$, we denote by $\mathcal{P}^k(\mathbb{C}^{1|2n})$ the space of homogeneous degree k superpolynomials.*

Note that $\mathcal{P}(\mathbb{C}^{1|2n})$ is simply the supersymmetric algebra $S(\mathbb{C}^{1|2n})$. In particular, as superalgebras

$$\mathcal{P}(\mathbb{C}^{1|2n}) \cong \mathcal{P}(\mathbb{C}) \otimes (\Lambda(\mathbb{C}^2))^{\otimes n}. \tag{4.1.2}$$

The explicit isomorphism is clear when we let $\mathcal{P}(\mathbb{C})$ be generated by y and the i^{th} $\Lambda(\mathbb{C}^2)$ be generated by x_i and x_{i+n}. Furthermore, the space of super polynomials can be naturally equipped with a $\mathfrak{gl}(1|2n)$-invariant supersymmetric bilinear form, $(\cdot,\cdot)$. For the details outlining this inner product, see Appendix B.

We define the partial derivatives ∂_y and $\partial_i := \partial_{x_i}$, for $1 \leq i \leq 2n$, for any $u, v \in \{y, x_1, \ldots, x_{2n}\}$ by

$$\partial_u(v) = \delta_{u,v},$$

where $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ otherwise. Furthermore, we extend these partial derivatives to all of $\mathcal{P}(\mathbb{C}^{1|2n})$ by the Leibniz formula given for $u, v, w \in \{y, x_1, \ldots, x_{2n}\}$ by

$$\partial_u(vw) = \partial_u(v)w + (-1)^{|u||v|} v\partial_u(w).$$

The subalgebra of $\mathrm{End}(\mathcal{P}(\mathbb{C}^{1|2n})$ generated by the ∂_u's is denoted by $\mathcal{D}(\mathbb{C}^{1|2n})$ and the subalgebra of $\mathrm{End}(\mathcal{P}(\mathbb{C}^{1|2n})$ generated by the ∂_u's and the left multiplication by elements of $\mathcal{P}(\mathbb{C}^{1|2n})$ is denoted by $\mathcal{PD}(\mathbb{C}^{1|2n})$. For more details on $\mathcal{PD}(\mathbb{C}^{1|2n})$ see [SS16, Sec. 2].

Define the following operators on $\mathcal{P}(\mathbb{C}^{1|2n})$

$$R^2 = y^2 - 2 \sum_{1 \leq i \leq n} x_{i+n} x_i, \tag{4.1.3}$$

$$\nabla^2 = \partial_y^2 - 2\sum_{i=1}^{n} \partial_{i+n}\partial_i, \text{ and}$$
$$\mathbb{E} = y\partial_y + \sum_{i=1}^{2n} x_i\partial_i.$$

We call ∇^2 the *super Laplace operator* and $\mathbb{E}$ the *super Euler operator*. We write

$$\theta^2 = -2\sum_{1\le i\le n} x_{i+n}x_i,$$
$$\nabla_{\mathrm{b}}^2 = \partial_y^2, \text{ and}$$
$$\nabla_{\mathrm{f}}^2 = -2\sum_{1\le i\le n} \partial_{i+n}\partial_i.$$

Furthermore, these operators admit the following relations:

$$\left[(1/2)R^2, -(1/2)\nabla^2\right] = \mathbb{E} + \frac{1-2n}{2}$$
$$\left[\mathbb{E} + \frac{1-2n}{2}, (1/2)R^2\right] = R^2$$
$$\left[\mathbb{E} + \frac{1-2n}{2}, -(1/2)\nabla^2\right] = \nabla^2$$

Thus $\mathfrak{sl}_2(\mathbb{C}) \cong \mathrm{span}\{H, E, F\}$, for

$$F = -(1/2)\nabla^2, \qquad (4.1.4)$$
$$E = (1/2)R^2 \text{ and}$$
$$H = \mathbb{E} + \frac{1-2n}{2}.$$

That is, we have an action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{1|2n})$. Now define the action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathcal{P}(\mathbb{C})$ and the i^{th} $\Lambda(\mathbb{C}^2)$ space by Table 4.1.4.

Let $\mathfrak{sl}_2(\mathbb{C})$ act on $\mathcal{P}(\mathbb{C}) \otimes (\Lambda(\mathbb{C}^2))^{\otimes n}$ as the inner tensor product module. Thus $\mathfrak{sl}_2(\mathbb{C})$ acts on both sides of equation (4.1.2). It is straightforward to see that these

$\mathfrak{sl}_2(\mathbb{C})$	$\mathcal{P}(\mathbb{C})$	$\Lambda(\mathbb{C}^2)$
H	$y\partial_y + \frac{1}{2}$	$x_i\partial_i + x_{i+n}\partial_{i+n} - 1$
F	$(-1/2)\partial_y^2$	$\partial_i\partial_{i+n}$
E	$(1/2)y^2$	$-x_ix_{i+n}$

Table 4.1: Action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathcal{P}(\mathbb{C})$ and $\Lambda(\mathbb{C}^2)$

actions commute with the isomorphism in Equation (4.1.2). That is, as $\mathfrak{sl}_2(\mathbb{C})$-modules

$$\mathcal{P}(\mathbb{C}^{1|2n}) \cong \mathcal{P}(\mathbb{C}) \otimes (\Lambda(\mathbb{C}^2))^{\otimes n}.$$

Notice that under the action of $\mathfrak{sl}_2(\mathbb{C})$ the space $\Lambda(\mathbb{C}^2)$ decomposes as

$$\begin{aligned}\Lambda(\mathbb{C}^2) &= \Lambda^1(\mathbb{C}^2) \oplus (\Lambda^0(\mathbb{C}^2) \oplus \Lambda^2(\mathbb{C}^2)) \\ &\cong (W_0)^{\oplus 2} \oplus W_{-1}\end{aligned}$$

and $\mathcal{P}(\mathbb{C})$ decomposes as

$$\mathcal{P}(\mathbb{C}) \cong W_{1/2} \oplus W_{3/2}.$$

Remark 4.1.4. The action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathcal{P}(\mathbb{C})$ is sometimes called the **oscillator representation**. The action defined on $\Lambda(\mathbb{C}^2)$ is not the action that one most naturally uses and can be seen as some 'odd' version of the oscillator module.

By complete reducibility of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ there are non-negative integers $m_0, \ldots, m_n$ such that

$$((W_0)^{\oplus 2} \oplus W_{-1})^{\otimes n} \cong \bigoplus_{i=0}^{n} m_i (W_0)^{\otimes i} \otimes (W_{-1})^{\otimes n-i}.$$

Thus we have an isomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules

$$\mathcal{P}(\mathbb{C}) \otimes (\Lambda(\mathbb{C}^2))^{\otimes n} \cong \bigoplus_{j\in\{1,3\}} \bigoplus_{i=1}^{n} m_i W_{j/2} \otimes (W_0)^{\otimes i} \otimes (W_{-1})^{\otimes n-i}.$$

Each term in the direct sum satisfies the hypothesis of Lemma 4.1.2. Therefore,

$$\mathcal{P}(\mathbb{C}) \otimes (\Lambda(\mathbb{C}^2))^{\otimes n} \cong \bigoplus_{i=0}^{\infty} m_i' W_{\lambda_i},$$

where the $m_i' \in \mathbb{Z}_{\geq 0}$ is the multiplicity and the λ_j are distinct half integers. These multiplicities can be computed explicitly, but we will not need them. Thus as $\mathfrak{sl}_2(\mathbb{C})$-modules we have $m_i' \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{P}(\mathbb{C}^{1|2n}) \cong \bigoplus_{i=0}^{\infty} W_{\lambda_i}^{\oplus m_i'}. \tag{4.1.5}$$

Theorem 4.1.5. *The map $\nabla^2\colon \mathcal{P}(\mathbb{C}^{1|2n}) \to \mathcal{P}(\mathbb{C}^{1|2n})$ is surjective.*

Proof: For convenience take the isomorphism in Equation (4.1.5) as identification. It suffices to show that the restriction $\nabla^2\big|_{\mathcal{P}^{k+2}(\mathbb{C}^{1|2n})}\colon \mathcal{P}^{k+2}(\mathbb{C}^{1|2n}) \to \mathcal{P}^{k}(\mathbb{C}^{1|2n})$ is surjective for all $k \in \mathbb{Z}_{\geq 0}$.

Take any $w \in \mathcal{P}^k(\mathbb{C}^{1|2n})$, then there are distinct weights λ_i, non-zero scalars c_i, and $w_i \in W_{\lambda_i}^{\oplus m_i'} \backslash \{0\}$ such that $w = \sum_{i=1}^{\ell} c_i w_i$. For each i we can write $w_i = E^{\ell_i} w_{\lambda_i}$, for w_{λ_i} the lowest weight vector of W_{λ_i}. We know that $Hw_i = k + \frac{1-2n}{2}$ and it also follows that

$$\begin{aligned} Hw_i &= E^{\ell_i} w_{\lambda_i} \\ &= (\lambda_i + 2\ell_i) w_i. \end{aligned}$$

Therefore $\ell_i = \frac{1}{2}\left(k + \frac{1-2n}{2} - \lambda_i\right)$. By Equation (4.1.1), we have that for all $j \in \mathbb{N}$

$$FE^j w_{\lambda_i} = -j(\lambda_i + j - 1)E^{j-1}w_{\lambda_i}.$$

Therefore

$$FEw_i = FE^{\ell_i+1}w_{\lambda_i} = \left(\frac{-1}{4}\right)\left(k + \frac{5-2n}{2} - \lambda_i\right)\left(k + \frac{1-2n}{2} + \lambda_i\right)w_i.$$

Note that each λ_i is a half integer and therefore $FEw_i \neq 0$ since otherwise Ew_i generates a non-zero proper submodule of W_{λ_i}. Furthermore, since $Ew_i \in \mathcal{P}^{k+2}(\mathbb{C}^{1|2n})$ we pick

$$w' = \sum_{i=1}^{\ell} \frac{-4c_i}{\left(k + \frac{5-2n}{2} - \lambda_i\right)\left(k + \frac{1-2n}{2} + \lambda_i\right)} Ew_i \in \mathcal{P}^{k+2}(\mathbb{C}^{1|2n})$$

and we have that $Fw' = -(1/2)\nabla^2 w' = w$. In particular, $\nabla^2\big|_{\mathcal{P}^{k+2}(\mathbb{C}^{1|2n})} \colon \mathcal{P}^{k+2}(\mathbb{C}^{1|2n}) \to \mathcal{P}^k(\mathbb{C}^{1|2n})$ is surjective for any $k \in \mathbb{Z}_{\geq 0}$ and therefore ∇^2 is surjective. ■

Remark 4.1.6. Although Theorem 4.1.5 is for ∇^2, a similar proof can be given for surjectivity of $\nabla_{\mathrm{f}}^2 \colon \mathcal{P}^k(\mathbb{C}^{0|2n}) \to \mathcal{P}^{k-2}(\mathbb{C}^{0|2n})$, where $2 \leq k \leq n$.

Recall that $\mathcal{P}(\mathbb{C}^{1|2n})$ is equipped with a bilinear form $(\cdot,\cdot)$. Then we call two operators $S, T \colon \mathcal{P}(\mathbb{C}^{1|2n}) \to \mathcal{P}(\mathbb{C}^{1|2n})$ **adjoint** to each other if for all $x, y \in \mathcal{P}(\mathbb{C}^{1|2n})$

$$(Tx, y) = (x, Sy).$$

Lemma 4.1.7. *The operators ∇^2 and R^2 are adjoint operators on $\mathcal{P}(\mathbb{C}^{1|2n})$.*

Proof: This follows from Equation (B.0.7) in Appendix B which, using the notation from Appendix B, says that for any $v, w \in \mathcal{P}(\mathbb{C}^{1|2n})$

$$(\partial_{X^i}\partial_{X_i}v, w) = (v, X^i X_i w),$$

where for $1 \leq i \leq 2n+1$

$$X_i = \begin{cases} y & \text{if } i = 1, \\ x_{i-1} & \text{if } i > 1, \end{cases}$$

$$X^i = \begin{cases} y & \text{if } i = 1, \\ -x_{i+n-1} & \text{if } 1 < i \leq n+1, \\ x_{i-n-1} & \text{if } n+1 < i \leq 2n+1. \end{cases}$$

It follows that ∇^2 is adjoint to R^2. ■

Lemma 4.1.8. *The image of $\mathfrak{osp}(1|2n)$ in $\mathcal{PD}(\mathbb{C}^{1|2n})$ commutes with $\mathfrak{sl}_2(\mathbb{C}) =$ span$\{H, E, F\}$. In particular, the image of $\mathfrak{sp}(2n) \subseteq \mathfrak{osp}(1|2n)$ in $\mathcal{PD}(\mathbb{C}^{1|2n})$ commutes with any multiplication operator by a polynomial expression in y^2 and θ^2.*

Proof: The subalgebra $\mathfrak{sp}(2n) \subseteq \mathfrak{osp}(1|2n)$ is generated by matrices of the form

$$\begin{aligned} &E_{i,j} - E_{j+n,i+n} \mapsto x_i\partial_j - x_{j+n}\partial_{i+n} \\ &E_{i+n,j} + E_{j+n,i} \mapsto x_{i+n}\partial_j + x_{j+n}\partial_i, \text{ and} \\ &E_{i,j+n} + E_{j,i+n} \mapsto x_i\partial_{j+n} + x_j\partial_{i+n}. \end{aligned}$$

where $\overline{1} \leq i, j \leq \overline{n}$. These generators will commute with any polynomial in y and therefore will commute with y^2. By a direct computation for $\overline{1} \leq i, j \leq \overline{n}$ we have

$$\begin{aligned} (x_i\partial_j - x_{j+n}\partial_{i+n})(\theta^2) &= (x_i\partial_j - x_{j+n}\partial_{i+n})\left(-2\sum_{\ell=\overline{1}}^{\overline{n}} x_{\ell+n}x_\ell\right) + \theta^2(x_i\partial_j - x_{j+n}\partial_{i+n}) \\ &= -2(-x_ix_{j+n} - x_{j+n}x_i) + \theta^2(x_i\partial_j - x_{j+n}\partial_{i+n}) \\ &= \theta^2(x_i\partial_j - x_{j+n}\partial_{i+n}), \end{aligned}$$

Analogous calculations for the other generators give the same results. Hence the elements of $\mathfrak{sp}(2n)$ commute with variables y^2 and θ^2. In particular, $\mathfrak{sp}(2n)$ will commute with any polynomial in y^2 and θ^2.

Similarly, the basis elements for $\mathfrak{osp}(1|2n)_{\overline{1}}$ are

$$E_{i,1} + E_{1,i+n} \mapsto x_i \partial_1 + y\partial_{i+n}$$
$$E_{i+n,1} - E_{1,i} \mapsto x_{i+n}\partial_y - y\partial_i,$$

for $\overline{1} \leq i \leq \overline{n}$. Again by a direct computation one can check that the image of $\mathfrak{osp}(1|2n)_{\overline{1}}$ in $\mathcal{PD}(\mathbb{C}^{1|2n})$ commutes with E. Thereby showing the image of $\mathfrak{osp}(1|2n)$ commutes with E.

Lemma 4.1.7 implies that E is adjoint to F. Therefore, using this and the fact that F is an even map we have for all $v, w \in \mathcal{P}(\mathbb{C}^{1|2n})$ and $g \in \mathfrak{osp}(1|2n)$

$$\begin{aligned}(gFv, w) &= (-1)^{|Fv||g|}(Fv, gw) \\ &= (-1)^{|Fv||g|}(v, Egw) \\ &= (-1)^{|Fv||g|}(v, gEw) \\ &= (gv, Ew) \\ &= (Fgv, w).\end{aligned}$$

Since this is for any $v, w \in \mathcal{P}(\mathbb{C}^{1|2n})$ it follows that the image of $\mathfrak{osp}(1|2n)$ in $\mathcal{PD}(\mathbb{C}^{1|2n})$ commutes with F and therefore also with $H = [E, F]$. ■

4.2 Harmonic Polynomials

In addition to the $\mathfrak{sl}_2(\mathbb{C})$-action described above, there is another natural action of $\mathfrak{gl}(1|2n)$ on $\mathcal{P}(\mathbb{C}^{1|2n})$. This action is defined for all $\overline{1} \leq i, j \leq \overline{2n}$ by

$$\begin{aligned}\mathfrak{gl}(1|2n) &\to \mathcal{PD}(\mathbb{C}^{1|2n})\\ E_{1,j} &\mapsto y\partial_{x_j}\\ E_{j,1} &\mapsto x_j\partial_y\\ E_{i,j} &\mapsto x_i\partial_{x_j}.\end{aligned}$$

The actions of $\mathfrak{osp}(1|2n)$ and $\mathfrak{sp}(2n)$ on $\mathcal{P}(\mathbb{C}^{1|2n})$ are indeed the restrictions of the action of $\mathfrak{gl}(1|2n)$ described above, with respect to the natural embeddings $\mathfrak{sp}(2n) \subseteq \mathfrak{osp}(1|2n) \subseteq \mathfrak{gl}(1|2n)$.

Definition 4.2.1 (Harmonic Superpolynomials)**.** *Let $P \in \mathcal{P}(\mathbb{C}^{1|2n})$. We say P is a* ***harmonic superpolynomial*** *if $\nabla^2 P = 0$. We denote the set of all such elements by $\mathcal{H} = \ker \nabla^2$ and $\mathcal{H}_k = \ker \nabla^2 \cap \mathcal{P}^k(\mathbb{C}^{1|2n})$, for all $k \in \mathbb{Z}_{\geq 0}$.*

The spaces $\mathcal{H}_k^{\mathrm{b}} \subseteq \mathcal{P}^k(\mathbb{C}^{1|0})$ and $\mathcal{H}_k^{\mathrm{f}} \subseteq \mathcal{P}^k(\mathbb{C}^{0|2n})$ are analogously defined using ∇_{b}^2 and ∇_{f}^2, respectively. Furthermore, we identify $\mathcal{H}_k^{\mathrm{b}}, \mathcal{H}_k^{\mathrm{f}}, \nabla_{\mathrm{b}}^2$, and ∇_{f}^2 with their respective embeddings into $\mathcal{P}(\mathbb{C}^{1|2n})$ and $\mathcal{PD}(\mathbb{C}^{1|2n})$. Since $\nabla_{\mathrm{b}}^2 = \partial_y^2$ it is easy to see that

$$\begin{aligned}\mathcal{H}_0^{\mathrm{b}} &= \mathbb{C},\\ \mathcal{H}_1^{\mathrm{b}} &= \mathbb{C}y, \quad \text{and}\\ \mathcal{H}_k^{\mathrm{b}} &= \{0\}, \text{ for all } k \in \mathbb{Z}_{\geq 2}.\end{aligned}$$

Corollary 4.2.2. *The spaces $\mathcal{H}_k$ have dimensions* $\dim(\mathcal{H}_0) = 1$, $\dim(\mathcal{H}_1) = 2n + 1$, *and for* $2 \le k \le 2n+1$

$$\dim(\mathcal{H}_k) = \dim(\mathcal{P}^k(\mathbb{C}^{1|2n})) - \dim(\mathcal{P}^{k-2}(\mathbb{C}^{1|2n})),$$

with

$$\dim \mathcal{P}^k(\mathbb{C}^{1|2n}) = \sum_{i=0}^{\min(k,2n)} \binom{2n}{i}.$$

Proof: The formula for dimension of $\mathcal{H}_k$ follows from Theorem 4.1.5 and Rank-Nullity. To calculate $\dim \mathcal{P}^k$, note that $\mathcal{P}^k(\mathbb{C}^{1|2n})$ has a basis of monomials and we count these possible monomials. For every monomial, let i be such that y^{k-i} is the largest power of y that appears in it. Then there are $\binom{2n}{i}$ choices of the odd variables. Clearly, $0 \le k-i$ and since we can only have a product of distinct odd variables we also have $i \le 2n$. That is, $0 \le i \le \min(k, 2n)$. Thus it follows that

$$\dim \mathcal{P}^k(\mathbb{C}^{1|2n}) = \sum_{i=0}^{\min(k,2n)} \binom{2n}{i}. \qquad \blacksquare$$

Remark 4.2.3. Corollary 4.2.2 is also true for the purely fermionic case. The proof is analogous but the result is $\dim(\mathcal{H}^{\mathrm{f}}_0) = 1$, $\dim(\mathcal{H}^{\mathrm{f}}_1) = 2n$, and for all $k \in \mathbb{N}_{\ge 2}$

$$\dim(\mathcal{H}^{\mathrm{f}}_k) = \begin{cases} \binom{2n}{k} - \binom{2n}{k-2} & \text{if } k \le n \text{ and} \\ 0 & \text{if } k > n. \end{cases}$$

Although this next example is seemingly out of place since it discusses non-super $\mathfrak{sp}(2n)$-modules it will give us a useful fact needed later.

Example 4.2.4. Take the triangular decomposition of $\mathfrak{sp}(2n)$ corresponding to Equation (2.3.4) for $\ell = 0$. Therefore

$$\Phi^+ = \{\delta_i \pm \delta_j, 2\delta_k \mid 1 \preceq i \prec j \preceq n, 1 \preceq k \preceq n\}$$

and the (non-super) Weyl vector is

$$\rho = \sum_{i=1}^{n} (n - i + 1)\delta_i.$$

In this example we will calculate the dimension of certain irreducible $\mathfrak{sp}(2n)$-modules using the Weyl Dimension formula. The statement of the Weyl Dimension Formula in full generality for any complex semisimple Lie algebras can be found in [Kna02, Theorem 5.84] with proof. For this example, let V be an irreducible finite-dimensional representation of $\mathfrak{sp}(2n)$ with highest weight λ. Then

$$\dim V = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$$

Now take V_{λ_k} to be the $\mathfrak{sp}(2n)$-highest weight module of weight $\lambda_k = \delta_1 + \cdots + \delta_k$. We can calculate $\dim(V_{\lambda_k})$ using the Weyl dimension formula as follows for $2 < k \leq n$

$$\begin{aligned}
\dim(V_{\lambda_k}) &= \prod_{\alpha \in \Phi^+} \frac{\langle \lambda_k + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\
&= \left(\prod_{1 \leq \ell \leq n} \frac{\langle \lambda_k + \rho, 2\delta_\ell \rangle}{\langle \rho, 2\delta_\ell \rangle} \right) \left(\prod_{1 \leq i < j \leq n} \frac{\langle \lambda_k + \rho, \delta_i - \delta_j \rangle}{\langle \rho, \delta_i - \delta_j \rangle} \right) \left(\prod_{1 \leq i < j \leq n} \frac{\langle \lambda_k + \rho, \delta_i + \delta_j \rangle}{\langle \rho, \delta_i + \delta_j \rangle} \right) \\
&= \left(\prod_{1 \leq \ell \leq k} \frac{n - \ell + 2}{n - \ell + 1} \right) \left(\prod_{1 \leq i \leq k < j \leq n} \frac{j - i + 1}{j - i} \right) \left(\prod_{1 \leq i \leq k < j \leq n} \frac{2n - i - j + 3}{2n - i - j + 2} \right) \\
&\qquad \left(\prod_{1 \leq i < j \leq k} \frac{2n - i - j + 4}{2n - i - j + 2} \right) \\
&= \frac{(2n + 1)!}{k!(2n - k + 2)!}(2n - 2k + 2).
\end{aligned}$$

For $k = 0, 1, 2$ we have $\dim(V_0) = 1$, $\dim(V_{\lambda_1}) = 2n$, and $\dim(V_{\lambda_2}) = (n-1)(2n+1)$.

Lemma 4.2.5. *For $0 \leq k \leq n$, the space $\mathcal{H}_k^f$ is an irreducible $\mathfrak{sp}(2n)$-highest weight module with highest weight $\delta_1 + \cdots + \delta_k$.*

Proof: Take the standard Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{sp}(2n)$ and triangular decomposition, $\mathfrak{sp}(2n) = (\mathfrak{n}^-)' \oplus \mathfrak{h}' \oplus (\mathfrak{n}^+)'$, associated to the positive system given in Equation (2.3.4) for $m = 0$. Thus

$$\mathfrak{h}' = \operatorname{span}\{E_{j,j} - E_{j+n,j+n} \mid 1 \leq j \leq n\} \text{ and}$$
$$(\mathfrak{n}')^+ = \operatorname{span}\{E_{i,j} - E_{j+n,i+n} \mid 1 \leq i < j \leq n\} \cup \{E_{i,j+n} + E_{j,i+n} \mid 1 \leq i \leq j \leq n\}.$$

For $0 \leq k \leq n$, consider the vector $x_1 x_2 \ldots x_k$. Observe that

$$\nabla_{\mathrm{f}}^2(x_1 x_2 \ldots x_k) = 0.$$

Thus $x_1 x_2 \ldots x_k \in \mathcal{H}_k^{\mathrm{f}}$. It is easy to verify that for all $g \in (\mathfrak{n}')^+$ and $h \in \mathfrak{h}'$

$$g \cdot (x_1 x_2 \ldots x_k) = 0 \text{ and}$$
$$h \cdot (x_1 x_2 \ldots x_k) = (\delta_1 + \cdots + \delta_k)(h)(x_1 x_2 \ldots x_k).$$

That is, $x_1 \cdots x_k$ is a $\mathfrak{sp}(2n)$-highest weight vector. and we write V_{λ_k} for the highest weight module generated by $x_1 \cdots x_k$. Note that $V_{\lambda_k} \subseteq \mathcal{H}_k^{\mathrm{f}}$ since Lemma 4.1.8 implies that that action of $\mathfrak{sp}(2n)$ commutes with ∇_{f}^2. Furthermore, Remark 4.2.3 tells us that for $2 \leq k \leq n$

$$\begin{aligned}\dim \mathcal{H}_k^{\mathrm{f}} &= \binom{2n}{k} - \binom{2n}{k-2} \\ &= \frac{(2n+1)!}{k!(2n-k+2)!}(2n+2-2k).\end{aligned}$$

But this is the same as the dimension found in Example 4.2.4. That is, $V_{\lambda_k} = \mathcal{H}_k^{\mathrm{f}}$. ■

For the next Corollary, recall that $\mathfrak{osp}(1|2n)_{\bar{0}} \cong \mathfrak{sp}(2n)$.

Corollary 4.2.6. *For any $p = 0, 1$ and $0 \leq q \leq n$, $\mathcal{H}^b_p \otimes \mathcal{H}^f_q$ is an $\mathfrak{osp}(1|2n)_{\bar{0}}$-highest weight module of weight $\delta_1 + \cdots + \delta_q$.*

Proof: Notice that if $H_q \in \mathcal{H}^{\mathrm{f}}_q$ is a $\mathfrak{sp}(2n)$-highest weight vector then a straightforward calculation shows that $y^p H_q$ will be a $\mathfrak{osp}(1|2n)_{\bar{0}}$-highest weight vector of weight $\delta_1 + \cdots + \delta_q$ and will generate $\mathcal{H}^{\mathrm{b}}_p \otimes \mathcal{H}^{\mathrm{f}}_q$. ■

Lemma 4.2.7. *For any $P \in \mathcal{P}^k(\mathbb{C}^{1|2n})$ and $t \in \mathbb{N}$ we have*

$$\nabla^2(R^{2t}P) = 2t(2k + (1-2n) + 2(t-1))R^{2t-2}P + R^{2t}\nabla^2 P.$$

Proof: By induction on t, for $t = 1$ we use the relation $[\nabla^2, R^2] = 4(\mathbb{E} + \frac{1-2n}{2})$ to obtain

$$\begin{aligned}\nabla^2(R^2P) &= 4\left(\mathbb{E} + \frac{1-2n}{2}\right)P + R^2\nabla^2 P \\ &= 2(2k + 1 - 2n)P + R^2\nabla^2 P.\end{aligned}$$

Let us assume that the lemma is true for some $t \in \mathbb{N}$. Then,

$$\begin{aligned}\nabla^2(R^{2t+2}P) &= \left(4\left(\mathbb{E} + \frac{1-2n}{2}\right) + R^2\nabla^2\right)(R^{2t}P) \\ &= 2(4t + 2k + 1 - 2n)R^{2t}P \\ &\qquad + R^2\left(2t(2k + 1 - 2n + 2(t-1))R^{2t-2}P + R^{2t}\nabla^2 P\right) \\ &= 2(t+1)(2k + 1 - 2n + 2t)R^{2t}P + R^{2t+2}\nabla^2 P.\end{aligned}$$

Thus the lemma follows by induction. ■

Remark 4.2.8. The above lemma is true in general for $\mathfrak{osp}(m|2n)$ and the proof is analogous, see [DBES09, Lemma 1] for details. In particular, for all $P \in \mathcal{P}^k(\mathbb{C}^{m|2n})$ and $t \in \mathbb{N}$ we have

$$\nabla^2(R^{2t}P) = 2t(2k + (m-2n) + 2(t-1))R^{2t-2}P + R^{2t}\nabla^2 P.$$

In this setting ∇^2 is the Laplace operator of $\mathbb{C}^{m|2n}$ and $[\nabla^2, R^2] = 4(\mathbb{E} + \frac{m-2n}{2})$.

Lemma 4.2.9. *For $0 \leq q \leq n$, $0 \leq \ell \leq n-q$, and $p \in \{0,1\}$ the super polynomial*

$$f_{\ell,p,q} = \sum_{i=0}^{\ell} a_i y^{2\ell-2i}\theta^{2i} \qquad \text{with} \qquad a_i = \binom{\ell}{i} \frac{(n-q-i)!\ \Gamma(\frac{1}{2}+p+\ell)}{\Gamma(\frac{1}{2}+p+\ell-i)\ (n-q-\ell)!}$$

is the unique (up to a scalar multiple) homogeneous super polynomial in y^2 and θ^2 of degree 2ℓ such that $f_{\ell,p,q}\mathcal{H}_p^{\mathrm{b}} \otimes \mathcal{H}_q^{\mathrm{f}} \neq 0$ and $\nabla^2(f_{\ell,p,q}\mathcal{H}_p^{\mathrm{b}} \otimes \mathcal{H}_q^{\mathrm{f}}) = 0$.

Proof: Suppose that

$$f_{\ell,p,q}(y^2, \theta^2) = \sum_{i=0}^{\ell} a_i y^{2\ell-2i}\theta^{2i}$$

is such that $\nabla^2(f_{\ell,p,q}\mathcal{H}_p^{\mathrm{b}} \otimes \mathcal{H}_q^{\mathrm{f}}) = 0$. We now find a recursion for the a_i. Take simple tensors $H_p^{\mathrm{b}} H_q^{\mathrm{f}} \in \mathcal{H}_p^{\mathrm{b}} \otimes \mathcal{H}_q^{\mathrm{f}}$. Then in the following calculations we use Lemma 4.2.7 for $n = 0$ and the analogous results for the classical ∇_{f}^2. In particular,

$$\begin{aligned}
\nabla^2(f_{\ell,p,q}H_p^{\mathrm{b}}H_q^{\mathrm{f}}) &= \sum_{i=0}^{\ell} a_i(\nabla_{\mathrm{b}}^2 + \nabla_{\mathrm{f}}^2) y^{2\ell-2i} H_p^{\mathrm{b}} \theta^{2i} H_q^{\mathrm{f}} \\
&= \sum_{i=0}^{\ell} a_i \nabla_{\mathrm{b}}^2 (y^{2\ell-2i} H_p^{\mathrm{b}}) \theta^{2i} H_q^{\mathrm{f}} + a_i y^{2\ell-2i} H_p^{\mathrm{b}} \nabla_{\mathrm{f}}^2(\theta^{2i} H_q^{\mathrm{f}}) \\
&= \sum_{i=0}^{\ell-1} a_i((2\ell-2i)(2p+1+2(\ell-i)-2)) y^{2\ell-2i-2} H_p^{\mathrm{b}} \theta^{2i} H_q^{\mathrm{f}}
\end{aligned}$$

$$+\sum_{i=1}^{\ell} a_i y^{2\ell-2i} H_p^{\mathrm{b}}(2i(2q-2n+2i-2))\theta^{2i-2} H_q^{\mathrm{f}}$$
$$=\sum_{i=0}^{\ell-1} 4b_i y^{2(\ell-i-1)}\theta^{2i} H_p^{\mathrm{b}} H_q^{\mathrm{f}},$$

where $b_i = a_i(\ell-i)\left(p+\frac{1}{2}+(\ell-i)-1\right)+a_{i+1}(i+1)(q-n+i)$ for $i=0,\dots,\ell-1$. Since $\nabla^2(f_{\ell,p,q}\mathcal{H}_p^{\mathrm{b}}\otimes\mathcal{H}_q^{\mathrm{f}})=0$ then $b_i=0$ for each $0\le i\le \ell-1$. Thus

$$a_{i+1}=\frac{(\ell-i)\left(\frac{1}{2}+p+(\ell-i)-1\right)}{(i+1)(n-q-i)}a_i$$

which implies that

$$a_i=\binom{\ell}{i}\frac{\Gamma(\frac{1}{2}+p+\ell)(n-q-i)!}{(n-q)!\,\Gamma(\frac{1}{2}+p+\ell-i)}a_0.$$

Since there are no conditions on a_0, we can set $a_0=\frac{(n-q)!}{(n-q-\ell)!}$ from which it follows that $a_i=\binom{\ell}{i}\frac{(n-q-i)!\;\Gamma(\frac{1}{2}+p+\ell)}{\Gamma(\frac{1}{2}+p+\ell-i)\;(n-q-\ell)!}$, for all $0\le i\le \ell$. ■

For $k\in\mathbb{N}$ we will be interested in the non-zero spaces $f_{\ell,p,q}\mathcal{H}_p^{\mathrm{b}}\otimes\mathcal{H}_q^{\mathrm{f}}$ contained in $\mathcal{H}_k$. Therefore we set $p=k-2\ell-q$ and introduce the notation

$$V_{\ell,q}=f_{\ell,k-2\ell-q,q}\mathcal{H}_{k-2\ell-q}^{\mathrm{b}}\otimes\mathcal{H}_q^{\mathrm{f}}\quad\text{and}\quad \varphi_{\ell,q}=f_{\ell,k-2\ell-q,q}. \tag{4.2.1}$$

For $0\le k\le 2n+1$ is easy to check that $V_{\ell,q}\subseteq\mathcal{H}_k$, for $0\le q\le\min(2n-k+1,k)$ and $\ell=\lfloor\frac{k-j}{2}\rfloor$.

Let C_{f} be the Casimir operator for $\mathfrak{sp}(2n)$. As in [DBES09, pg. 11], we define the operators

$$\mathbf{P}_s=\prod_{j=0,j\neq s}^{n}\frac{C_{\mathrm{f}}+j(j-2n+1)}{(j-s)(j+s-2n+1)}\in\mathfrak{U}(\mathfrak{sp}(2n))\subseteq\mathfrak{U}(\mathfrak{g}),$$

for integers $0 \leq s \leq n$.

For the following Corollary, recall that $\mathfrak{osp}(1|2n)_{\bar{0}} = \mathfrak{sp}(2n)$.

Corollary 4.2.10. *The space $V_{\ell,q} \subseteq \mathcal{H}_k$, for $0 \leq q \leq \min(2n-k+1, k)$ and $\ell = \lfloor \frac{k-j}{2} \rfloor$, is an irreducible $\mathfrak{osp}(1|2n)_{\bar{0}}$-module with highest weight $\delta_1 + \cdots + \delta_q$. Furthermore, under the $\mathfrak{osp}(1|2n)_{\bar{0}}$ action*

$$\mathbf{P}_s(V_{\ell,q}) = \delta_{s,q} V_{\ell,q},$$

for all integers $0 \leq s \leq n$.

Proof: Lemma 4.2.9 implies that left multiplication by $f_{\ell,k-2\ell-q,q}$ is a surjective $\mathfrak{osp}(1|2n)_{\bar{0}}$-intertwining map $\mathcal{H}^{\mathrm{b}}_{k-2\ell-q} \otimes \mathcal{H}^{\mathrm{f}}_q \to V_{\ell,q} = f_{\ell,k-2\ell-q,q}\mathcal{H}^{\mathrm{b}}_{k-2\ell-q} \otimes \mathcal{H}^{\mathrm{f}}_q$. Since $\mathcal{H}^{\mathrm{b}}_{k-2\ell-q} \otimes \mathcal{H}^{\mathrm{f}}_q$ is an irreducible $\mathfrak{osp}(1|2n)_{\bar{0}}$-module the kernel of this multiplication is 0 and $\mathcal{H}^{\mathrm{b}}_{k-2\ell-q} \otimes \mathcal{H}^{\mathrm{f}}_q \cong V_{\ell,q}$. That is, $V_{\ell,q}$ is an irreducible $\mathfrak{osp}(1|2n)_{\bar{0}}$-module with highest weight $\delta_1 + \cdots + \delta_q$.

Since $\mathcal{H}^{\mathrm{f}}_q$ is an irreducible $\mathfrak{sp}(2n)$-module with highest weight $\delta_1 + \cdots + \delta_q$, Proposition 3.3.4 implies that C_{f} acts on $\mathcal{H}^{\mathrm{f}}_q$ by the scalar

$$\langle \delta_1 + \cdots + \delta_q + \rho, \delta_1 + \cdots + \delta_q + \rho \rangle - \langle \rho, \rho \rangle = q(2n + 1 - q),$$

with $\rho = \frac{1}{2}(\rho_0 - \rho_1)$. Thus by direct computation, for $0 \leq q \leq \min(n, k)$

$$\begin{aligned}
\mathbf{P}_s(\mathcal{H}^{\mathrm{f}}_q) &= \left(\prod_{j=0, j\neq s}^{\min(n,k)} \frac{C_{\mathrm{f}} + j(j-2n+1)}{(j-s)(j+s-2n+1)} \right) \mathcal{H}^{\mathrm{f}}_q \\
&= \left(\prod_{j=0, j\neq s}^{\min(n,k)} \frac{-q(q-2n+1) + j(j-2n+1)}{(j-s)(j+s-2n+1)} \right) \mathcal{H}^{\mathrm{f}}_q \\
&= \delta_{s,q} \mathcal{H}^{\mathrm{f}}_q.
\end{aligned}$$

The last equality follows because if $q = s$ then each term in the second line is

$$\frac{-q(q-2n+1)+j(j-2n+1)}{(j-q)(j+q-2n+1)} = 1$$

and if $q \neq s$ then for $j = q$ in the product on the second line we have

$$\frac{-q(q-2n+1)+q(q-2n+1)}{(q-s)(q+s-2n+1)} = 0.$$

Thus it follows that

$$\begin{aligned}\mathbf{P}_s(V_{\ell,q}) &= f_{\ell,k-2\ell-q,q}\mathcal{H}^{\mathrm{b}}_{k-2\ell-q} \otimes \mathbf{P}_s(\mathcal{H}^{\mathrm{f}}_q) \\ &= \delta_{s,q}V_{\ell,q}.\end{aligned}$$ ■

The next theorem is a special case of [Cou13, Theorem 2.3] for $\mathfrak{osp}(1|2n)$.

Theorem 4.2.11 (Decomposition of $\mathcal{H}_k$)**.** *For $0 \leq k \leq 2n+1$ the space $\mathcal{H}_k$ decomposes under the action of $\mathfrak{osp}(1|2n)_{\bar{0}}$ such that*

$$\mathcal{H}_k = \bigoplus_{j=0}^{\min(2n-k+1,k)} V_{\lfloor\frac{k-j}{2}\rfloor,j}, \tag{4.2.2}$$

where $V_{\lfloor\frac{k-j}{2}\rfloor,j}$ is defined in Equation (4.2.1).

Proof: First, Corollary 4.2.10 gives use that the RHS of Equation (4.2.2) is a direct sum because each summand $V_{\lfloor\frac{k-j}{2}\rfloor,j}$ is an irreducible $\mathfrak{osp}(1|2n)_{\bar{0}}$-module of distinct highest weights $\delta_1 + \cdots + \delta_j$. Furthermore, Lemma 4.2.9 implies that the RHS is a subspace of the LHS. By Corollary 4.2.2,

$$\dim(\mathcal{H}_k) = \sum_{i=\min(k-1,2n)}^{\min(k,2n)} \binom{2n}{i} = \binom{2n+1}{k}.$$

Since $\mathcal{H}^{\mathrm{b}}_{k-2\lfloor\frac{k-j}{2}\rfloor-j} \otimes \mathcal{H}^{\mathrm{f}}_j \cong V_{\lfloor\frac{k-j}{2}\rfloor,j}$ and $\dim\left(\mathcal{H}^{\mathrm{b}}_{k-2\lfloor\frac{k-j}{2}\rfloor-j} \otimes \mathcal{H}^{\mathrm{f}}_j\right) = \dim(\mathcal{H}^{\mathrm{f}}_j)$, it follows that

$$\dim\left(V_{\lfloor\frac{k-j}{2}\rfloor,j}\right) = \dim(\mathcal{H}_j^{\mathrm{f}}).$$

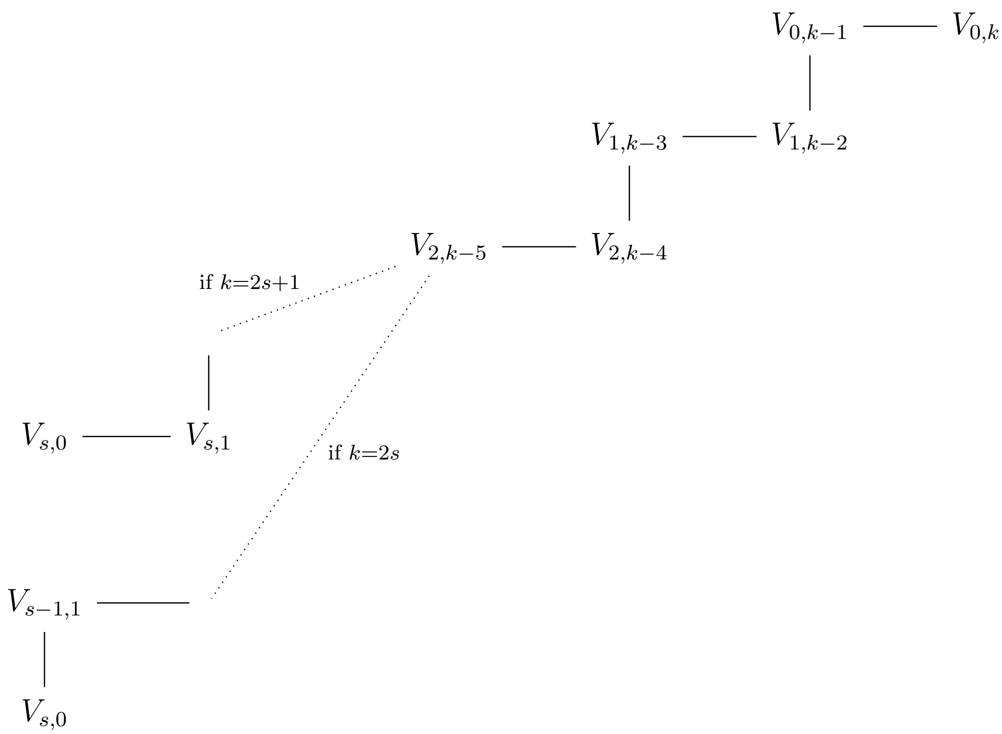
Therefore by Remark 4.2.3

$$\begin{aligned}\sum_{j=0}^{\min(2n-k+1,k)} \dim(\mathcal{H}_j^{\mathrm{f}}) &= 1 + 2n + \sum_{j=2}^{\min(2n-k+1,k)} \binom{2n}{j} - \binom{2n}{j-2} \\ &= \binom{2n}{\min(2n-k+1,k)} + \binom{2n}{\min(2n-k,k-1)} \\ &= \binom{2n+1}{k}.\end{aligned}$$

Therefore we have that the RHS is a subspace of the LHS of Equation (4.2.2) and both sides have the same dimension. That is, the spaces are equal. ■

Our ambient goal is to show that $\mathcal{H}_k$ is an irreducible $\mathfrak{osp}(1|2n)$-module. To this end, we will show that for any two summands in the decomposition (4.2.2) there is an element in $\mathfrak{U}(\mathfrak{g})$ which, when considered as a map between these summands, is a non-zero map. Since every finite-dimensional $\mathfrak{osp}(1|2n)$-module is completely irreducible, the latter statement implies that indeed $\mathcal{H}_k$ is irreducible.

For $0 \leq k \leq n$ we represent our decomposition by the following diagram.



(4.2.3)

A similar diagram can be drawn for $n < k \leq 2n+1$ with the top right corner starting with $V_{k-n-1,2n-k+1}$ then alternating directions starting with drawing downwards and left. While drawing downwards we add one unit to the first component while subtracting one unit from the second component and while drawing left we subtract one unit from the second component. In each of these diagrams the edges represent the actions of the elements of $\mathfrak{U}(\mathfrak{g})$ and the dotted lines represent the two possibilities for the diagram depending on the parity of k. We want to show that the action of $\mathfrak{U}(\mathfrak{g})$ is non-zero in both directions.

Throughout the remainder of this section we will make use of particular elements from the image of $\mathfrak{osp}(1|2n)$ in $\mathcal{PD}(\mathbb{C}^{1|2n})$. We denote these elements by $K_{1,j+n}$, for all $1 \leq j \leq n$. They are given as the image of the following basis elements from

$\mathfrak{osp}(1|2n)_{\overline{1}}$

$$E_{1,j+1} - E_{j+n+1,1} \mapsto K_{1,j+n} := y\partial_j - x_{j+n}\partial_y.$$

Lemma 4.2.12. *For any* $\overline{1} \le j \le \overline{n}$*,*

$$K_{1,j+n}\varphi_{\ell,q} = 2\ell\left(\frac{1-2n}{2} + k - \ell - 1\right)\varphi_{\ell-1,q+1}yx_{j+n}$$

where $\varphi_{\ell,q}$ *is defined in Equation* (4.2.1)*.*

Proof: To avoid any confusion within this proof, we denote the coefficients of $\varphi_{\ell,q}$ as defined in Proposition 4.2.9 by $a_{s,\ell,q}$ and similarly for $\varphi_{\ell-1,q+1}$ by $b_{s,\ell-1,q+1}$. By direct computation

$$\begin{aligned} K_{1,j+n}\varphi_{\ell,q} &= (y\partial_j - x_{j+n}\partial_y)\left(\sum_{s=0}^{\ell} a_{s,\ell,q}y^{2\ell-2s}\theta^{2s}\right) \\ &= 2yx_{j+n}\left(\sum_{s=1}^{\ell} sa_{s,\ell,q}x^{2\ell-2s}\theta^{2s-2}\right) - 2yx_{j+n}\left(\sum_{s=0}^{\ell-1}(\ell-s)a_{s,\ell,q}x^{2\ell-2s-2}\theta^{2s}\right) \\ &= 2yx_{j+n}\left(\sum_{s=0}^{\ell-1}\left((s+1)a_{s+1,\ell,q} - (\ell-s)a_{s,\ell,q}\right)x^{2\ell-2s-2}\theta^{2s}\right). \end{aligned} \tag{4.2.4}$$

Consider the coefficients in the final sum. Using the property that $\Gamma(x+1) = x\Gamma(x)$ for $x = \frac{1}{2} + k - \ell - q - s - 1$ we can see that these coefficients satisfy

$$(s+1)a_{s+1,\ell,q} - (\ell-s)a_{s,\ell,q} = \ell\left(\frac{1-2n}{2} + k - \ell - 1\right)b_{s,\ell-1,q+1}.$$

Thus we can apply this to Equation (4.2.4) to conclude the proof. ■

Lemma 4.2.13. *Fix any* $0 \le k \le 2n+1$*. We have the following identities:*

1. *For all* $0 \leq q \leq \min(k, 2n-k+1)$ *with* $q+1 \equiv k \mod 2$*,*

$$(q-k)\varphi_{\frac{k-q-1}{2},q+1} = g_1(y^2,\theta^2) := \frac{(k-q-1)}{2}\,(k+q-2n)\,\varphi_{\frac{k-q-3}{2},q+1}y^2 - \varphi_{\frac{k-q-1}{2},q}. \tag{4.2.5}$$

2. *For all* $1 \leq q \leq \min(k, 2n-k+1)$ *with* $q+1 \equiv k \mod 2$*,*

$$\frac{\varphi_{\frac{k-q+1}{2},q-1}}{n-q+1} = g_2(y^2,\theta^2) := \frac{(q-k)\varphi_{\frac{k-q-1}{2},q+1}}{2(q-n-1)}\theta^2 + \varphi_{\frac{k-q-1}{2},q}y^2. \tag{4.2.6}$$

3. *For* $1 \leq q \leq \min(k, 2n-k+1)$ *with* $q \equiv k \mod 2$*,*

$$\left(\frac{2n+2-k-q}{2(n-q+1)}\right)\varphi_{\frac{k-q}{2},q-1} = g_3(y^2,\theta^2) := \varphi_{\frac{k-q}{2},q} + \frac{(k-q)\,(k+q-2n-1)}{4(q-n-1)}\varphi_{\frac{k-q}{2}-1,q+1}\theta^2 \tag{4.2.7}$$

Proof:

1. Let $0 \leq q \leq \min(k, 2n-k+1)$ with $q+1 \equiv k \mod 2$. Take any $H_q^{\mathrm{f}} \in \mathcal{H}_q^{\mathrm{f}}$ such that H_q^{f} does not contain any x_1 nor x_{1+n}. Such a H_q^{f} can be chosen since $q \leq n-1$. For example, $H_q^{\mathrm{f}} = x_2 \cdots x_{q+1}$. Using the Leibniz rule and Lemma 4.2.12 we obtain

$$K_{1,1+n}(\varphi_{\frac{k-q-1}{2},q}yH_q^{\mathrm{f}}) = g_1(y^2,\theta^2)x_{1+n}H_q^{\mathrm{f}}.$$

The LHS of the above is harmonic since the action of ∇^2 commutes with the $\mathfrak{U}(\mathfrak{osp}(1|2n))$ action. Furthermore, $x_{1+n}H_q^{\mathrm{f}} \in \mathcal{H}_0^{\mathrm{b}} \otimes \mathcal{H}_{q+1}^{\mathrm{f}} \backslash \{0\}$ and therefore Lemma 4.2.9 implies that there is $\lambda_1 \in \mathbb{C}$ such that $g_1(y^2,\theta^2) = \lambda_1\varphi_{\frac{k-q-1}{2},q+1}$. To show that $\lambda_1 = q-k$, compare the coefficients of θ^{k-q-1} in $g_1(y^2,\theta^2)$ and

$\lambda_1 \varphi_{\frac{k-q-1}{2},q+1}$. In particular,

$$\lambda_1 \frac{\Gamma\left(\frac{k-q}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = -\frac{\Gamma\left(\frac{k-q}{2}\right)\left(\frac{k-q}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}.$$

This implies that $\lambda_1 = q-k$.

For parts 2 and 3 we define the following for any $1 \le q \le \min(k, 2n-k+1)$

$$\begin{aligned} H^{\mathrm{f}}_{q-1} &= x_2 \cdots x_q \in \mathcal{H}^{\mathrm{f}}_{q-1}, \\ H^{\mathrm{f}}_{q} &= x_1 H^{\mathrm{f}}_{q-1} \in \mathcal{H}^{\mathrm{f}}_{q}, \text{ and} \\ H^{\mathrm{f}}_{q+1} &= 2x_{1+n}x_1 H^{\mathrm{f}}_{q-1} - \frac{1}{q-n-1}\theta^2 H^{\mathrm{f}}_{q-1} \in \mathcal{H}^{\mathrm{f}}_{q+1}. \end{aligned}$$

Then for any $1 \le q \le \min(k, 2n-k+1)$ and $0 \le p \le 1$ such that $q+p \equiv k \mod 2$ we use the Leibniz rule to get

$$\begin{aligned} K_{1,1+n}\left(\varphi_{\frac{k-q-p}{2},q} y^p H^{\mathrm{f}}_q\right) = \left(\frac{k-q-p}{2}\right)(k+q+p-2n-1)\,\varphi_{\frac{k-q-p}{2}-1,q+1} y^{p+1} x_{1+n} H^{\mathrm{f}}_q \\ + \varphi_{\frac{k-q-p}{2},q} y^{p+1} H^{\mathrm{f}}_{q-1} - p\varphi_{\frac{k-q-p}{2},q} x_{1+n} H^{\mathrm{f}}_q. \end{aligned} \tag{4.2.8}$$

We can now give the arguments for parts 2 and 3 below.

2. Take $1 \le q \le \min(k, 2n-k+1)$, $p=1$, and $q+p \equiv k \mod 2$ in Equation (4.2.8) then substitute in H^{f}_{q+1} and apply equation (4.2.5) to get

$$K_{1,1+n}\left(\varphi_{\frac{k-q-1}{2},q} y H^{\mathrm{f}}_q\right) = \frac{(q-k)\varphi_{\frac{k-q-1}{2},q+1}}{2} H^{\mathrm{f}}_{q+1} + g_2(y^2,\theta^2) H^{\mathrm{f}}_{q-1}.$$

But the LHS and the first term on the RHS are in $\mathcal{H}_k$, hence the second term on the RHS is in $\mathcal{H}_k$. Therefore $g_2(y^2,\theta^2)H^{\mathrm{f}}_{q-1} \in V_{\frac{k-q+1}{2},q-1} \backslash \{0\}$. Furthermore, since Lemma 4.1.8 tells us that the action of $\mathfrak{sp}(2n)$ commutes with $g_2(y^2,\theta^2)$

and ∇^2 it follows that $g_2(y^2,\theta^2)\mathcal{H}_0^{\mathbf{b}}\otimes\mathcal{H}_{q-1}^{\mathrm{f}}\neq 0$ and $\nabla^2(g_2(y^2,\theta^2)\mathcal{H}_0^{\mathbf{b}}\otimes\mathcal{H}_{q-1}^{\mathrm{f}})=0$. Thus Lemma 4.2.9 implies that $g_2(y^2,\theta^2)=\lambda_2\varphi_{\frac{k-q+1}{2},q-1}$, for some $\lambda_2\in\mathbb{C}$.

Similar to part 1, by a direct comparison of the coefficients of y^{k-q+1} in $g_2(y^2,\theta^2)$ and $\lambda_2\varphi_{\frac{k-q+1}{2},q-1}$ we have

$$\lambda_2\frac{(n-q+1)!}{(n-\frac{k+q-1}{2})!}=\frac{(n-q)!}{(n-\frac{k+q-1}{2})!}.$$

This implies that $\lambda_2=\frac{1}{n-q+1}$.

3. Lastly, let $1\leq q\leq\min(k,2n-k+1)$, $p=0$, and $q+p\equiv k \mod 2$ in Equation (4.2.8) and write

$$K_{1,1+n}\left(\varphi_{\frac{k-q}{2},q}H_q^{\mathrm{f}}\right)=\left(\frac{k-q}{2}\right)\left(\frac{k+q-2n-1}{2}\right)\varphi_{\frac{k-q}{2}-1,q+1}yH_{q+1}^{\mathrm{f}} + g_3(y^2,\theta^2)yH_{q-1}^{\mathrm{f}}.$$

The LHS and the first term on the RHS are harmonic then $g_3(y^2,\theta^2)=\lambda_3\varphi_{\frac{k-q}{2},q-1}$, for $\lambda_3\in\mathbb{C}\backslash\{0\}$, by an argument analogous to part 2.

Again by comparing the coefficients of y^{k-q} in $g_3(y^2,\theta^2)$ and $\lambda_3\varphi_{\frac{k-q}{2},q-1}$ we have

$$\lambda_3\frac{(n-q+1)!}{(n-\frac{k+q}{2}+1)!}=\frac{(n-q)!}{(n-\frac{k+q}{2})!}.$$

Therefore by isolating for λ_3 we have $\lambda_3=\frac{2n+2-k-q}{2(n-q+1)}$. ■

Definition 4.2.14 (Admissible Direction)**.** *For any integers $0\leq k\leq 2n+1$, $0\leq q\leq\min(k,2n-k+1)$, $0\leq p\leq 1$, and $j\in\{\rightarrow,\leftarrow,\uparrow,\downarrow\}$, we say j is an* ***admissible direction for*** $V_{\frac{k-q-p}{2},q}$ *if in the Diagram (4.2.3) for $\mathcal{H}_k$ there is a space adjacent to $V_{\frac{k-q-p}{2},q}$ in direction j.*

Theorem 4.2.15. *Fix any $0 \leq k \leq 2n+1$, $0 \leq q \leq \min(k, 2n-k+1)$, $0 \leq p \leq 1$, and admissible direction $j \in \{\rightarrow, \leftarrow, \uparrow, \downarrow\}$ for $V_{\frac{k-q-p}{2},q}$. There exists $u_j \in \mathfrak{U}(\mathfrak{osp}(1|2n))$ such that u_j is a non-zero map from $V_{\frac{k-q-p}{2},q}$ in the j direction.*

Proof: Fix any $0 \leq k \leq 2n+1$. This proof is split into cases for $j \in \{\rightarrow, \leftarrow, \uparrow, \downarrow\}$.

$\uparrow$: For $0 \leq q < \min(k, 2n-k+1)$ such that $k \equiv q \mod 2$ the up direction is an admissible direction for $V_{\frac{k-q}{2},q}$. Take any $H_q^{\mathrm{f}} \in \mathcal{H}_q^{\mathrm{f}}$ not containing x_1 nor x_{1+n}, this is possible since $q < n$. Then using the Leibniz rule we have

$$K_{1,1+n}(\varphi_{\frac{k-q}{2},q} H_q^{\mathrm{f}}) = (k-q)\left(\frac{1-2n}{2} + \frac{k+q}{2} - 1\right)\varphi_{\frac{k-q}{2}-1,q+1} y x_{1+n} H_q^{\mathrm{f}} \in V_{\frac{k-q}{2}-1,q+1} \backslash \{0\}.$$

Therefore, $u_\uparrow = K_{1,1+n}$ will always be a non-zero map from $V_{\frac{k-q}{2},q}$ in the up direction.

$\rightarrow$: When $0 \leq q < \min(k, 2n-k+1)$ and $q+1 \equiv k \mod 2$ then the right direction is an admissible direction for $V_{\frac{k-q-1}{2},q}$. Again take any $H_q^{\mathrm{f}} \in \mathcal{H}_q^{\mathrm{f}}$ not containing x_1 nor x_{1+n} and using Equation (4.2.5) from Lemma 4.2.13 we get

$$\begin{aligned} K_{1,1+n}(\varphi_{\frac{k-q-1}{2},q} y H_q^{\mathrm{f}}) &= \left(\frac{(k-q-1)}{2}(k+q-2n)\,\varphi_{\frac{k-q-3}{2},q+1} y^2 - \varphi_{\frac{k-q-1}{2},q}\right) x_{1+n} H_q^{\mathrm{f}} \\ &= (q-k)\varphi_{\frac{k-q-1}{2},q+1} x_{1+n} H^{\mathrm{f}}_{k-2\frac{k-q-1}{2}-1} \in V_{\frac{k-q-1}{2},q+1} \backslash \{0\}. \end{aligned}$$

Hence, set $u_\rightarrow = K_{1,1+n}$.

For the remaining cases, let H_{q-1}^{f}, H_q^{f}, and H_{q+1}^{f} be defined as in the proof of Lemma 4.2.13.

$\downarrow$: For $1 \leq q \leq \min(k, 2n-k+1)$ and $q+1 \equiv k \mod 2$ the downward direction is an admissible direction of $V_{\frac{k-q-1}{2},q}$. Similarly to the last cases, using Equations

(4.2.5) and (4.2.6) from Lemma 4.2.13 then

$$K_{1,1+n}\left(\varphi_{\frac{k-q-1}{2},q} y H_q^{\mathrm{f}}\right) = \frac{(q-k)\varphi_{\frac{k-q-1}{2},q+1}}{2} H_{q+1}^{\mathrm{f}} + \frac{\varphi_{\frac{k-q+1}{2},q-1}}{n-q+1} H_{q-1}^{\mathrm{f}}.$$

Both of the terms in the above sum are elements of different summands from Theorem 4.2.11 and the second term in particular is always non-zero. Therefore $u_{\downarrow} = \mathbf{P}_{q-1}^{k} K_{1,1+n}$ is a non-zero map.

$\leftarrow$: Lastly, when $1 \leq q \leq \min(k, 2n-k+1)$ and $q \equiv k \mod 2$ then left is an admissible direction for $V_{\frac{k-q}{2},q}$. Using Equation (4.2.7) from Lemma 4.2.13 we have the following calculations

$$\begin{aligned} K_{1,1+n}\left(\varphi_{\frac{k-q}{2},q} H_q^{\mathrm{f}}\right) = &\frac{k-q}{2}\left(\frac{k+q-2n-1}{2}\right)\varphi_{\frac{k-q}{2}-1,q+1} y H_{q+1}^{\mathrm{f}} \\ &+ \frac{(2n-k-q+2)}{2(n-q+1)}\varphi_{\frac{k-q}{2}-1,1} y H_{q-1}^{\mathrm{f}} \end{aligned}$$

Again both of the terms in the RHS are elements of different summands from Theorem 4.2.11 and the second term is non-zero. Thus $u_{\leftarrow} = \mathbf{P}_{q-1}^{k} K_{1,1+n}$ is a non-zero map. ■

We now have all the necessary tools to show that $\mathcal{H}_k$ is an irreducible $\mathfrak{osp}(1|2n)$-module.

Corollary 4.2.16. *For $0 \leq k \leq 2n+1$ the space $\mathcal{H}_k$ is an irreducible $\mathfrak{osp}(1|2n)$-module of highest weight*

$$\lambda_k = \sum_{j=1}^{\min(k,2n-k+1)} \delta_j.$$

Proof: Using Theorem 4.2.11 write

$$\mathcal{H}_k = \bigoplus_{j=0}^{\min(2n-k+1,k)} V_{\lfloor\frac{k-j}{2}\rfloor,j}.$$

Let $\ell = \min(2n-k+1, k)$ and define

$$H_k = \begin{cases} \varphi_0 x_1 \cdots x_\ell & \text{if } \ell = k, \\ \varphi_{k-n-1} y x_1 \cdots x_\ell & \text{if } \ell = 2n-k+1. \end{cases}$$

We will show that H_k is an $\mathfrak{osp}(1|2n)$-highest weight vector of $\mathcal{H}_k$. Notice that $H_k \in V_{\lfloor \frac{k-\ell}{2} \rfloor, \ell}$ and it follows from Theorem 4.2.15 that H_k generates $\mathcal{H}_k$. Furthermore, using Lemma 4.1.8 it is straightforward to see that for all $h \in \mathfrak{h}$

$$hH_k = (\delta_1 + \cdots + \delta_\ell)(h) H_k.$$

Thus we need only show that $gH_k = 0$, for all $g \in \mathfrak{n}^+$. Lemma 4.1.8 implies that $g(H_k) = 0$ for all $g \in (\mathfrak{n}^+)_{\overline{0}}$. Therefore it suffices to show that for all $\overline{1} \le j \le \overline{n}$ that $(E_{j,1} + E_{1,j+n})(H_k) = 0$. For $\ell = k$ this is easy to see. Let $\ell = 2n-k+1$, $t = k-n-1$, and for any $\overline{1} \le j \le \overline{n}$ consider the following

$$\begin{aligned} (E_{j,1} + E_{1,j+n})(H_k) &= (x_j \partial_y + y \partial_{j+n})(\varphi_t y x_1 \cdots x_\ell) \\ &= \varphi_t x_j x_1 \cdots x_\ell + (x_j \partial_y + y \partial_{j+n})(\varphi_t) y x_1 \cdots x_\ell \\ &= \varphi_t x_j x_1 \cdots x_\ell \\ &\quad + \left(\sum_{s=0}^{t-1} a_s (2t-2s) y^{2t-2s-1} - y \sum_{s=1}^{t} a_s 2s y^{2t-2s} \theta^{2s-2} \right) x_j x_1 \cdots x_\ell \\ &= \left(\varphi_t - \sum_{s=0}^{t-1} a_s y^{2t-2s} \theta^{2s} \right) x_j x_1 \cdots x_\ell \\ &= a_t \theta^{2t} x_j x_1 \cdots x_\ell, \end{aligned}$$

where the a_s are the coefficients of φ_{k-n-1} defined in Lemma 4.2.9. But each monomial of θ^{2t} has exactly t distinct x_i, for $1 \le i \le n$, and there are $\ell + 1$ such terms in $x_j x_1 \cdots x_\ell$. Thus $\theta^{2t} x_j x_1 \cdots x_\ell = 0$ since each monomial has $t + \ell + 1 = n + 1$ terms with x_i for $1 \leq i \leq n$.

Therefore, H_k is an $\mathfrak{osp}(1|2n)$-highest weight vector of $\mathcal{H}_k$ with weight $\delta_1 + \cdots + \delta_\ell$ and we can conclude that $\mathcal{H}_k$ is an irreducible $\mathfrak{osp}(1|2n)$-module. ■

4.3 Joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$ Highest Weights

With the goal of writing an alternate expression for the joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vectors we define an element $E^{k-n-(1/2)}$, for $n+1 \leq k \leq 2n+1$. Here E is as defined in Equation (4.1.4) and it is worthwhile to notice that the exponent of E will be a half-integer. The elements $E^{k-n-(1/2)}$, for $n+1 \leq k \leq 2n+1$, will be elements of the localization of $\mathcal{P}(\mathbb{C}^{1|2n})$ by $S = \{y^i \mid i \in \mathbb{Z}_{\geq 0}\}$. See Appendix A for more details on the definition of the localization of a non-commutative ring by a central element. Furthermore, since y is not a zero divisor, we have that the map $a \mapsto \frac{a}{1}$ is an injection of $\mathcal{P}(\mathbb{C}^{1|2n})$ into $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$.

For $n+1 \leq \ell \leq 2n+1$, consider the Taylor expansion

$$\begin{aligned} f(x) &= (1+x)^{k-n-1/2} \\ &= \sum_{j=0}^{\infty} \binom{k-n-1/2}{j} x^j. \end{aligned}$$

Using this expansion we write

$$\begin{aligned} E^{k-n-1/2} &= \left(\frac{y^2}{2}\right)^{k-n-1/2} \left(1 - 2\sum_{i=1}^{n} \frac{x_i x_{i+n}}{y^2}\right)^{k-n-1/2} \\ &= \left(\frac{y^2}{2}\right)^{k-n-1/2} \left(\sum_{j=0}^{\infty} \left(\frac{-2}{y^2}\right)^j \binom{k-n-1/2}{j} \left(\sum_{i=1}^{n} x_i x_{i+n}\right)^j\right). \end{aligned}$$

Note that $\sum_{i=1}^{n} x_i x_{i+n}$ is nilpotent of order $n+1$ and therefore

$$E^{k-n-1/2} = \sum_{j=0}^{n} (-1)^j \left(\frac{y^{2k-2n-1-2j}}{2^{k-n-j-1/2}} \right) \binom{k-n-1/2}{j} \left(\sum_{i=1}^{n} x_i x_{i+n} \right)^j . \tag{4.3.1}$$

Thus $E^{k-n-1/2}$ is a well-defined element of $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$.

Now we extend the definition of ∂_y and ∂_i, for $1 \le i \le 2n$, to $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$. In particular, for any $a \in \mathcal{P}(\mathbb{C}^{1|2n})$, $n \in \mathbb{Z}_{\ge 0}$, and $1 \le i \le 2n$ define the following

$$\partial_y \left(\frac{a}{y^n} \right) = \frac{\partial_y(a)}{y^n} - n \frac{a}{y^{n+1}} \text{ and}$$
$$\partial_i \left(\frac{a}{y^n} \right) = \frac{\partial_i(a)}{y^n}.$$

Lemma 4.3.1. *The extension of ∂_y and ∂_i, for $1 \le i \le 2n$, to $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$ is well-defined.*

Proof: Suppose that $\frac{a}{y^n} = \frac{b}{y^m}$ then $y^{p+m}a = y^{p+n}b$, for some $p \in \mathbb{Z}_{\ge 0}$, and we can compute the following in $\mathcal{P}(\mathbb{C}^{1|2n})$

$$\begin{aligned} 0 &= \partial_y(y^{p+m}a - y^{p+n}b) \\ &= y^p(y^{m-1}((p+m)y^{m-1}a + y^m\partial_y(a)) - y^{n-1}((p+n)b + y\partial_y(b))). \end{aligned} \tag{4.3.2}$$

Equation (4.3.2) then implies that in $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$ we have

$$\frac{(p+m)a + y\partial_y(a)}{y^{n-1}} = \frac{(p+n)b + y\partial_y(b)}{y^{m-1}}. \tag{4.3.3}$$

Using our assumption that $\frac{a}{y^n} = \frac{b}{y^m}$ on Equation (4.3.3) we then have

$$\partial_y \left(\frac{a}{y^n} \right) = \frac{\partial_y(a)}{y^n} - n \frac{a}{y^{n+1}} = \frac{\partial_y(b)}{y^m} - m \frac{b}{y^{m+1}} = \partial_y \left(\frac{b}{y^m} \right).$$

Similarly for ∂_i we have

$$\begin{aligned} 0 &= \partial_i(y^{p+m}a - y^{p+n}b) \\ &= y^p(y^m\partial_i(a) - y^n\partial_i(b)). \end{aligned} \tag{4.3.4}$$

Equation (4.3.4) then implies that

$$\partial_i\left(\frac{a}{y^n}\right) = \frac{\partial_i(a)}{y^n} = \frac{\partial_i(b)}{y^m} = \partial_i\left(\frac{b}{y^m}\right).$$

Thus the extension of ∂_y and the ∂_is from $\mathcal{P}(\mathbb{C}^{1|2n})$ to $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$ is well-defined. ■

Lemma 4.3.1 implies that we can extend the action of $\mathfrak{osp}(1|2n)$ on $\mathcal{P}(\mathbb{C}^{1|2n})$ to $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$.

Lemma 4.3.2. *The element $E^{k-n-(1/2)}$ defined as in Equation (4.3.1), for $n+1 \leq k \leq 2n+1$, is invariant under the action of $\mathfrak{osp}(1|2n)$.*

Proof: Consider

$$\begin{aligned} E^{1/2} &= \left(\frac{y^2}{2}\right)^{1/2}\left(\sum_{j=0}^{\infty}\left(\frac{-2}{y^2}\right)^j\binom{1/2}{j}\left(\sum_{i=1}^{n}x_ix_{i+n}\right)^j\right) \\ &= \left(\frac{y^2}{2}\right)^{1/2}\left(\sum_{j=0}^{n}\left(\frac{-2}{y^2}\right)^j\binom{1/2}{j}\left(\sum_{i=1}^{n}x_ix_{i+n}\right)^j\right). \end{aligned}$$

Furthermore, since E is invariant under the action of $\mathfrak{osp}(1|2n)$ and $E = (E^{1/2})^2$ then for any basis element $x \in \mathfrak{osp}(1|2n)$ we have

$$\begin{aligned} 0 &= x(E^{1/2})^2 \\ &= 2E^{1/2}x(E^{1/2}). \end{aligned}$$

But y invertible in $S^{-1}\mathcal{P}(\mathbb{C}^{1|2n})$ and $\sum_{i=1}^{n} x_i x_{i+n}$ is nilpotent, thus we have that $E^{1/2}$ is a unit and therefore $x(E^{1/2}) = 0$. That is, $E^{1/2}$ is invariant under the action of $\mathfrak{osp}(1|2n)$. Thus $E^{k-n-1/2}$ can be written as a product of $\mathfrak{osp}(1|2n)$-invariant elements and is therefore $\mathfrak{osp}(1|2n)$-invariant. ■

For the next theorem, we would like to thank V. Serganova for the related e-mail correspondence.

Theorem 4.3.3. *1. For every $0 \leq k \leq 2n+1$, there is a unique (up to scalar multiple) joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vector in $\mathcal{P}(\mathbb{C}^{1|2n})$. We denote this unique vector by H_k.*

2. Let $\ell = \min(k, 2n-k+1)$. Then H_k is given by the equation

$$H_k = \begin{cases} x_1 \cdots x_\ell & \text{if } \ell = k, \\ x_1 \cdots x_\ell (E)^{k-n-(1/2)} & \text{if } \ell = 2n-k+1. \end{cases}$$

and is of highest weight

$$\lambda_k = \sum_{i=1}^{\ell} \delta_i,$$

where $(E)^{k-n-(1/2)}$ is defined in Equation (4.3.1).

Proof:

1. Existences and uniqueness follows from the Corollary 4.2.16. In particular, for each $0 \leq k \leq 2n+1$ the space $\mathcal{H}_k$ is an irreducible $\mathfrak{osp}(1|2n)$-highest weight module and it then follows that there is a unique (up to scalar) highest weight vector.

2. When $0 \leq k \leq n$ it is straight-forward to show that $H_k = x_1 \cdots x_k$ is a joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vector of weight $\delta_1 + \cdots + \delta_\ell$.

   For $n+1 \leq k \leq 2n+1$, let $\ell = 2n - k + 1$ and we consider H_k. Since we are multiplying $E^{k-n-(1/2)}$ by $x_1 \cdots x_\ell$ then we can truncate the outer sum of Equation (4.3.1) to end at $j = n - \ell$ because any summand after this would be zero. Furthermore, we factor y from the term $E^{k-n-(1/2)}$ to get

$$H_k = yx_1 \cdots x_\ell \sum_{j=0}^{n-\ell} (-1)^j \left(\frac{y^{2(k-n-1-j)}}{2^{k-n-j-1/2}} \right) \binom{k-n-1/2}{j} \left(\sum_{i=1}^{n} x_i x_{i+n} \right)^j . \tag{4.3.5}$$

   It is straightforward to check from Equation 4.3.5 that $H_k \in \mathcal{P}^k$. Using Lemma 4.1.8 it is easy to verify by a direct calculation that for all $h \in \mathfrak{h}$

$$hH_k = (\delta_1 + \cdots + \delta_\ell)(h)H_k.$$

   Furthermore, by Lemma 4.3.2 and the Leibniz rule we have that for any basis vector $g \in \mathfrak{n}^+$

$$\begin{aligned} g(x_1 \cdots x_\ell (E)^{k-n-1/2}) &= g(x_1 \cdots x_\ell)(E)^{k-n-1/2} + x_1 \cdots x_\ell g((E)^{k-n-1/2}) \\ &= 0. \end{aligned}$$

   Therefore we can conclude that H_k is a joint $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vector of weight $\delta_1 + \cdots + \delta_\ell$. ■

Example 4.3.4. For $n = 1, 2$ we compute all the possible $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vectors. First, we calculate the possible $E^{k-n-1/2}$ that occur in both of these cases. For $n = 1$,

$$E^{2-1-1/2} = \frac{1}{\sqrt{2}}\left(y - \frac{x_1 x_2}{y}\right) \text{ and}$$
$$E^{3-1-1/2} = \frac{1}{2^{3/2}}\left(y^3 - 3y x_1 x_2\right).$$

For $n = 2$,

$$E^{3-2-1/2} = \frac{1}{\sqrt{2}}\left(y - \frac{x_1x_3 + x_2x_4}{y} + \frac{x_1x_2x_3x_4}{y^3}\right)$$
$$E^{4-2-1/2} = \frac{1}{2^{3/2}}\left(y^3 - 3y(x_1x_3 + x_2x_4) - 3\frac{x_1x_2x_3x_4}{y}\right) \text{ and}$$
$$E^{5-2-1/2} = \frac{1}{2^{5/2}}\left(y^5 - 5y^3(x_1x_3 + x_2x_4) - 15yx_1x_2x_3x_4\right).$$

Therefore we have the following $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{osp}(1|2n)$-highest weight vectors for $n = 1, 2$

k	H_k
0	1
1	x_1
2	yx_1
3	$y^3 - 3yx_1x_2$

Table 4.2: Harmonic $\mathfrak{osp}(1|2n)$-Highest Weight Vectors for $n = 1$

k	H_k
0	1
1	x_1
2	x_1x_2
3	yx_1x_2
4	$y^3x_1 - 3yx_1x_2x_4$
5	$y^5 - 5y^3(x_1x_3 + x_2x_4) - 15yx_1x_2x_3x_4$

Table 4.3: Harmonic $\mathfrak{osp}(1|2n)$-Highest Weight Vectors for $n = 2$

Chapter 5

Capelli Operators and Results

In this chapter we utilize the irreducibility of $\mathcal{H}_k$ shown in Chapter 4 to decompose $\mathcal{P}^k(\mathbb{C}^{1|2n})$ into irreducible $\mathfrak{osp}(1|2n)$-modules (Lemma 5.1.2). Subsequently this gives a decomposition of $\mathcal{P}(\mathbb{C}^{1|2n})$ into irreducible $\mathfrak{osp}(1|2n)$-modules. With the goal of decomposing $\mathcal{P}(\mathbb{C}^{1|2n})$ into multiplicity-free modules we introduce the action of $\mathfrak{gosp}(1|2n) = \mathbb{C} \oplus \mathfrak{osp}(1|2n)$ and define the Capelli Operators.

In the latter part of this chapter we explore a problem closely related to the Capelli Eigenvalue problem. Following the footsteps of Howe and Umeda we consider the abstract Capelli problem which asks:

Problem 5.0.1. Is the map $\mathcal{Z}(\mathfrak{gosp}(1|2n)) \mapsto \mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{gosp}(1|2n)}$ a surjective map?

We relate the surjectivity of the map from $\mathcal{Z}(\mathfrak{gosp}(1|2n))$ to $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{gosp}(1|2n)}$ to the non-vanishing of certain determinants. The main new result in this chapter is Theorem 5.2.10 where we give a complete factorization of many of the determinants with roots and their multiplicity.

5.1 Capelli Operators

The space $\mathcal{P}(\mathbb{C}^{1|2n})$ decomposes as

$$\mathcal{P}(\mathbb{C}^{1|2n}) = \bigoplus_{k=0}^{\infty} \mathcal{P}^k(\mathbb{C}^{1|2n}).$$

Recall $E = (1/2)R^2$ as defined in Equation (4.1.4) and $\mathcal{H}_k$ as defined in Definition 4.2.1.

Proposition 5.1.1. *The space $E^i\mathcal{H}_k$ is an irreducible $\mathfrak{osp}(1|2n)$-module of highest weight*

$$\lambda_k = \sum_{j=1}^{\min(k,2n-k+1)} \delta_j.$$

Proof: By commutativity of the action of $\mathfrak{g}$ with E, injectivity of E, and Schur's Lemma we have that as $\mathfrak{osp}(1|2n)$-modules $E^i\mathcal{H}_k \cong \mathcal{H}_k$. The result then follows from Corollary 4.2.16. ∎

Lemma 5.1.2 (Fischer decomposition)**.** *For $k \in \mathbb{Z}_{\geq 0}$ and $\ell = \lfloor \frac{k}{2} \rfloor$. Then $\mathcal{P}^k(\mathbb{C}^{1|2n})$ can be decomposed as a direct sum of irreducible $\mathfrak{osp}(1|2n)$-modules*

$$\mathcal{P}^k(\mathbb{C}^{1|2n}) = \begin{cases} \mathcal{H}_k \oplus E\mathcal{H}_{k-2} \oplus \cdots \oplus E^\ell \mathcal{H}_{k-2\ell} & \text{for } k \leq 2n+1 \text{ and} \\ E\mathcal{P}^{k-2}(\mathbb{C}^{1|2n}) & \text{for } k > 2n+1. \end{cases}$$

Proof: Let $k \in \mathbb{Z}_{\geq 0}$ and $\ell = \lfloor \frac{k}{2} \rfloor$. We will prove this for $0 \leq k \leq 2n+1$ then an analogous argument can be done for $k > 2n+1$.

First, note that $E^i\mathcal{H}_{k-2i} \subseteq \mathcal{P}_k(\mathbb{C}^{1|2n})$, for each $0 \leq i \leq \ell$. Furthermore, for each $0 \leq i < j \leq \ell$ we have $E^i\mathcal{H}_{k-2i} \cap E^j\mathcal{H}_{k-2j} = 0$, since Proposition 5.1.1 implies that $E^i\mathcal{H}_{k-2i} \cong E^j\mathcal{H}_{k-2j}$ if and only if $k-2i = 2n-(k-2j)+1$. This leads to a contradiction since we would then have that $-2(i+j) = -2k+1$. Therefore,

$$\mathcal{H}_k \oplus E\mathcal{H}_{k-2} \oplus \cdots \oplus E^\ell \mathcal{H}_{k-2\ell} \subseteq \mathcal{P}_k. \tag{5.1.1}$$

Since $E^i\mathcal{H}_{k-2i} \cong \mathcal{H}_{k-2i}$ we use Corollary 4.2.2 to show that the dimension of the LHS of Equation (5.1.1) is

$$\begin{aligned}\sum_{i=0}^{\ell} \dim \mathcal{H}_{k-2i} &= \left(\sum_{i=0}^{\ell-1} \dim \mathcal{P}^{k-2i} - \dim \mathcal{P}^{k-2(i+1)} \right) + \dim \mathcal{H}_{k-2\ell} \\ &= \dim \mathcal{P}^k.\end{aligned}$$

Thus $\mathcal{P}^k(\mathbb{C}^{1|2n}) = \mathcal{H}_k \oplus E\mathcal{H}_{k-2} \oplus \cdots \oplus E^\ell\mathcal{H}_{k-2\ell}$, for $0 \le k \le 2n+1$. An analogous argument applies for $k \ge 2n+2$ and therefore

$$\mathcal{P}^k(\mathbb{C}^{1|2n}) = \begin{cases} \mathcal{H}_k \oplus E\mathcal{H}_{k-2} \oplus \cdots \oplus E^\ell \mathcal{H}_{k-2\ell} & \text{for } k \le 2n+1 \text{ and} \\ E\mathcal{P}^{k-2}(\mathbb{C}^{1|2n}) & \text{for } k > 2n+1. \end{cases}$$

■

Lemma 5.1.2 gives us a complete decomposition of $\mathcal{P}(\mathbb{C}^{1|2n})$ into $\mathfrak{osp}(1|2n)$-modules but notice that this decomposition is not multiplicity-free. Thus we introduce the Lie superalgebra $\mathfrak{gosp}(1|2n) = \mathbb{C}\mathbb{E} \oplus \mathfrak{osp}(1|2n)$, where $\mathbb{E}$ generates $\mathbb{C}\mathbb{E}$ and $\mathbb{C}\mathbb{E}$ is even and commutes with $\mathfrak{osp}(1|2n)$. The action of $\mathfrak{osp}(1|2n)$ is consistent with our originally defined action and the action of $\mathbb{C}\mathbb{E} \subseteq \mathfrak{gosp}(1|2n)$ on $\mathcal{P}(\mathbb{C}^{1|2n})$ is defined by

$$\mathbb{E} \mapsto y\partial_y + \sum_{p=1}^{2n} x_p \partial_p. \tag{5.1.2}$$

Thus the decomposition of $\mathcal{P}(\mathbb{C}^{1|2n})$ given by Lemma 5.1.2 is a multiplicity-free decomposition into $\mathfrak{gosp}(1|2n)$-modules because we can distinguish irreducible representations using their $\mathfrak{osp}(1|2n)$-highest weight and using $\mathbb{E}$ we can distinguish the $\mathcal{P}^k(\mathbb{C}^{1|2n})$ which they are contained in.

We define $\mathcal{V}_{k,i} = E^i\mathcal{H}_{k-2i}$. Then it turns out that the irreducible $\mathfrak{gosp}(1|2n)$-submodules of $\mathcal{P}(\mathbb{C}^{1|2n})$ are parametrized by

$$\Omega = \left\{(k,i) \middle| k,i \in \mathbb{Z}_{\geq 0},\ \left\lfloor \frac{k}{2} \right\rfloor - n \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \right\}. \tag{5.1.3}$$

Since $\mathbb{C}^{1|2n} \cong (\mathbb{C}^{1|2n})^*$ it follows that $\mathcal{P}((\mathbb{C}^{1|2n})^*) \cong \mathcal{D}(\mathbb{C}^{1|2n})$ also decomposes into a multiplicity-free direct sum of dual $\mathfrak{gosp}(1|2n)$-modules. These modules will also be indexed by Ω and as $\mathfrak{gosp}(1|2n)$-modules we have

$$\begin{aligned}\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{gosp}(1|2n)} &\cong \left(\mathcal{P}(\mathbb{C}^{1|2n}) \otimes \mathcal{D}(\mathbb{C}^{1|2n})\right)^{\mathfrak{gosp}(1|2n)} \\ &\cong \bigoplus_{\mu,\nu\in\Omega} (\mathcal{V}_\mu \otimes \mathcal{V}_\nu^*)^{\mathfrak{gosp}(1|2n)} \\ &\cong \bigoplus_{\mu,\nu\in\Omega} \mathrm{Hom}_{\mathfrak{gosp}(1|2n)}(\mathcal{V}_\nu, \mathcal{V}_\mu). \end{aligned} \tag{5.1.4}$$

Definition 5.1.3 (Capelli Operators)**.** *The element $D_\mu \in \mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{gosp}(1|2n)}$ that maps to $\mathrm{id}_{\mathcal{V}_\mu} \in \mathrm{Hom}_{\mathfrak{gosp}(1|2n)}(\mathcal{V}_\mu, \mathcal{V}_\mu)$ by the isomorphism in Equation (5.1.4) is called a* ***Capelli Operator****.*

5.2 Results

For the remainder of this thesis let $\mathfrak{g} = \mathfrak{gosp}(1|2n)$ and the action of $\mathfrak{g}$ on $\mathcal{P}(\mathbb{C}^{1|2n})$ be as in the previous section. Recall that Lemma 5.1.2 gives a multiplicity-free decomposition of $\mathcal{P}(\mathbb{C}^{1|2n})$, indexed by Ω defined in Equation (5.1.3), as a $\mathfrak{g}$-module.

We would like to show that $\mathcal{Z}(\mathfrak{g})$ maps surjectively $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{g}}$. Recall the Casimir element of $\mathfrak{osp}(1|2n)$, given in Proposition 3.3.5 and denoted by C, and the Euler operator, given by the RHS of Equation 5.1.2, denoted by $\mathbb{E}$.

Lemma 5.2.1. *The elements* $\mathbb{E}, C \in \mathcal{Z}(\mathfrak{g})$.

Proof: Proposition 3.3.3 tells us that $C \in \mathcal{Z}(\mathfrak{osp}(1|2n))$ and since $\mathbb{E}$ commutes with $\mathfrak{osp}(1|2n)$ we have that $\mathbb{E}, C \in \mathcal{Z}(\mathfrak{g})$. ■

Thus we will use the image of $\mathbb{E}$ and C in $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{g}}$. In particular, we will look at the action of $C^p\mathbb{E}^q$, for $p, q \in \mathbb{Z}_{\geq 0}$, on the irreduicible representations of $\mathcal{P}(\mathbb{C}^{1|2n})$.

Lemma 5.2.2. *If* V_{λ_k} *is an* $\mathfrak{osp}(1|2n)$*-highest weight module of weight* $\lambda_k = \delta_1 + \cdots + \delta_k$*, for* $0 \leq k \leq n$*, then* C *acts on* V_{λ_k} *by* $(2n+1)k - k^2$.

Proof: Proposition 3.3.4 tells us that C acts on V_{λ_k} by

$$\langle \lambda_k + \rho, \lambda_k + \rho \rangle - \langle \rho, \rho \rangle.$$

Recall ρ is the Weyl vector defined in Equation (2.5.1) as $\rho = \frac{1}{2}(\rho_0 - \rho_1)$, where ρ_i is the sum of all positive roots of parity i. Equation (2.3.4) gives us the positive roots for $\mathfrak{osp}(1|2n)$:

$$\{\delta_i \pm \delta_j, \delta_k, 2\delta_k \mid i, j, k \in I(0|n),\ i < j\}$$

Therefore we have that

$$\rho = \sum_{i=1}^{n} (n - i + \frac{1}{2})\delta_i.$$

Thus by a direct calculation it follows that

$$\langle \lambda_k + \rho, \lambda_k + \rho \rangle - \langle \rho, \rho \rangle = (2n+1)k - k^2.$$ ■

Corollary 5.2.3. *Let* $(k, i) \in \Omega$*,* $p, q \in \mathbb{Z}_{\geq 0}$*, and* $\ell = \min(k - 2i, 2n - k + 2i + 1)$.

Then $C^p\mathbb{E}^q$ acts on $\mathcal{V}_{k,i}$ by the scalar

$$\lambda_{(p,q),(k,i)} = ((2n+1)\ell - \ell^2)^p k^q.$$

Proof: This follows from Proposition 5.1.1, Lemma 5.2.2 and the fact that $\mathbb{E}$ acts by degree on homogeneous polynomials. ■

Note that the D_μ form a basis for $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{gosp}(1|2n)}$. The order of any D_ν is given by d where $V_\nu \subseteq \mathcal{P}^d(\mathbb{C}^{1|2n})$ and $C^i\mathbb{E}^j$ is of order $2i+j$. For D_ν, of order d, we would like to write D_ν as a linear combination of central elements of the form $C^i\mathbb{E}^j$, where $2i+j \leq d$. If this can be done it then follows that $\mathcal{Z}(\mathfrak{g})$ maps surjectively onto $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{g}}$ since the D_ν form a basis of $\mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{g}}$ and each $C^i\mathbb{E}^j \in \mathcal{Z}(\mathfrak{g})$. This together with the Harish-Chandra homomorphism should lead to a formula for eigenvalues of D_μ with respect to the eigenvalues of C and $\mathbb{E}$.

For $d \in \mathbb{Z}_{\geq 0}$ define the set

$$\Omega_d = \{(k,i) \in \Omega \mid k \leq d\}. \tag{5.2.1}$$

Notice that for $d \leq 2n+1$ that $|\Omega_d| = \left(\lfloor \frac{d}{2} \rfloor + 1\right)\left(d - \lfloor \frac{d}{2} \rfloor + 1\right)$. Furthermore, define

$$\Theta_d = \{(i,j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid 2i+j \leq d\}. \tag{5.2.2}$$

It is straightforward to show that $|\Theta_d| = \left(\lfloor \frac{d}{2} \rfloor + 1\right)\left(d - \lfloor \frac{d}{2} \rfloor + 1\right)$, $d \in \mathbb{Z}_{\geq 0}$.

Definition 5.2.4. *For $0 \leq d \leq 2n+1$, define the square matrix*

$$M_d = \left[\lambda_{(p,q),(k,i)}\right]_{(p,q)\in\Theta_d,(k,i)\in\Omega_d}.$$

Example 5.2.5. For $d = 2, 3$ we have the following matrices with entries which are polynomials in a parameter n.

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 4 & 4 \\ 0 & 2n & 0 & 4n-2 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 1 & 4 & 4 & 9 & 9 \\ 0 & 1 & 8 & 8 & 27 & 27 \\ 0 & 2n & 0 & 4n-2 & 2n & 6n-6 \\ 0 & 2n & 0 & 8n-4 & 6n & 18n-18 \end{pmatrix}$$

Furthermore, we calculate the determinants of the first several matrices:

$$\det(M_2) = 8\left(n - \frac{1}{2}\right)$$
$$\det(M_3) = -192\left(n - \frac{3}{2}\right)\left(n - \frac{1}{2}\right)$$
$$\det(M_4) = -1179648\left(n - \frac{5}{2}\right)\left(n - \frac{3}{2}\right)^2\left(n - \frac{1}{2}\right)^2$$
$$\det(M_5) = 108716359680\left(n - \frac{7}{2}\right)\left(n - \frac{5}{2}\right)^2\left(n - \frac{1}{2}\right)^2\left(n - \frac{3}{2}\right)^3$$

For the remainder of this section fix $d \in \mathbb{Z}_{\geq 2}$ and consider M_d as matrix with coefficients a polynomials in the parameter n.

Lemma 5.2.6. *Let $p, q \in \mathbb{Z}_{\geq 0}$ with $p \geq 1$ and $(k,i), (k,i') \in \Omega$. Then*

$$\lambda_{(p,q),(k,i)} - \lambda_{(p,q),(k,i')}$$

is divisible by $\left(n-\frac{\ell+\ell'-1}{2}\right)$, *where* $\ell=k-2i$ *and* $\ell'=k-2i'$.

Proof: Let $p,q\in\mathbb{Z}_{\geq 0}$ with $p\geq 1$ and distinct $(k,i),(k,i')\in\Omega$. Set $\ell=k-2i$ and $\ell'=k-2i'$. Consider the following

$$\begin{aligned}\lambda_{(p,q),(k,i)}-\lambda_{(p,q),(k,i')}&=k^q\left(((2n+1)\ell-\ell^2)^p-((2n+1)\ell'-\ell'^2)^p\right)\\&=\frac{k^q}{2}(\ell-\ell')\left(n-\frac{\ell+\ell'-1}{2}\right)\\&\qquad\left(\sum_{j=0}^{p-1}((2n+1)\ell-\ell^2)^{p-1-j}((2n+1)\ell'-\ell'^2)^j\right).\end{aligned}$$

Thus it is clear that $\left(n-\frac{\ell+\ell'-1}{2}\right)$ divides $\lambda_{(p,q),(k,i)}-\lambda_{(p,q),(k,i')}$. ■

Note that as (p,q) varies in Lemma 5.2.6 the term $\left(n-\frac{\ell+\ell'-1}{2}\right)$ can always be factored. That is, we can apply column operations to $\det(M_d)$ and factor terms from columns to determine the linear factors of $\det(M_d)$. This leads us to our next lemma where we will fix $2\leq k\leq d$ and explore the possible roots that may be factored from the columns of M_d indexed by $(k,i)\in\Omega_d$.

Lemma 5.2.7. *Let* $2\leq k\leq d$. *The term*

$$\left(n-\frac{2\ell+1}{2}\right)^{g_k(2\ell+1)}$$

divides $\det(M_d)$, *where* $\ell=0,1,2\ldots,k-2$ *and*

$$g_k(2\ell+1)=\begin{cases}\left\lfloor\frac{\ell+2-\left(k-2\lfloor\frac{k}{2}\rfloor\right)}{2}\right\rfloor & \text{if } 1\leq 2\ell+1\leq k\\ \left\lfloor\frac{k-\ell}{2}\right\rfloor & \text{if } k<2\ell+1\leq 2k-3\\ 0 & \text{otherwise.}\end{cases}$$

Proof: Take distinct $(k,i),(k,i')\in\Omega_d$ and let $j=k-2i$ and $j'=k-2i'$. The possible values for j and j' are

$$k - 2\left\lfloor \frac{k}{2} \right\rfloor, k - 2\left\lfloor \frac{k}{2} \right\rfloor + 2, \ldots, k.$$

Since i and i' are distinct we may without loss of generality assume that $j' < j$. Lemma 5.2.6 tells us that when we subtract the column indexed by (k, i') from (k, i) we can factor a linear term with root $\frac{j+j'-1}{2}$.

Table 5.1 gives the possible values for $\frac{j+j'-1}{2}$ when k is even and Table 5.2 does the same for odd k. In particular, the vertical axis gives the value of j' and the horizontal gives the value of j.

$j' < j$	2	4	6	$\cdots$	$k-2$	k
0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\cdots$	$\frac{k-3}{2}$	$\frac{k-1}{2}$
2		$\frac{5}{2}$	$\frac{7}{2}$	$\cdots$	$\frac{k-1}{2}$	$\frac{k+1}{2}$
4			$\frac{9}{2}$	$\cdots$	$\frac{k+1}{2}$	$\frac{k+3}{2}$
$\vdots$				$\ddots$	$\vdots$	$\vdots$
$k-4$					$\frac{2k-7}{2}$	$\frac{2k-5}{2}$
$k-2$						$\frac{2k-3}{2}$

Table 5.1: Values of $\frac{j+j'-1}{2}$ for k even

$j' < j$	3	5	7	$\cdots$	$k-2$	k
1	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\cdots$	$\frac{k-2}{2}$	$\frac{k}{2}$
3		$\frac{7}{2}$	$\frac{9}{2}$	$\cdots$	$\frac{k}{2}$	$\frac{k+2}{2}$
5			$\frac{11}{2}$	$\cdots$	$\frac{k+2}{2}$	$\frac{k+4}{2}$
$\vdots$				$\ddots$	$\vdots$	$\vdots$
$k-4$					$\frac{2k-7}{2}$	$\frac{2k-5}{2}$
$k-2$						$\frac{2k-3}{2}$

Table 5.2: Values of $\frac{j+j'-1}{2}$ for k odd

Thus the the linear terms $\left(n - \frac{j+j'-1}{2}\right)$ have possible roots

$$\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2k-3}{2}$$

when k is even and

$$\frac{3}{2}, \frac{5}{2}, \ldots, \frac{2k-3}{2}$$

when k is odd.

Fix a root $\frac{2\ell+1}{2}$. The number of columns that $\left(n - \frac{2\ell+1}{2}\right)$ can be factored from corresponds to the number columns of Tables 5.1 or 5.2 that contain $\frac{2\ell+1}{2}$. By a direct calculation, the term $\left(n - \frac{2\ell+1}{2}\right)$ can be simultaneously factored from the columns of $\det(M_d)$ indexed by $\{(k,i) \mid (k,i) \in \Omega\}$ with multiplicity given by

$$g_k\left(2\ell+1\right) = \begin{cases} \left\lfloor \frac{\ell+2-(k-2\lfloor \frac{k}{2} \rfloor)}{2} \right\rfloor & \text{if } 1 \leq 2\ell+1 \leq k \\ \left\lfloor \frac{k-\ell}{2} \right\rfloor & \text{if } k < 2\ell+1 \leq 2k-3 \\ 0 & \text{otherwise.} \end{cases}$$

That is, for fixed k the term

$$\left(n - \frac{2\ell+1}{2}\right)^{g_k(2\ell+1)}$$

can be factored from the columns indexed by $(k,i) \in \Omega_d$ in $\det(M_d)$. ■

Lemma 5.2.8. *For $d \in \mathbb{Z}_{\geq 2}$ and $\ell = 0, 1, 2, \cdots, d-2$, the term*

$$\left(n - \frac{2\ell+1}{2}\right)^{f(d,2\ell+1)}$$

factors from $\det(M_d)$*, where*

$$f(d,2\ell+1)=\begin{cases} f(d-1,2\ell+1)+\left\lfloor\frac{\ell+2-(d-2\lfloor\frac{d}{2}\rfloor)}{2}\right\rfloor & \text{if } 1\le 2\ell+1\le d-2,\\ \frac{1}{2}\left(\left\lfloor\frac{d-\ell}{2}\right\rfloor\left(\left\lfloor\frac{d-\ell}{2}\right\rfloor+1\right)+\left\lfloor\frac{d-\ell-1}{2}\right\rfloor\left(\left\lfloor\frac{d-\ell-1}{2}\right\rfloor+1\right)\right) & \text{if } d-1\le 2\ell+1\le 2d-3,\\ 0 & \text{if } 2d-2\le 2\ell+1.\end{cases}$$

Proof: Applying Lemma 5.2.7 for all possible $2 \le k \le d$ implies that $\det(M_d)$ has linear factors with roots:

$$\frac{1}{2},\frac{3}{2},\ldots,\frac{2d-5}{2},\frac{2d-3}{2}. \tag{5.2.3}$$

Fix $\frac{2\ell+1}{2}$ from the list in Equation (5.2.3). We can factor $\left(n-\frac{2\ell+1}{2}\right)^{g_k(2\ell+1)}$ for each $2\le k\le d$ using Lemma 5.2.7. Therefore $\left(n-\frac{2\ell+1}{2}\right)$ can be factored from $\det(M_d)$ with multiplicity

$$\sum_{k=2}^{d} g_k\left(2\ell+1\right). \tag{5.2.4}$$

Next we show that Equation (5.2.4) is equal to $f(d,2\ell+1)$ as defined in the statement of the Lemma. We split into two cases: 1. when $d-1\le 2\ell+1\le 2d-3$ and 2. when $1\le 2\ell+1\le d-2$.

1. For the case $d-1\le 2\ell+1\le 2d-3$. Notice that $g_k(2\ell+1)=0$ for all $k<\ell+2$ and therefore

$$\begin{aligned}\sum_{k=2}^{d} g_k\left(2\ell+1\right) &= \sum_{k=\ell+2}^{d} g_k\left(2\ell+1\right)\\ &= g_d(2\ell+1)+g_{d-1}(2\ell+1)+\sum_{k=\ell+2}^{d-2}\left\lfloor\frac{k-\ell}{2}\right\rfloor\end{aligned}$$

$$= g_d(2\ell+1) + g_{d-1}(2\ell+1) + \sum_{i=2}^{d-\ell-2} \left\lfloor \frac{i}{2} \right\rfloor.$$

By direct calculation it is easy to verify that when $2\ell+1 = d-1$ then

$$g_{d-1}(2\ell+1) = \left\lfloor \frac{\ell+2-(d-1-2\lfloor\frac{d-1}{2}\rfloor)}{2} \right\rfloor = \left\lfloor \frac{d-\ell-1}{2} \right\rfloor$$
$$g_d(2\ell+1) = \left\lfloor \frac{\ell+2-(d-2\lfloor\frac{d}{2}\rfloor)}{2} \right\rfloor = \left\lfloor \frac{d-\ell}{2} \right\rfloor.$$

Similarly, when $2\ell+1 = d$ then

$$g_{d-1}(2\ell+1) = \left\lfloor \frac{d-1-\ell}{2} \right\rfloor$$
$$g_d(2\ell+1) = \left\lfloor \frac{\ell+2-(d-2\lfloor\frac{d}{2}\rfloor)}{2} \right\rfloor = \left\lfloor \frac{d-\ell}{2} \right\rfloor.$$

Therefore

$$\begin{aligned}\sum_{k=2}^{d} g_k(2\ell+1) &= \sum_{i=2}^{d-\ell} \left\lfloor \frac{i}{2} \right\rfloor \\ &= \frac{1}{2}\left(\left\lfloor \frac{d-\ell}{2} \right\rfloor \left(\left\lfloor \frac{d-\ell}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{d-\ell-1}{2} \right\rfloor \left(\left\lfloor \frac{d-\ell-1}{2} \right\rfloor + 1 \right) \right) \\ &= f(d, 2\ell+1).\end{aligned}$$

2. For the case when $1 \le 2\ell+1 \le d-2$, we fix $\ell \in \mathbb{Z}_{\ge 0}$ and write $s = 2\ell+1$. Then by induction on $d \in \mathbb{Z}_{\ge s+2}$ we show that

$$\sum_{k=2}^{d} g_k(s) = f(d-1, s) + \left\lfloor \frac{\ell+2-(d-2\lfloor\frac{d}{2}\rfloor)}{2} \right\rfloor.$$

First note that, when $d = s + 2$ then from the previous case we have

$$f(d-1, d-2) = \sum_{k=2}^{d-1} g_k(d-2). \tag{5.2.5}$$

Furthermore, for any $d \in \mathbb{Z}_{\geq s+2}$ we have

$$g_d(s) = \left\lfloor \frac{\ell + 2 - (d - 2\lfloor \frac{d}{2} \rfloor)}{2} \right\rfloor. \tag{5.2.6}$$

By summing Equation (5.2.6) and (5.2.5) we have

$$\sum_{k=2}^{d} g_k(s) = f(d-1, s) + \left\lfloor \frac{\ell + 1 - (d - 2\lfloor \frac{d}{2} \rfloor)}{2} \right\rfloor = f(d, s).$$

By induction it follows that $f(d, s) = \sum_{k=2}^{d} g_k(s)$ for any $1 \leq s \leq d-2$. ■

In the next lemma we prove a stronger version of Lemma 5.2.8. Indeed, the proof of Lemma 5.2.9 depends on Lemma 5.2.8.

Lemma 5.2.9. *For $d \in \mathbb{Z}_{\geq 2}$ then*

$$\sum_{(p,q)\in\Theta_d} p = \sum_{\ell=0}^{d-2} f(d, 2\ell + 1) \tag{5.2.7}$$

where Θ_d is defined in Equation (5.2.2).

Proof: We prove this by induction on $d \in \mathbb{Z}_{\geq 2}$. For the base case when $d = 2$, we have

$$\begin{aligned}\sum_{(p,q)\in\Theta_2} p &= 1 \\ &= \frac{1}{2}\left(\left\lfloor \frac{2-0}{2} \right\rfloor \left(\left\lfloor \frac{2-0}{2} \right\rfloor + 1\right) + \left\lfloor \frac{2-0-1}{2} \right\rfloor \left(\left\lfloor \frac{2-0-1}{2} \right\rfloor + 1\right)\right)\end{aligned}$$

$$= f(d, 1).$$

Thus we assume that there is some $d \in \mathbb{Z}_{\geq 2}$ such that

$$\sum_{(p,q)\in\Theta_{d-1}} p = \sum_{\ell=0}^{d-3} f(d-1, 2\ell+1).$$

Notice that for any $d \in \mathbb{Z}_{\geq 2}$

$$\Theta_d = \Theta_{d-1} \cup \left\{ (p, d-2p) \,\middle|\, p = 0, 1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor \right\}.$$

Now consider the following calculations

$$\begin{aligned}
\sum_{(p,q)\in\Theta_d} p &= \sum_{(p,q)\in\Theta_{d-1}} p + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} i \\
&= \sum_{\ell=0}^{d-3} f(d-1, 2\ell+1) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} i \\
&= \sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor - 1} f(d-1, 2\ell+1) + \sum_{\ell=\lfloor \frac{d-2}{2} \rfloor}^{d-3} f(d-1, 2\ell+1) + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} i \\
&= \left(\sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor - 1} f(d-1, 2\ell+1) + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} i \right) + \sum_{\ell=\lfloor \frac{d-2}{2} \rfloor + 1}^{d-2} f(d, 2\ell+1),
\end{aligned}$$

where the last equality is given by the fact that $f(d-1, 2(\ell-1)+1) = f(d, 2\ell+1)$ when $d-1 \leq 2\ell+1$. Thus we will show that

$$\sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor - 1} f(d-1, 2\ell+1) + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} i = \sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor} f(d, 2\ell+1). \tag{5.2.8}$$

We do this by first manipulating the RHS of Equation (5.2.8) using the definition of

$f(d, 2\ell+1)$ for $1 \leq 2\ell+1 \leq d-2$

$$\sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor} f(d, 2\ell+1) = f\left(d, 2\left\lfloor \frac{d-2}{2} \right\rfloor + 1\right) + \sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor - 1} f(d-1, 2\ell+1) + \left\lfloor \frac{\ell + 2 - (d - 2\lfloor \frac{d}{2} \rfloor)}{2} \right\rfloor.$$

That is, we will show that

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} i = f\left(d, 2\left\lfloor \frac{d-2}{2} \right\rfloor + 1\right) + \sum_{\ell=0}^{\lfloor \frac{d-2}{2} \rfloor - 1} \left\lfloor \frac{\ell + 2 - (d - 2\lfloor \frac{d}{2} \rfloor)}{2} \right\rfloor. \tag{5.2.9}$$

Finally we show Equation (5.2.9) by considering the two cases d even or odd. When $d = 2k+1$, for some $k \in \mathbb{Z}$, then $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} i = \frac{k(k+1)}{2}$ and the RHS of Equation (5.2.9) is equal to

$$\begin{aligned} f(d-1, 2k-1) + \sum_{\ell=1}^{k-1} \left\lfloor \frac{\ell+1}{2} \right\rfloor &= \frac{1}{2}\left(\left\lfloor \frac{d-k}{2} \right\rfloor \left(\left\lfloor \frac{d-k}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{d-k-1}{2} \right\rfloor \left(\left\lfloor \frac{d-k-1}{2} \right\rfloor + 1 \right) \right. \\ &\quad \left. + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{k-1}{2} \right\rfloor \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1 \right) \right) \\ &= \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1 \right)^2 + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \\ &= \frac{k(k+1)}{2}, \end{aligned}$$

where the last equality is given by simply checking the two cases of k either even or odd.

Similarly, when $d = 2k$ then again $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} i = \frac{k(k+1)}{2}$ and the RHS of Equation (5.2.9) is equal to

$$\begin{aligned} f(d, 2k-1) + \sum_{\ell=1}^{k-2} \left\lfloor \frac{\ell+2}{2} \right\rfloor &= \frac{1}{2}\left(\left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{k-1}{2} \right\rfloor \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1 \right) \right. \\ &\quad \left. + \left\lfloor \frac{k+1}{2} \right\rfloor \left(\left\lfloor \frac{k+1}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \right) \end{aligned}$$

$$= \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1\right)^2 + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)$$
$$= \frac{k(k+1)}{2}.$$

Thus Equation (5.2.9) follows which in turn implies Equation (5.2.8) and therefore Equation (5.2.7). ■

Theorem 5.2.10. *For integers $0 \leq d \leq n$ the matrices from Definition 5.2.4 are such that*

$$\det M_d = c_d \left(n - \frac{1}{2}\right)^{f(d,1)} \left(n - \frac{3}{2}\right)^{f(d,3)} \cdots \left(n - \frac{2d-3}{2}\right)^{f(d,2d-3)},$$

where c_d is some scalar and f defined by

$$f(d, 2\ell+1) = \begin{cases} f(d-1, 2\ell+1) + \left\lfloor \frac{\ell+2-(d-2\lfloor \frac{d}{2} \rfloor)}{2} \right\rfloor & \text{if } 1 \leq 2\ell+1 \leq d-2, \\ \frac{1}{2}\left(\left\lfloor \frac{d-\ell}{2} \right\rfloor \left(\left\lfloor \frac{d-\ell}{2} \right\rfloor + 1\right) + \left\lfloor \frac{d-\ell-1}{2} \right\rfloor \left(\left\lfloor \frac{d-\ell-1}{2} \right\rfloor + 1\right)\right) & \text{if } d-1 \leq 2\ell+1 \leq 2d-3, \\ 0 & \text{if } 2d-2 \leq 2\ell+1. \end{cases}$$

Proof: Using Lemma 5.2.8 for all possible roots $s = 1, 3, 5, \ldots, 2d-3$ we have that if $\det(M_d) \neq 0$ then

$$\sum_{\ell=0}^{d-2} f(d, 2\ell+1) \leq \deg(\det(M_d)).$$

It is easy to check that

$$\deg(\lambda_{(p,q),(k,i)}) = p$$

for all $(p, q) \in \Theta_d$ and $(k, i) \in \Omega$. Any monomial term of the determinant of M_d is the product of terms from each row of M_d. Therefore each monomial is of degree at most

$$\sum_{(p,q)\in\Theta_d} p.$$

Therefore we have that

$$\deg(\det(M_d)) \leq \sum_{(p,q)\in\Theta_d} p.$$

But Lemma 5.2.9 shows that $\sum_{(p,q)\in\Theta_d} p = \sum_{\ell=0}^{d-2} f(d, 2\ell+1)$. Therefore it follows that if $\det(M_d) \neq 0$ then $\deg(\det(M_d)) = \sum_{\ell=0}^{d-2} f(d, 2\ell+1)$. That is, there is some $c_d \in \mathbb{C}$ such that

$$\det(M_d) = c_d \left(n - \frac{1}{2}\right)^{f(d,1)} \left(n - \frac{3}{2}\right)^{f(d,3)} \cdots \left(n - \frac{2d-3}{2}\right)^{f(d,2d-3)}. \qquad \blacksquare$$

To complete the Capelli Eigenvalue problem there are still a number of results that must be confirmed. In particular, Theorem 5.2.10 will give us the full results we are looking for once it is shown that $c_d \neq 0$. Furthermore, we are developing a theorem analogous to Theorem 5.2.10 for $n+1 \leq d \leq 2n+1$. The matrices in these cases will be analogously defined and will still be square matrices. Lastly, for $2n+2 \leq d$ the M_d will no longer be square. Therefore in this case we will show that this matrix has full rank.

Once these final results are confirmed then we can conclude that for Problem 5.0.1 the answer is: Yes, the map $\mathcal{Z}(\mathfrak{gosp}(1|2n)) \mapsto \mathcal{PD}(\mathbb{C}^{1|2n})^{\mathfrak{gosp}(1|2n)}$ is surjective.

Appendix A

Localization

Let R be a ring, not necessarily commutative, with unity. Our goal is to review the construction of the localization of R with respect to a central element. To this end we have the following:

Definition A.1 (Multiplicative Subset)**.** *A set $S \subseteq Z(R)$ is called **multiplicative** if:*

1. $0 \notin S$,

2. $1 \in S$, *and*

3. *for any* $a, b \in S$, *then* $ab \in S$.

For a multiplicative set $S \subseteq R$, we define a relation on $R \times S$ by

$$(a, b) \sim (c, d) \leftrightarrow s(ad - cb) = 0, \text{ for some } s \in S.$$

Lemma A.2. *The relation $\sim$ on $R \times S$ is an equivalence relation.*

Proof: Reflexivity follows from the fact that $1 \in S$ and symmetry is a simple algebraic manipulation. For transitivity, take any $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.

Then there is $s, t \in S$ such that

$$s(ad - cb) = 0 \quad \text{and } t(cf - ed) = 0.$$

Then multiple the left equation by tf and the right equation by sb then sum both of these equations together to get

$$\begin{aligned} 0 &= stf(ad - cb) + stb(cf - ed) \\ &= std(af - eb). \end{aligned}$$

That is, $(a, b) \sim (e, f)$ and we have transitivity. ■

For any $(a, b) \in R \times S$ we write $\frac{a}{b}$ to be the equivalence class of (a, b) by $\sim$.

Definition A.3 (Localization of R)**.** *Let $S \subseteq Z(R)$ be multiplicative. The **localization of** R **by** S is the set*

$$S^{-1}R = \left\{ \frac{a}{b} \middle| (a, b) \in R \times S \right\}.$$

Proposition A.4. *The set $S^{-1}R$ is a ring with the following defined for $a, c \in R$ and $b, d \in S$*

1. *Addition:* $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$
2. *Multiplication:* $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$
3. *Additive Identity:* $\frac{0}{1}$
4. *Additive Inverse:* $-\left(\frac{a}{b}\right) = \frac{-a}{b}$
5. *Multiplicative Identity:* $\frac{1}{1}$

Proof: It is straightforward to verify that addition and multiplication are well-defined. An equally straightforward calculation can be done to verify additive and multiplicative identity and additive inverse. ■

Appendix B

Adjointness of R^2 and ∇^2

Let $\mathfrak{osp}(m|2n)$ have matrix realization with respect to a matrix J, such that J is invertible, $J = \text{diag}(J_1, J_2)$ with $J_1 = J_1^T$ and $J_2 = -J_2^T$. We define a supersymmetric bilinear form on $V = \mathbb{C}^{m|2n}$ defined for any standard basis elements $e_i, e_i \in V$ by

$$(e_i, e_j) = e_j^T J e_i = J_{ji}.$$

This, in turn, induces a $\mathfrak{osp}(m|2n)$-invariant bilinear form on $V^{\otimes k}$ defined by

$$(e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_k} \otimes \cdots \otimes e_{j_1}) = (e_{i_1}, e_{j_1}) \cdots (e_{i_k}, e_{j_k}). \tag{B.0.1}$$

Let $\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ and let $\mathcal{J} \subseteq T(V)$ be the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$, for $x, y \in V_{\bar{0}} \cup V_{\bar{1}}$. The **symmetric algebra** of V is

$$\mathcal{S}(V) = \mathcal{T}(V)/\mathcal{J}.$$

We write $\mathcal{S}^k(V) = \mathcal{T}^k(V)/\mathcal{J} \cap \mathcal{T}^k(V)$ and define the symmetrization map on homogeneous elements of $\mathcal{S}(V)$ by

$$\begin{aligned} \mathcal{S}(V) &\to \mathcal{T}(V) \\ v_1 \cdots v_k &\mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \beta(v_1 \cdots v_k \to v_{\sigma(1)} \cdots v_{\sigma(k)}) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, \end{aligned}$$

where $\beta(v_1 \cdots v_k \to v_{\sigma(1)} \cdots v_{\sigma(k)})$ is the resulting sign from transforming $v_1 \cdots v_k$ to $v_{\sigma(1)} \cdots v_{\sigma(k)}$. The symmetrization map is used alongside Equation B.0.1 to define an inner product on $\mathcal{S}(V)$. In particular, we equip $\mathcal{S}^k(V)$ with $(\cdot, \cdot) := k!(\cdot, \cdot)$, where the inner product on the RHS is induced by $\mathcal{T}(V)$.

Definition B.1 (Superpolynomials)**.** *Denote $\mathcal{P}(\mathbb{C}^{m|2n})$ by the space of all polynomials in indeterminate $\{X_i \mid 1 \leq i \leq m + 2n\}$, such that for all $1 \leq i, j \leq m$ and $m + 1 \leq k, \ell \leq m + 2n$*

$$X_i X_j = X_j X_i, \quad X_i X_k = X_k X_i, \quad \textit{and} \quad X_k X_\ell = -X_\ell X_k.$$

Furthermore, we denote the parity of X_i by

$$|X_i| = \begin{cases} 0 & \text{if } i \leq m \text{ or} \\ 1 & \text{otherwise.} \end{cases}$$

Remark B.0.1. It is easy to verify that $\mathcal{P}(V) \cong \mathcal{S}(V)$ where X_i is identified with e_i, for $1 \leq i \leq m + 2n$. Thus the inner product of $\mathcal{S}(V)$ induces one on $\mathcal{P}(V)$.

The action of $\mathfrak{gl}(m|2n)$ on $\mathcal{P}(\mathbb{C}^{m|2n})$ is defined for all $1 \leq i, j \leq m + 2n$ by

$$E_{i,j} \mapsto X_i \partial_{X_j}$$

and induces an action of $\mathfrak{osp}(m|2n)$.

Write $J^{-1} = [u_{ij}]$ and for $1 \leq i \leq m+2n$ define

$$X^i = \sum_{j=1}^{m+2n} u_{ji} X_j.$$

It then follows that

$$X_i = \sum_{j=1}^{m+2n} J_{ji} X^j.$$

For $1 \leq i \leq m+2n$, we define the partial derivatives ∂_{X_i} and ∂_{X^i} to be

$$\partial_{X_i}(v) = (X^i, v) \quad \text{and} \quad \partial_{X^i}(v) = (v, X_i), \tag{B.0.2}$$

for any $v \in V$. Furthermore, these partial derivatives satisfy the following Leibniz rule for any homogeneous $u, v, w \in V$

$$\partial_u(vw)\partial_u(v)w + (-1)^{|u||v|} v\partial_u(w).$$

Lemma B.2. *The following relations hold for all $1 \leq i, j \leq m+2n$:*

(a) $\partial_{X_i}(X^j) = u_{ij}$ *and* $\partial_{X^i}(X_j) = J_{ij}$,

(b) $\partial_{X^i} = \sum_{\ell=1}^{m+2n} J_{i\ell} \partial_{X_\ell}$.

Define the following maps

$$D_{X_i} \colon V^{\otimes(k+1)} \to V^{\otimes k}$$

$$v_1 \otimes \cdots \otimes v_{k+1} \mapsto \sum_{j=1}^{k+1} (-1)^{|X_i|(|v_1|+\cdots+|v_{j-1}|)} (X^i, v_j) v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes \cdots v_{k+1}$$

and

$$\begin{aligned} T_{X^i} : V^{\otimes k} &\to V_{\otimes(k+1)} \\ v_1 \otimes \cdots \otimes v_k &\mapsto \sum_{j=1}^{k+1} (-1)^{|X^i|(|v_1+\cdots+|v_{j-1}|)} v_1 \otimes \cdots \otimes v_{j-1} \otimes X^i \otimes v_j \otimes \cdots \otimes v_k. \end{aligned}$$

Proposition B.3. *For any $v \in V^{\otimes(k+1)}$ and $w \in V^{\otimes k}$*

$$(D_{X_i} v, w) = (-1)^{|X_i||v|}(v, T_{X^i} w).$$

Proof: Suppose that $v = v_1 \otimes \cdots \otimes v_{k+1}$ and $w = w_k \otimes \cdots \otimes w_1$ are simple tensors. To verify this, we need only check the non-zero terms of both sides of the form

$$(v_{k+1}, w_k) \cdots (v_{j+1}, w_j)(X^i, v_j)(v_{j-1}, w_{j-1}) \cdots (v_1, w_1).$$

By a direct computation, the LHS has coefficient

$$(-1)^{|X_i|(|v_1|+\cdots+|v_{j-1}|)}$$

and the RHS has coefficient

$$(-1)^{|X^i|(|w_k|+\cdots+|w_j|)+|X^i||v_j|+|X_i||v|}.$$

This is for non-zero terms and therefore

$$|v_\ell| = \begin{cases} |w_\ell| & \text{for } 1 \le \ell \le j-1, \\ |X^i| & \text{for } \ell = j, \\ |w_{\ell-1}| & \text{for } j+1 \le \ell \le k. \end{cases}$$

That is, the LHS is equal to the RHS. ∎

Furthermore, one can check that D_{X_i} and T_{X^i} leave the ideal $\mathcal{J} \subseteq \mathcal{T}(V)$ invariant. Therefore D_{X_i} and T_{X^i} are well-defined on $\mathcal{S}(V)$.

Proposition B.4. *When restricted we have the following identities:* $D_{X_i}|_{\mathcal{S}(V)} = \partial_{X_i}$ *and* $T_{X^i}|_{\mathcal{S}^k(V)} = (k+1)m_{X^i}$*, where* ∂_{X_i} *is defined in Equation (B.0.2) and for any* $k \in \mathbb{Z}_{\geq 0}$ *the map* $m_{X^i} \colon \mathcal{S}^k(V) \to \mathcal{S}^{k+1}(V)$ *is defined for all* $x \in \mathcal{S}^k(V)$ *by* $m_{X^i}(x) = X^i x$.

Proof: For symmetrized monomials of $\mathcal{S}^k(V)$ we have the following computation

$$D_{X_i}\left(\frac{1}{(k+1)!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \beta(v_1\cdots v_{k+1} \to v_{\sigma(1)}\cdots v_{\sigma(k+1)})v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k+1)}\right) \tag{B.0.3}$$

$$= \frac{1}{(k+1)!}\sum_{\sigma\in\mathfrak{S}_{k+1}}\sum_{j=1}^{k+1}\beta(v_1\cdots v_{k+1}\to v_{\sigma(1)}\cdots v_{\sigma(k+1)})\beta(X^i v_{\sigma(1)}\cdots v_{\sigma(j-1)}\to v_{\sigma(1)}\cdots v_{\sigma(j-1)}X^i)$$
$$(X^i, v_{\sigma(j)})v_{\sigma(1)}\otimes\cdots\otimes\widehat{v_{\sigma(j)}}\otimes\cdots v_{\sigma(k+1)}.$$

Define $\mathfrak{S}_k^j$ as the group of permutations on the set $\{1,\cdots,\widehat{j},\cdots,k\}$ and for ∂_{X_i} we have

$$\partial_{X_i}(v_1\cdots v_{k+1}) = \sum_{\ell=1}^{k+1}(X^i, v_\ell)\beta(X^i v_1\cdots v_{\ell-1}\to v_1\cdots v_{\ell-1}X^i)v_1\cdots\widehat{v_\ell}\cdots v_{k+1} \tag{B.0.4}$$

$$= \frac{1}{k!}\sum_{\ell=1}^{k+1}\sum_{\tau\in\mathfrak{S}_{k+1}^{\ell}}(X^i, v_\ell)\beta(X^i v_1\cdots v_{\ell-1}\to v_1\cdots v_{\ell-1}X^i)$$
$$\beta(v_1\cdots v_{\ell-1}v_{\ell+1}\cdots v_{k+1}\to v_{\tau(1)}\cdots v_{\tau(\ell-1)}v_{\tau(\ell+1)}\cdots v_{\tau(k+1)})$$
$$v_{\tau(1)}\otimes\cdots\otimes v_{\tau(\ell-1)}\otimes v_{\tau(\ell+1)}\otimes\cdots\otimes v_{\tau(k+1)}$$

The term indexed by $1 \leq \ell \leq k+1$ and $\tau \in \mathfrak{S}_{k+1}^{\ell}$ on the RHS of Equation (B.0.4) corresponds to the terms on the RHS of Equation (B.0.4) indexed (σ, j) for $1 \leq j \leq k+1$ and σ defined in the following when $j < \ell$

$$\sigma(r) = \begin{cases} \tau(r) & \text{if } 1 \leq r < j, \\ \ell & \text{if } r = j, \\ \tau(r-1) & \text{if } j < r \leq \ell, \\ \tau(r) & \text{if } \ell < r \leq k+1, \end{cases}$$

when $j > \ell$

$$\sigma(r) = \begin{cases} \tau(r) & \text{if } 1 \leq r < j, \\ \ell & \text{if } r = j, \\ \tau(r+1) & \text{if } j < r \leq \ell, \\ \tau(r) & \text{if } \ell < r \leq k+1, \end{cases}$$

when $j = \ell$

$$\sigma(r) = \begin{cases} \tau(r) & \text{if } 1 \leq r < j, \\ \ell & \text{if } r = j, \\ \tau(r) & \text{if } j < r \leq k+1. \end{cases}$$

Furthermore, using the fact that if $(X^i, v_\ell) \neq 0$, then $\beta(X^i v_\ell v_s \to v_s X^i v_\ell) = 1$ it is straightforward to verify that the coefficients of all these terms are the same. That is, $D_{X_i}|_{\mathcal{S}(V)} = \partial_{X_i}$.

Similarly for T_{X^i} we have the following computations

$$\begin{aligned} T_{X^i} & \left(\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \beta(v_1 \cdots v_k \to v_{\sigma(1)} \cdots v_{\sigma(k)}) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right) \\ & = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \beta(v_1 \cdots v_k \to v_{\sigma(1)} \cdots v_{\sigma(k)}) \\ & \quad \left(\sum_{j=1}^{k+1} \beta(X^i v_{\sigma(1)} \cdots v_{\sigma(j-1)} \to v_{\sigma(1)} \cdots v_{\sigma(j-1)} X^i) \right. \\ & \quad \left. v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j-1)} \otimes X^i \otimes v_{\sigma(j)} \otimes \cdots \otimes v_{\sigma(k)} \right). \end{aligned} \tag{B.0.5}$$

For $v_{k+1} := X^i$ we have

$$\begin{aligned} (k+1) m_{X^i}(v_1 \cdots v_k) &= (k+1)\beta(X^i v_1 \cdots v_k \to v_1 \cdots v_{k+1}) v_1 \cdots v_{k+1} \\ &= \left(\frac{\beta(X^i v_1 \cdots v_k \to v_1 \cdots v_{k+1})}{k!} \right) \end{aligned} \tag{B.0.6}$$

$$\left(\sum_{\tau \in \mathfrak{S}_{k+1}} \beta(v_1 \cdots v_{k+1} \to v_{\tau(1)} \cdots v_{\tau(k+1)}) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k+1)} \right).$$

Take any term on the RHS of Equation (B.0.6) indexed by τ where $\tau(j) = k+1$. This term corresponds to the term on the RHS of Equation (B.0.5) indexed by σ and j, where σ is such that

$$\sigma(\ell) = \begin{cases} \tau(r) & \text{if } 1 \leq \ell \leq j-1 \\ \tau(r+1) & \text{if } j \leq \ell \leq k. \end{cases}$$

Knowing this, it is an easy computation to verify that the coefficients of each of these terms are equal. Therefore, $T_{X^i}|_{\mathcal{S}^k(V)} = (k+1)m_{X^i}$. ■

Using the adjointness shown in Proposition B.3 and Proposition B.4 we have the following corollary.

Corollary B.5. *For any $v, w \in \mathcal{S}(V)$*

$$(\partial_{X_i} v, w) = (-1)^{|v||X_i|}(v, X^i w).$$

The next corollaries follow from Corollary B.5 and Lemma B.2.

Corollary B.6. *For all $v, w \in \mathcal{S}(V)$*

$$(\partial_{X^i} v, w) = (-1)^{(1+|v|)|X^i|}(v, X_i w).$$

Corollary B.7. *For all $v, w \in \mathcal{S}(V)$*

$$(\partial_{X^i} \partial_{X_i} v, w) = (v, X^i X_i w). \tag{B.0.7}$$

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