Symmetric Polynomials, Interpolation Polynomials and Their Positivity

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NSERC USRA Summer 2022

1 Introduction

The purpose of this paper is to review the techniques used in recent work done by Naqvi, Sahi, and Sergel [2] to prove the positivity conjecture of interpolation polynomials. The positivity conjecture for Jack polynomials was proved over 25 years ago; the coefficients of these polynomials with respect to their monomial expansions have non-negative integer values. Generalizing this result to interpolation polynomials relies on a reduction technique to analogous nonsymmetric polynomials. The problem can be reduced to proving *bar monomials* — the result of playing a combinatorial algorithm called the *bar game* — have non-negative integer coefficients. We begin by reviewing some properties of symmetric polynomials and five well-known bases for this vector space. Next, we introduce Jack polynomials and see interpolation polynomials as a sum of a Jack polynomial and lower degree terms. We build up to the proof strategy by first understanding a few key tools: recursions on nonsymmetric polynomials, intertwining operators, the dehomogenization operator, and the bar game.

2 Preliminaries

Symmetric polynomials have connections to a wide variety of topics such as algebra, geometry, topology, graph theory, and combinatorics. This section reviews the fundamentals of the vector space of symmetric polynomials and five well-known bases. In particular, we will be expressing interpolation polynomials as a linear combination of monomial symmetric polynomials and investigating properties of their coefficients. Throughout this paper we assume the base field for the coefficients is $\mathbb{Q}$.

2.1 Symmetric Polynomials

A symmetric polynomial is a polynomial in n variables that can be written as a sum of terms where each term is a distinct permutation of the indices of the variables. Applying any permutation in S_n does not change the polynomial, but may rearrange the order of the terms. We write $\Lambda(X_n)$ to denote the set of all symmetric polynomials which forms a vector space over $\mathbb{Q}$ for all $n \in \mathbb{N}$.

Example 2.1.1 The polynomial $f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$ is a symmetric polynomial.

Here we are working in three variables $n = 3$, so we must apply every permutation in the symmetric group S_3 and check that the polynomial remains the same. Recall $S_3 = \{123, 213, 321, 132, 231, 312\}$. By applying the permutation $\pi = 213$ to the indices of the variables, we get $f(x_1, x_2, x_3) = x_2x_1+x_2x_3+x_1x_3$. By applying $\pi = 321$ we get $f(x_1, x_2, x_3) = x_3x_2 + x_3x_1 + x_2x_1$. Applying any permutation of S_3 to the polynomial returns the same polynomial, and hence this polynomial is symmetric.

We write $\pi(f)$ to denote a permutation applied to a polynomial and define this by the permutations action on the indices of the variables. For some permutation $\pi \in S_n$ and some polynomial $f(x_1, x_2, ..., x_n)$, we have

$$\pi(f) := f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$$

Example 2.1.2 If $f(x_1, x_2, x_3, x_4) = 3x_3x_4^2$ and $\pi = 2413$, then by applying the permutation to the subscripts of the variables we get

$$\begin{aligned}\pi(f) &= f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)})\\ &= 3x_{\pi(3)}x_{\pi(4)}^2\\ &= 3x_1x_3^2\end{aligned}$$

Properties 2.1.3 *Suppose $f(X_n)$ and $g(X_n)$ are polynomials in n variables, c is constant and $\pi, \sigma \in S_n$. Then we have the following properties [1]:*

(i) $\pi(cf) = c\pi(f)$;

(ii) $\pi(f + g) = \pi(f) + \pi(g)$;

(iii) $\pi(fg) = \pi(f)\pi(g)$;

(iv) $(\pi\sigma)(f) = \pi(\sigma(f))$.

Symmetric polynomials occur when $\pi(f) = f$ for every $\pi \in S_n$. When $\pi(f) = f$ for some $\pi \in S_n$, we say f is *invariant under* π. Every permutation $\pi \in S_n$ has its own set of invariant polynomials, but the symmetric polynomials are invariant under every permutation.

Notice that no matter how much a term of a polynomial is changed by permuting the variables, the terms total degree does not change. Looking back to Example 2.1.1 we see $\pi(f)$ and f are both the sum of three terms that each have degree 2. In Example 2.1.2 we see $\pi(f)$ and f are both one term of degree 3. When every term of a polynomial is exactly degree k, we say that polynomial is *homogeneous degree* k and we write $\Lambda_k(X_n)$ to denote the set of all symmetric polynomials in n variables which are homogeneous degree k.

Example 2.1.4 Consider the symmetric polynomial

$$f(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

This polynomial is a sum of three terms where each term has a total degree of 4. Therefore $f \in \Lambda_4(X_3)$.

Any polynomial can be rewritten as a sum of terms arranged in order of decreasing degree. Hence we can consider any polynomial as a sum of homogeneous degree k polynomials for varying k.

Example 2.1.5 The following nonsymmetric polynomial can be expressed as a sum of homogeneous polynomials

$$f(x_1, x_2, x_3, x_4) = 7x_2 x_3^2 x_4 + 9x_3^2 x_4^2 + x_1^3 + x_1 x_2 x_4 + x_2^2 + 15$$

The first two terms are degree 4, the next two terms are degree 3, there is one term degree 2, no terms of degree 1 and one term that is constant (degree 0). We can write f as a sum of homogeneous degree polynomials since

$$f = f_4 + f_3 + f_2 + f_1 + f_0$$

where

$$\begin{aligned} f_4 &= 7x_2 x_3^2 x_4 + 9x_3^2 x_4^2 \\ f_3 &= x_1^3 + x_1 x_2 x_4 \\ f_2 &= x_2^2 \\ f_1 &= 0 \\ f_0 &= 15 \end{aligned}$$

Notice if f is a **symmetric** polynomial in $x_1, x_2, ..., x_n$ and $f = f_0 + f_1 + ...$ is the decomposition of f into its homogeneous parts, then each f_j is also a symmetric polynomial.

2.2 Monomial Symmetric Polynomials

One way to find symmetric polynomials is to construct them from a monomial. Given a monomial in n variables, we apply all permutations of S_n to the subscripts of the variables, and take the sum of distinct terms to obtain a symmetric polynomial.

Example 2.2.1 Let us construct a symmetric polynomial given the monomial

$$f(x_1, x_2, x_3) = x_1 x_2^3$$

There are $n = 3$ variables, so we apply all permutations of the symmetric group $S_3 = \{123, 213, 321, 132, 231, 312\}$. Our new symmetric polynomial is a sum of the distinct terms. We get

$$f(x_1, x_2, x_3) = x_1 x_2^3 + x_1 x_3^3 + x_2 x_3^3 + x_2 x_1^3 + x_3 x_1^3 + x_3 x_2^3$$

Recall the following definitions.

Definition 2.2.2 A *partition* λ is a sequence of non-negative integers in weakly decreasing order. We say λ partitions an integer k when the entries of the partition sum to k. We write $l(\lambda)$ to denote the length of a partition, the numbers of entries in the sequence.

Definition 2.2.3 A *composition* η is a sequence of non-negative integers without order.

We write $m_\lambda(X_n)$ to denote the monomial symmetric polynomial indexed by a partition λ. Throughout this paper, entries of λ and η correspond the the exponents of a term.

Example 2.2.4 If $\lambda = (2, 1)$ and $n = 3$, then the monomial symmetric polynomial indexed by λ is given by

$$m_{2,1}(x_1, x_2, x_3) = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_2 x_1^2 + x_3 x_1^2 + x_3 x_2^2$$

Example 2.2.5 If $\lambda = (3, 3, 1, 1)$ and $n = 4$, then the monomial symmetric polynomial indexed by λ is given by

$$\begin{aligned} m_{3,3,1,1}(x_1, x_2, x_3, x_4) = x_1^3 x_2^3 x_3 x_4 + x_1 x_2^3 x_3^3 x_4 + x_1 x_2^3 x_3 x_4^3 \\ + x_1^3 x_2 x_3^3 x_4 + x_1^3 x_2 x_3 x_4^3 + x_1 x_2 x_3^3 x_4^3 \end{aligned}$$

Definition 2.2.6 A *monomial symmetric polynomial* is the sum of the monomial $\prod_{j=1}^{l(\lambda)} x_j^{\lambda_j}$ and all of its distinct images under the elements S_n, where $l(\lambda)$ is the length of the partition.

Note that $m_\lambda(X_n) = 0$ by convention when the length of the partition exceeds the number of variables. This makes sense because, for example, if we try to write the monomial $m_{3,3,1,1}(X_3)$ we have $x_1^3 x_2^3 x_3^1 x_*^1 + ...$ We do not have enough variables to complete the term since $l(\lambda) = 4 > 3 = n$.

The proof of the next theorem can be found in [1].

Theorem 2.2.7 *The following statements hold.*

(i) *If λ is a partition of k, then the monomial symmetric polynomial $m_\lambda(X_n)$ is homogeneous degree k, and hence $m_\lambda(X_n) \in \Lambda_k(X_n)$.*

(ii) *If $n \geq k$, then for all λ partitions of k, $m_\lambda(X_n)$ is a basis for $\Lambda_k(X_n)$ and $dim\Lambda_k(X_n) = p(k)$, the number of partitions of k.*

(iii) *If $n \geq k$, then the algebraic properties of $\Lambda_k(X_n)$ does not depend on how many variables we have. The dimension of $\Lambda_k(X_n)$ is independent of n.*

(iv) *There is exactly one monomial symmetric polynomial in $\Lambda_k(X_n)$ for each partition of k.*

2.3 Elementary Symmetric Polynomials

The *elementary symmetric polynomials* form our second basis for $\Lambda(X_n)$. To understand these polynomials, we first consider a monic function $f(t)$ whose roots are $x_1, x_2, ..., x_n$. Then we can express f as a product of its roots

$$f(t) = (t - x_1)(t - x_2)...(t - x_n)$$

If we multiply and expand, we get the following expression

$$\begin{aligned} f(t) = {} & (1)t^n \\ & + (x_1 + x_2 + ... + x_n)t^{n-1} \\ & + (x_1x_2 + x_1x_3 + ... + x_1x_n + x_2x_3 + x_2x_4 + ... + x_2x_n + ... + x_{n-1}x_n)t^{n-2} \\ & + ... + (-1)^n(x_1x_2...x_n)t^0 \end{aligned}$$

The elementary symmetric polynomials are those polynomials attached to each distinct t^k. The above equation can be written as

$$f(t) = e_0t^n + e_1t^{n-1} + ... + e_nt^0$$

Each e_i is the sum of all products of exactly k distinct x_j's. Note that $(-1)^k e_k$ is the coefficient of t^{n-k}.

Equivalently, we have that the ordinary generating function for the sequence of elementary symmetric functions is given by

$$\sum_{n=0}^{\infty} e_n t^n = \prod_{j=1}^{\infty}(1 + x_j t)$$

We can see that the vector space $\Lambda_k(X_n)$ contains e_k. For example the set of all homogeneous symmetric polynomials of degree 3 contains the elementary symmetric polynomial

$$e_3 = x_1x_2x_3 + x_1x_2x_4 + ... + x_1x_2x_n + x_2x_3x_4 + ... + x_{n-2}x_{n-1}x_n$$

but $\Lambda_3(X_n)$ does not contain $e_2 = x_1x_2+x_1x_3+...+x_2x_3+x_2x_4+...+x_{n-1}x_n$. We want to find a set of elementary symmetric polynomials that form a basis for $\Lambda_k(X_n)$. To do this, we need to define a symmetric function e_λ for every partition λ of k. If λ partitions k, then $e_\lambda \in \Lambda_k(X_n)$ and we have that $\Lambda_k(X_n)$ contains exactly $p(k)$ elementary symmetric functions. For some partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$, we have

$$e_\lambda = e_{\lambda_1} \cdot e_{\lambda_2} \cdots e_{\lambda_m}$$

Recall the monomial symmetric functions span $\Lambda_k(X_n)$, and so any elementary symmetric polynomial e_k can be written as a linear combination of monomial symmetric polynomials m_μ, where μ is also a partition of k. Now we want to write e_λ elementary symmetric polynomials *partitioned by* λ as a linear combination of monomials.

Example 2.3.1 Suppose $\lambda = (2, 1)$ and $n = 3$, then

$$\begin{aligned}
e_\lambda(X_3) &= e_{2,1} \\
&= e_2e_1 \\
&= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\
&= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_3^2x_2 + 3x_1x_2x_3 \\
&= m_{2,1}(X_3) + 3m_{1,1,1}(X_3)
\end{aligned}$$

Note that $k = 3$, and so for any $n \geq 3$, the coefficients do not change. When k is large enough, the coefficients do not depend on n as we see below:

$$\begin{aligned}
e_{2,1}(X_4) &= m_{2,1}(X_4) + 3m_{1,1,1}(X_4) \\
e_{2,1}(X_5) &= m_{2,1}(X_5) + 3m_{1,1,1}(X_5) \\
e_{2,1}(X_6) &= m_{2,1}(X_6) + 3m_{1,1,1}(X_6) \\
e_{2,1}(X_n) &= m_{2,1}(X_n) + 3m_{1,1,1}(X_n)
\end{aligned}$$

Note that if λ partitions k and $\lambda = (k, 0, ..., 0)$, then we have

$$e_\lambda = e_k = m_{1^k} = m_{1,1,...,1}$$

Recall that a *Ferrer Diagram* is obtained from a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ by putting down a row of squares equal in number to the first entry λ_1, then immediately below it a row of squares equal in number to the next entry λ_2, and the nth row having the same number of squares as the nth term in the partition.

The following statements [1] are equivalent ways of determining the coefficients of m_μ in the linear combination for some e_λ, where $|\lambda| = |\mu| = k$:

(i) The coefficient of m_μ is the number of fillings of the Ferrer Diagram of λ with positive integers for which the entries in each row are strictly increasing from left to right, and each integer j appears exactly μ_j times.

(ii) The coefficient of m_μ is the number of $k \times k$ matrices in which every entry is 0 or 1, the sum of the entries in row m is μ_m for all m, and the sum of the entries in column j is λ_j for all j.

2.4 Complete Homogeneous Symmetric Polynomials

The *complete homogeneous symmetric polynomials* form another basis for $\Lambda(X_n)$. As the name implies, each term of a complete homogeneous symmetric polynomial will be degree k and it is the sum of all possible products of x_i resulting in degree k.

Example 2.4.1 The complete homogeneous symmetric polynomials h_1 and h_2 in n variables are given by
$h_1(X_n) = x_1 + x_2 + ... + x_n$
$h_2(X_n) = x_1^2 + x_2^2 + ... + x_n^2 + x_1x_2 + x_1x_3 + ... + x_1x_n + ... + x_{n-1}x_n$

We note that for every partition λ of k, there will exist a term in h_k of the form $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_{l(\lambda)}^{l(\lambda)}$ and hence we have

$$h_k = \sum_{\lambda \vdash k} m_\lambda$$

Similar to the elementary symmetric polynomials, the complete homogeneous symmetric polynomials h_k do not form a basis for $\Lambda_k(X_n)$, but the complete homogeneous symmetric polynomials *partitioned by* λ do form a basis, where

$$h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_m}$$

We can also express h_λ as a linear combination of monomial symmetric polynomials.

Example 2.4.2 Suppose $k = 3$, $\lambda = (2, 1)$ and $n = 3$, then

$$\begin{aligned} h_{2,1}(X_3) &= (x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\ &= (x_1^3 + x_2^3 + x_3^3) + 2(x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2) + 3(x_1x_2x_3) \\ &= m_3(X_3) + 2m_{2,1}(X_3) + 3m_{1,1,1}(X_3) \end{aligned}$$

There is a useful relationship between the complete homogeneous symmetric polynomials and the elementary symmetric polynomials. Recall the generating function for elementary symmetric polynomials is given by

$$E(t) = \sum_{n=0}^{\infty} e_n t^n = \prod_{j=1}^{\infty}(1 + x_j t)$$

and the generating function for the homogeneous symmetric polynomials is given by

$$H(t) = \sum_{n=0}^{\infty} h_n t^n = \prod_{j=1}^{\infty} \frac{1}{(1 - x_j t)}$$

From these equations we can see that $H(t)$ and $E(-t)$ are multiplicative inverses. Proving directly that h_λ is a basis for $\Lambda_k(X_n)$ is challenging, but since we know $\dim(h_\lambda) = p(k)$ and we can show h_λ spans $\Lambda_k(X_n)$, we have that h_λ is a basis for $\Lambda_k(X_n)$.

2.5 Power Sum Symmetric Polynomials

If we consider all possible partitions of k, there are two extreme cases. One in which $\lambda = (1, 1, 1, ..., 1)$ up to k times, and the other $\lambda = (k, 0, ..., 0)$. We have seen how the elementary symmetric polynomials gives the relationship $e_k = m_{1^k}$, and power sum symmetric polynomials are on the other end of the spectrum where $p_k = m_k$. The power sum symmetric polynomials partitioned by λ form another basis for $\Lambda_k(X_n)$, where λ partitions k and

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{l(\lambda)}}$$

Example 2.5.1 The power sum symmetric polynomials and p_2 and $p_{3,1}$ in n variables are given by

$$\begin{aligned} p_2(X_n) &= m_2(X_n) \\ &= x_1^2 + x_2^2 + ... + x_n^2 \end{aligned}$$

$$\begin{aligned} p_{3,1}(X_n) &= p_3(X_n) p_1(X_n) \\ &= m_3(X_n) m_1(X_n) \\ &= (x_1^3 + x_2^3 + ... + x_n^3)(x_1 + x_2 + ... + x_n) \end{aligned}$$

Similarly to our other bases, if n is large enough, the coefficients of m_μ are independent of n.

2.6 Schur Symmetric Polynomials

Any term in an elementary symmetric polynomial can be represented by the filling of a Young Tableau with rows strictly increasing from left to right. Any term in a complete homogeneous symmetric polynomial can be represented by the filling of a Young Tableau with rows weakly increasing. For example, the term $x_1^2 x_2^1 x_4^3 x_6^1$ in $e_{3,3,1}(X_7)$ is given by either

$$\begin{array}{|c|cc} \hline 4 & & \\ \hline 1 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad OR \quad \begin{array}{|c|cc} \hline 4 & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

Now we consider the filling of a Young Tableau of shape λ where columns are strictly increasing from bottom to top, and rows are weakly increasing from left to right. We will find that this corresponds to terms of a *Schur symmetric polynomial.* If the partition is at one extreme end where $\lambda = (1, 1, 1, ..., 1)$, then this corresponds to the diagram with just one strictly increasing column and we find that

$$s_{1^k}(X_n) = e_k(X_n)$$

On the other hand, if $\lambda = (k, 0, ..., 0)$ then this corresponds to one weakly increasing row and hence

$$s_k(X_n) = h_k(X_n)$$

For any partition λ, we can write s_λ as a linear combination of monomial symmetric polynomials where the coefficients are called *Kostka numbers* $K_{\lambda,\mu}m_\mu$. In addition to a combinatorial interpretation of Schur polynomials using Young Diagrams, they can also be represented as ratios of determinants. Consider the Vandermonde matrix

$$V = \begin{bmatrix} x_0^n & x_1^n & x_2^n & ... & x_n^n \\ ... & ... & ... & ... & ... \\ x_0^2 & x_1^2 & x_2^2 & ... & x_n^2 \\ x_0^1 & x_1^1 & x_2^1 & ... & x_n^1 \\ x_0^0 & x_1^0 & x_2^0 & ... & x_n^0 \end{bmatrix} = \begin{bmatrix} x_0^n & x_1^n & x_2^n & ... & x_n^n \\ ... & ... & ... & ... & ... \\ x_0^2 & x_1^2 & x_2^2 & ... & x_n^2 \\ x_0 & x_1 & x_2 & ... & x_n \\ 1 & 1 & 1 & ... & 1 \end{bmatrix}$$

The determinant of the Vandermonde matrix is

$$\det(V) = (x_1 - x_2)(x_1 - x_3) \cdots (x_i - x_j)$$

for $1 \leq i < j \leq n$, and we note that $\det(V) \neq 0$ if and only if each x_i are distinct.

Before we can establish the relationship between Schur polynomials and determinants, we need to be able to transform a given partition λ that is weakly decreasing, into a strictly decreasing sequence. We can do this by simply adding a strictly decreasing sequence to it. Consider the strictly decreasing sequence $\delta_n = (n-1, n-2, ..., 2, 1, 0)$. We can add this to any partition λ of length n, and obtain another strictly decreasing sequence.

Example 2.6.1 Take $\lambda = (4, 4, 3, 3, 2, 1)$ and since $l(\lambda) = 6$, we add to it $\delta_6 = (5, 4, 3, 2, 1, 0)$ to get $\lambda + \delta_n = (9, 8, 6, 5, 3, 1)$, a strictly decreasing sequence.

We use this to express Schur polynomials as a ratio of determinants. The numerator is the determinant of a matrix with entries in x_n and exponents that correspond to the sequences $\lambda + \delta_n$. The denominator is the determinant of the Vandermonde matrix with exponents that correspond to the sequence δ_n.This means we find the following relationship

$$\begin{aligned} s_\lambda(X_n) &= a_{\lambda+\delta_n}(X_n)/a_{\delta_n}(X_n) \\ &= \det(M)/\det(V) \end{aligned}$$

Example 2.6.2 Suppose $\lambda = (3, 3, 1)$ and $n = 3$, then $\delta_3 = (2, 1, 0)$ and $\lambda + \delta_3 = (5, 4, 1)$. We have $a_{\lambda+\delta_3}(X_3) = \det(M)$ and $a_{\delta_3}(X_3) = \det(V)$ where

$$M = \begin{bmatrix} x_1^5 & x_2^5 & x_3^5 \\ x_1^4 & x_2^4 & x_3^4 \\ x_1 & x_2 & x_3 \end{bmatrix} \qquad V = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{aligned} \det(M) &= x_1^5(x_2^4 x_3 - x_3^4 x_2) - x_2^5(x_1^4 x_3 - x_3^4 x_1) + x_3^5(x_1^4 x_2 - x_2^4 x_1) \\ \det(V) &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \end{aligned}$$

After simplifying, we find the Schur symmetric polynomial partitioned by $\lambda = (3,3,1)$ is given by

$$\begin{aligned} s_{3,3,1}(X_3) &= a_{(3,3,1)+(2,1,0)}(X_3)/a_{(2,1,0)}(X_3) \\ &= \det(M)/\det(V) \\ &= x_1^3x_2^3x_3 + x_1^3x_2x_3^3 + x_1x_2^3x_3^3 \\ &+ x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^2x_2^2x_3^3 \\ &= m_{3,3,1}(X_3) + m_{3,2,2}(X_3) \end{aligned}$$

3 Jack Polynomials

In this section we introduce another family of symmetric polynomials that are closely related to the Schur polynomials. Unlike the Schur polynomials, the coefficients of Jack polynomials are not integers, but elements of the field $\mathbb{Q}(\alpha)$. Setting $\alpha = 1$, the Jack polynomials specialize to Schur polynomials. Despite this close connection, our strategy to define Jack polynomial is quite different from the one used in Section 2 in the case of Schur polynomials. Rather than giving a determinantal formula, we characterize Jack polynomials as eigenvectors of a certain differential operator that comes from physics.

3.1 Jack polynomials as eigenfunctions of the CMS operator

The deformed Calogero–Moser–Sutherland (CMS) operators are given by

$$\begin{aligned} L_{n,m,\alpha} = &-(\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}) - k(\frac{\partial^2}{\partial y_1^2} + ... + \frac{\partial^2}{\partial y_n^2}) \\ &+ \sum_{i<j}^{n} \frac{2k(k+1)}{sin^2(x_i - x_j)} + \sum_{i<j}^{m} \frac{2(k^{-1}+1)}{sin^2(y_i - y_j)} + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{2(k+1)}{sin^2(x_i - y_j)} \end{aligned}$$

It is related to the Laplace–Beltrami operator used by Macdonald

$$\mathcal{L}_\alpha^{(N)} = 2\alpha \square_N^{\frac{1}{\alpha}} - \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} \tag{1}$$

If α is a non-negative rational number or zero, then for any partition λ where $l(\lambda) \leq N$ there is a unique polynomial, called a *Jack polynomial* $P_\lambda^{(\alpha)}(X_n)$ that satisfies the following [4]:

(i) $P_\lambda^{(\alpha)}(X_n) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu$

(ii) $P_\lambda^{(\alpha)}(X_n)$ is an eigenfunction of the CMS operator $\mathcal{L}_\alpha^{(N)}$

Jack polynomials are eigenfunctions obtained by applying a symmetric polynomial to the CMS operator. When the Jack polynomial is written as a linear combination of monomial symmetric polynomials, the coefficients of that expansion are the eigenvalues of the system.

Looking back at the Laplace–Beltrami operator used by Macdonald (1), the last term of this equation is the Euler Operator

$$\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i}$$

Note that when we apply a homogeneous degree polynomial, it returns a homogeneous degree polynomial.

Example 3.1.1 Let us apply Euler's Operator to the term $x_1 x_2^2$

$$\begin{aligned} \sum_{i=1}^{N} x_i \frac{\partial x_1 x_2^2}{\partial x_i} &= x_1 x_2^2 + x_2(2x_1x_2) \\ &= 3x_1x_2^2 \end{aligned}$$

Here we find the eigenfunction is $x_1x_2^2$ and the eigenvalue is 3.

The Euler Operator distinguishes the eigenfunctions and eigenvalues. Similarly, Jack polynomials are the eigenfunctions of the CMS operator. Eigenvalues are coefficients needed to normalize the Jack polynomial. The CMS operator gives **distinct** eigenvalues that form a basis, and there exists a linear transformation (scalar transformation) to the normalized basis of eigenvalues.

Interpolation polynomials $P_\lambda^\rho(X_n)$ are inhomogeneous symmetric polynomials partitioned by λ and parameterized by $\rho = (\rho_1, \rho_2, ..., \rho_n)$. In particular, we study the one-parameter family where $\rho = r\delta$ and $\delta = (n-1, n-2, ..., 1, 0)$. Since this family of symmetric polynomials $P_\lambda^{r\delta}(X_n)$ is inhomogeneous, we can collect all terms of highest degree and find that those top degree terms are actually Jack polynomials.

$$P_\lambda^{r\delta}(X_n) = P_\lambda^{(\alpha)}(X_n) + (\text{terms of degree} < |\lambda|)$$

Jack polynomials are given by $P_\lambda^{(\alpha)}(X_n)$ where $\alpha = 1/r$. If we let $r = 1$, then this gives us the Schur polynomials. Interpolation polynomials $P_\lambda^\rho(X_n)$ are uniquely characterized by the vanishing property described in the following theorem.

Theorem 3.1.2 *(Thm 2.1 [2])There is a unique symmetric polynomial $P_\lambda^\rho(X_n)$ of total degree $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$ such that*
(i) $P_\lambda^\rho(\mu + \rho) = 0$ *for all $\mu \in \mathcal{P}_n$ where $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$*
(ii) *the coefficient of the symmetric monomial m_μ in P_λ^ρ is 1*

Example 3.1.3 Let us find the unique interpolation polynomial $P^{\rho}_{\lambda}(X_n)$ for $\lambda = (1, 1)$.

We have $|\lambda| = 2$, so we write all possible partitions μ such that $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$. We find that

$$\mu_1 = \emptyset, \qquad \mu_2 = (1, 0), \qquad \mu_3 = (2, 0)$$

Here $n = 2$ and recall $\delta = (n - 1, ..., 1, 0)$, so then $\rho = r\delta = r(1, 0) = (r, 0)$. Consider the interpolation polynomial

$$P^{(r,0)}_{1,1}(X_2) = x_1 x_2$$

We need to verify $P^{\rho}_{\lambda}(\mu + \rho) = 0$ for all μ_i, and $P^{\rho}_{\lambda}(\mu + \rho) \neq 0$ for $\mu = \lambda$.

$$\begin{aligned}
P_{1,1}(\mu_1 + \rho) &= P_{1,1}(\emptyset + (r, 0)) = P_{1,1}(r, 0) = r(0) = 0 \\
P_{1,1}(\mu_2 + \rho) &= P_{1,1}((1, 0) + (r, 0)) = P_{1,1}((r + 1, 0)) = (r + 1)(0) = 0 \\
P_{1,1}(\mu_3 + \rho) &= P_{1,1}((2, 0) + (r, 0)) = P_{1,1}((r + 2, 0)) = (r + 2)(0) = 0 \\
P_{1,1}(\lambda + \rho) &= P_{1,1}((1, 1) + (r, 0)) = P_{1,1}((r + 1, 1)) = (r + 1)(1) = r + 1 \neq 0
\end{aligned}$$

Hence $P^{(r,0)}_{1,1}(X_2) = x_1 x_2$ satisfies the conditions of Theorem 3.0.2, it is the unique interpolation polynomial for $\lambda = (1, 1)$.

So far we have seen that interpolation polynomials $P^{r\delta}_{\lambda}$ can be expressed as a sum of Jack polynomials and lower degree terms. These polynomials can be expressed as *normalized* polynomials

$$J^{r\delta}_{\lambda} = J^{\alpha}_{\lambda} + (\text{terms of lower degree})$$

where

$$J^{(\alpha)}_{\lambda} = c_{\lambda}(\alpha) P^{(\alpha)}_{\lambda}$$

We call $E^{r\delta}_{\eta}$ and $E^{(\alpha)}_{\eta}$ the nonsymmetric analogues of $P^{r\delta}_{\lambda}$ and $P^{(\alpha)}_{\lambda}$ respectively, where η is a *composition* rather than a partition λ (defined in Section 2.2). We call $F^{r\delta}_{\eta}$ and $F^{(\alpha)}_{\eta}$ the normalized nonsymmetric analogues of $J^{r\delta}_{\lambda}$ and $J^{(\alpha)}_{\lambda}$.

4 Recursions and Operators

Previous work by Knop and Sahi [3] has shown that the coefficients of $J^{(\alpha)}_{\lambda}$ with respect to m_{μ} belong to $\mathbb{N}[\alpha]$. Recent work by Naqvi, Yusra, and Sahi [2] extends this result to prove the positivity conjecture of interpolation polynomials $J^{r\delta}_{\lambda}$ and $P^{r\delta}_{\lambda}$. To do this, a few extra tools are required. This proof uses certain recursions and intertwining properties that are used on *nonsymmetric* polynomials, described in this section.

4.1 Generating monomials by recursion

Any monomial can be expressed recursively from a sequence of elementary transpositions s_i and affine intertwiners Φ, starting from $\mathbf{x^0} = x_1^0 x_2^0 \cdots x_n^0 = 1$. The elementary transposition works on a composition η in the following way

$$s_1(\eta) = s_1(\eta_1, \eta_2, \eta_3, ..., \eta_n) = (\eta_2, \eta_1, \eta_3, ..., \eta_n)$$

Similarly, we can write the action of s_i on a function in n variables. For example take $i = 2$, then

$$s_2(f(X_n)) = f(s_2(x_1, x_2, x_3, ..., x_n)) = f(x_1, x_3, x_2, ..., x_n)$$

The affine intertwiner Φ works on a composition η by moving the first entry to the last position and adding 1

$$\Phi(\eta) = \Phi(\eta_1, \eta_2, \eta_3, ..., \eta_n) = (\eta_2, \eta_3, ..., \eta_n, \eta_1 + 1)$$

We also have that Φ works on a function in n variables by moving the last variable to the first position and multiplying the function by that entry

$$\Phi f(x_1, x_2, x_3, ..., x_n) = x_n f(x_n, x_1, x_2, x_3, ..., x_{n-1})$$

Note that $\Phi(\eta)$ increases the length of a composition with each recursion since $|\eta|$ will increase by 1. Also note $\Phi(f)$ increases the degree of a monomial with each recursion since the function is multiplied by x_n.

Example 4.1.1 We consider the two variable case $n = 2$ and our goal is to show that $f(x, y) = x^4 y^8$ can be obtained recursively.

We start with the composition $\eta = (4, 8)$ and we want to write a sequence of s_i and Φ such that we arrive at the zero composition in two variables $(0, 0)$ which corresponds to $x^0y^0 = 1$.

$$\begin{aligned}
(4,8) &= \Phi(7,4)\\
(7,4) &= \Phi(3,7)\\
(3,7) &= \Phi(6,3)\\
(6,3) &= \Phi(2,6)\\
(2,6) &= \Phi(5,2)\\
(5,2) &= \Phi(1,5)\\
(1,5) &= \Phi(4,1)\\
(4,1) &= \Phi(0,4)\\
(0,4) &= \Phi(3,0)\\
s_1(3,0) &= (0,3)\\
(0,3) &= \Phi(2,0)\\
s_1(2,0) &= (0,2)\\
(0,2) &= \Phi(1,0)\\
s_1(1,0) &= (0,1)\\
(0,1) &= \Phi(0,0)
\end{aligned}$$

Hence we obtain the monomial x^4y^8 from x^0y^0 recursively since

$$(4,8) = \Phi^9 s_1 \Phi s_1 \Phi s_1 \Phi(0,0)$$

4.2 Generating nonsymmetric interpolation polynomials by recursion

Theorem 4.2.1 *(Thm 2.13 [2]) Nonsymmetric interpolation polynomials satisfy the recursion*

$$E^{r\delta}_{\phi\eta} = \phi^- E^{r\delta}_{\eta}, \qquad (\sigma_i^- + c_i^\eta)E^{r\delta}_{\eta} = d_i^\eta E^{r\delta}_{s_i\eta}$$

where

$$\phi^- f(x) = x_n f(x_n - 1, x_1, ..., x_{n-1}), \qquad \sigma_i^- = s_i - r\partial_i$$

and the scalars c_i^η and d_i^η are given by

$$c_i^\eta = \frac{r}{\overline{\eta}_i - \overline{\eta}_{i+1}}, \qquad d_i^\eta = \begin{cases} 1 & \text{if } \eta_i < \eta_{i+1} \\ 1 - (c_i^\eta)^2 & \text{if } \eta_i \geq \eta_{i+1} \end{cases}$$

These recursions suffice to generate all $E^{r\delta}_{\eta}$ from $E^{r\delta}_{0} = 1$.

4.3 The Dehomogenization Operator

We want to find a way to relate homogeneous and inhomogeneous polynomials, and we can do this using the *dehomogenization operator* Ξ. Recall that the set

$\{1, x, x^2\}$ forms a basis for $\mathcal{P}_2$, but so does $\{1+x, 2x+x^2, -3+5x+x^2\}$. The first basis is a set of homogeneous polynomials, whereas the second basis is a set of inhomogeneous polynomials and there is a unique linear map that relates them. Similarly, the homogeneous polynomials $F_\eta^{(\alpha)}$ and the inhomogeneous polynomials $F_\eta^{r\delta}$ are both basis for the vector space of polynomials $\mathcal{P}_n$. The dehomogenization operator Ξ is the unique linear operator that maps $F_\eta^{(\alpha)}$ to $F_\eta^{r\delta}$.

$$\Xi(F_\eta^{(\alpha)}) = F_\eta^{r\delta}$$

Recall $F_\eta^{(\alpha)}$is the normalized version of $E_\eta^{(\alpha)}$ and $F_\eta^{r\delta}$ is the normalized version of $E_\eta^{r\delta}$.The operator Ψ is the unique linear operator that maps $E_\eta^{(\alpha)}$ to $E_\eta^{r\delta}$ where

$$\Psi = S^{-1}\Xi S = S\Xi S$$

and S is the sign change operator. Ξ and Ψdo not commute with Φ and s_i, but we do find the following relationships

$$\Xi\Phi = \Phi^+\Xi, \qquad \Xi s_i = \sigma_i^+\Xi$$
$$\Psi\Phi = \Phi^-\Psi, \qquad \Psi s_i = \sigma_i^-\Psi$$

where $\sigma_i^+ = s_i + r\partial_i$ and $\Phi^+ f(x) = x_n f(x_n+1, x_1, ..., x_{n-1})$.

The following propositions are proved in [2] and included here for reference.

Proposition 4.3.1 *If f is a homogeneous polynomial, then $\Psi(f)$ is characterized by the properties*
(i)$\Psi(f(x)) = f(x) +$ *terms of degree* $<$ *deg*(f)
(ii)$\Psi(f(\overline{\eta})) = 0$ *for all compositions* η *with* $|\eta| <$ *deg*(f)

Proposition 4.3.2 *The operator Ψ preserves the space of symmetric polynomials.*

Proposition 4.3.3 *If f is a homogeneous symmetric polynomial, then $\Psi(f)$ is characterized by the properties*
(i)$\Psi(f)$*is symmetric*
(ii)$\Psi(f(x)) = f(x) +$ *terms of degree* $<$ *deg*(f)
(iii)$\Psi(f(\mu + r\delta)) = 0$ *for all partitions* μ *with* $|\mu| <$ *deg*(f)

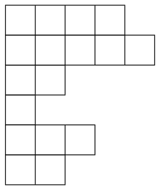
Proposition 4.3.4 *The operator Ψ maps $P_\lambda^{(\alpha)}$ to $P_\lambda^{r\delta}$.*

Proposition 4.3.5 *If f is a homogeneous polynomial, then $\Xi(f)$ is characterized by the properties*
(i)$\Xi(f(x)) = f(x) +$ *terms of degree* $<$ *deg*(f)
(ii)$\Psi(f(-\overline{\eta})) = 0$ *for all compositions* η *with* $|\eta| <$ *deg*(f)

Theorem 4.3.6 *The operator Ξ preserves the space of symmetric polynomials and maps $J_\lambda^{(\alpha)}$ to $J_\lambda^{r\delta}$. If f is a homogeneous symmetric polynomial, then $\Xi(f)$ is characterized by the properties*
(i)$\Xi(f)$*is symmetric*
(ii)$\Xi(f(x)) = f(x) +$ *terms of degree* $< deg(f)$
(iii)$\Xi(f(-\mu - r\delta)) = 0$ *for all partitions* μ *with* $|\mu| < deg(f)$

5 The Bar Game

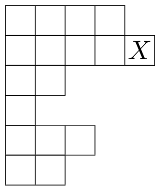
Given any composition η, we can create a diagram by stacking rows of boxes where the length of each row corresponds to the entries of a composition η. If $\eta = (4, 5, 2, 1, 3, 2)$, then the associated diagram would be



where we call row 1 the uppermost row. A *bar game*, denoted by $G(\eta)$ is of a sequence of moves $g_1, g_2, ...g_{|\eta|}$, where each move g_i is called a *glissade*.

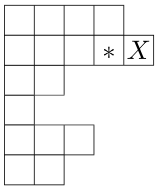
5.1 Performing a glissade

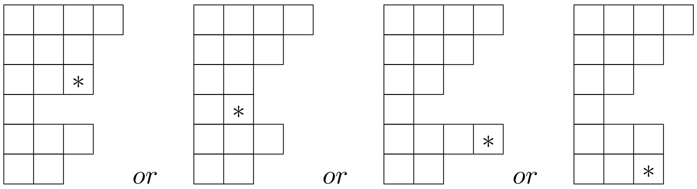
First identify the critical box - the rightmost and uppermost box of the diagram. We label the critical box by X.



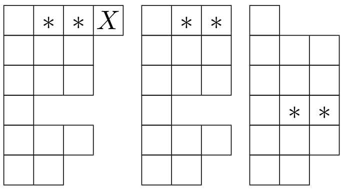
Let m be the length of a row, and let k be the row number where $k = 1$ is the uppermost row. Above, the critical box has row length $m = 5$ and is in row $k = 2$. The critical box, denoted by $s[\eta]$, is given by $s[\eta] = (k, m) = (2, 5)$. Once we identify the critical box, we delete it and choose l boxes to the left of the critical box. Move these l boxes to the end of another row - either above and strictly left, or below and weakly left of their original position.

Note that $l \geq 0$ and so there can be more than one possible game for each composition η. We call $\mathcal{G}(\eta)$ the set of all possible bar games for the composition η. In the previous example, suppose $l = 1$ (marked by $*$), since moving the box to the end of the above row would not be strictly to the left, we can only move the box to a row below and weakly to the left. There are 4 possible places to move the box.

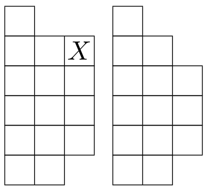




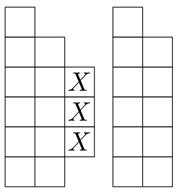
Suppose we choose the first option, then our new composition becomes $\eta_2 = (4, 3, 3, 1, 3, 2)$. This completes our first glissade of this bar game, denoted G_1. We repeat this process until no boxes remain. Let us complete this example and choose the second glissade g_2 to be the deletion of the critical box and movement of $l = 2$ to row 4.



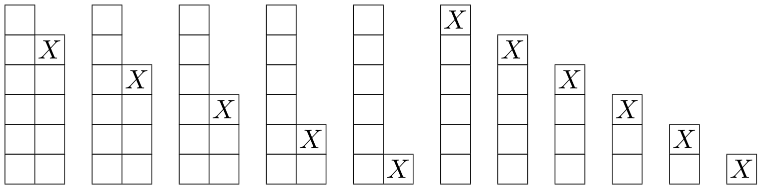
Let the third glissade g_3 be the deletion of the critical box, and movement of $l = 0$ (do not move any boxes).



Let glissades g_4, g_5, g_6 be the deletion of the critical box, and movement of $l = 0$ (do not move any boxes).



At this point in the game, we notice that there is only one possible option for subsequent glissades. There is no possible way to delete the critical box and move another box above and strictly left or below and weakly left. Our only option is to delete the critical box and make no movements, hence $l = 0$ for glissades $g_7, ..., g_{17}$.



This constitutes one bar game G_1 in the set of all possible bar games $\mathcal{G}(\eta)$ where

$$G_1 = g_1, g_2, ..., g_{17} \tag{2}$$

5.2 Weight of a glissade

Now that we understand the bar game, we can use it to describe a *bar monomial.* First, we need to understand the *weight of a glissade* and the *weight of a bar game.* Recall the critical box is $s[\eta] = (k, m)$ where k is the row number and m is the length of the row. Let n be the number of rows remaining. Let $l[\eta]$ be the leg of the critical box (k, m), this is the number of rows below row k that have length m, in addition to the number of rows above row k that have length $m - 1$.Then the weight of a glissade $w(g)$ is given by

$$w(g) = \begin{cases} x_k + (m-1) + r(n - 1 - l[\eta]), & l = 0 \\ r, & l > 0 \end{cases}$$

There are 17 glissades in the bar game outlined above (2). Let us calculate the weight of each glissade.

$$
\begin{aligned}
w(g_1) &= r, \text{since } l > 0 \\
w(g_2) &= r, \text{since } l > 0 \\
w(g_3) &= x_2 + (3-1) + r(6-1-3) = x_2 + 2 + 2r \\
w(g_4) &= x_3 + (3-1) + r(6-1-3) = x_3 + 2 + 2r \\
w(g_5) &= x_4 + (3-1) + r(6-1-1) = x_4 + 2 + 2r \\
w(g_6) &= x_5 + (3-1) + r(6-1-3) = x_5 + 2 + 2r \\
w(g_7) &= x_2 + (2-1) + r(6-1-5) = x_2 + 1 \\
w(g_8) &= x_3 + (2-1) + r(6-1-5) = x_3 + 1 \\
w(g_9) &= x_4 + (2-1) + r(6-1-5) = x_4 + 1 \\
w(g_{10}) &= x_5 + (2-1) + r(6-1-5) = x_5 + 1 \\
w(g_{11}) &= x_6 + (2-1) + r(6-1-5) = x_6 + 1 \\
w(g_{12}) &= x_1 + (1-1) + r(6-1-5) = x_1 \\
w(g_{13}) &= x_2 + (1-1) + r(6-1-5) = x_2 \\
w(g_{14}) &= x_3 + (1-1) + r(6-1-5) = x_3 \\
w(g_{15}) &= x_4 + (1-1) + r(6-1-5) = x_4 \\
w(g_{16}) &= x_5 + (1-1) + r(6-1-5) = x_5 \\
w(g_{17}) &= x_6 + (1-1) + r(6-1-5) = x_6
\end{aligned}
$$

Definition 5.2.1 The *weight of a bar game* is the product of all the weights of each glissade in that bar game.

$$w(G) = \prod_{g \in G} w(g)$$

Example 5.2.2 The weight of the bar game $G_1 = g_1, g_2, ..., g_{17}$ is given by

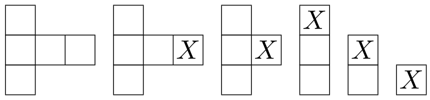
$$
\begin{aligned}
w(G_1) = &\prod_{g \in G} w(g) \\
=&(x_2 + 2 + 2r)(x_3 + 2 + 2r)(x_4 + 2 + 2r)(x_5 + 2 + 2r) \\
&(x_2 + 1)(x_3 + 1)(x_4 + 1)(x_5 + 1)x_1x_2x_3x_4x_5x_6 \\
=&\prod_{i=2}^{5}(x_i + 2 + 2r)(x_i + 1) \prod_{j=1}^{6} x_j
\end{aligned}
$$

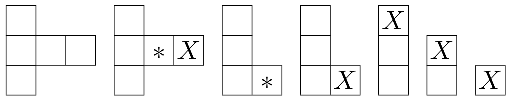
The *bar monomial* denoted by $x^{\underline{\eta}}$ can now be expressed as the sum of weights of all possible bar games for the composition η.

$$x^{\underline{\eta}} = \sum_{G \in \mathcal{G}(\eta)} w(G)$$

Example 5.2.3 Let us determine the bar monomial for the composition $\eta = (1, 3, 1)$.

There are two possible bar games for this composition.

G_1 :


G_2 :


The weight of the first bar game G_1 is the product of the weight of each glissade
$w(g_1) = x_2 + (3-1) + r(3-1-0) = x_2 + 2 + 2r$
$w(g_2) = x_2 + (2-1) + r(3-1-1) = x_2 + 1 + r$
$w(g_3) = x_1 + (1-1) + r(3-1-2) = x_1$
$w(g_4) = x_2 + (1-1) + r(2-1-1) = x_2$
$w(g_5) = x_3 + (1-1) + r(1-1-0) = x_3$

$$w(G_1) = (x_2 + 2 + 2r)(x_2 + 1 + r)x_1x_2x_3$$

Similarly we can calculate the weight of the second bar game G_2.
$w(g_1) = r$
$w(g_2) = x_3 + (2-1) + r(3-1-2) = x_3 + 1$
$w(g_3) = x_1 + (1-1) + r(3-1-2) = x_1$
$w(g_4) = x_2 + (1-1) + r(2-1-1) = x_2$
$w(g_5) = x_3 + (1-1) + r(1-1-0) = x_3$

$$w(G_2) = r(x_3 + 1)x_1x_2x_3$$

The bar monomial $x^{\underline{\eta}}$ is the sum of the weights of each bar game

$$x^{1,3,1} = x_1x_2x_3(x_2 + 2 + 2r)(x_2 + 1 + r) + r(x_3 + 1)x_1x_2x_3$$

6 Uniqueness of bar monomials

We have now seen that given a composition, we can find an expression for the corresponding bar monomial by playing bar games and summing their weights. It remains to be proved that bar monomials are uniquely defined by these expressions.

6.1 Bar monomials and the dehomogenization operator

Proving uniqueness of bar monomials uses the dehomogenization operator Ξ, by constructing a map from the monomials indexed by composition $x^\eta = x_1^{\eta_1} x_2^{\eta_2} \cdots x_n^{\eta_n}$ to the bar monomials $x^{\underline{\eta}}$. We can also do the same for partitions.

$$x^{\underline{\eta}} = \Xi(x^\eta), \qquad m_{\underline{\lambda}} = \Xi(m_\lambda)$$

where the expansion coefficients of the bar monomials are defined by

$$x^{\underline{\eta}} = \sum_\gamma c_{\eta,\gamma}(r) x^\gamma, \qquad m_{\underline{\lambda}} = \sum_\mu d_{\lambda,\mu}(r) m_\mu$$

Theorem 6.1.1 *(Thm 3.11 [2]) The bar monomial $x^{\underline{\eta}}$ is the unique polynomial $g(x)$ satisfying*
(i) $g(x) = x^\eta +$ *terms of degree* $< |\eta|$
(ii) $g(-\overline{\gamma}) = 0$ *if* $|\gamma| < |\eta|$

Recall $\overline{\gamma} = \gamma + w_\gamma(\rho)$ where w_γ is the shortest permutation that sorts the composition γ into a partition γ^+ and $\rho = r(n-1, ..., 1, 0)$.

Example 6.1.2 Suppose $n = 3$ and $\eta = (1, 3, 1)$. Let us use Theorem 6.0.1 to show that the bar monomial $x^{\underline{1,3,1}}$ is the unique polynomial

$$g(x_1 x_2 x_3) = x_1 x_2 x_3 (x_2 + 2 + 2r)(x_2 + 1 + r) + r(x_3 + 1) x_1 x_2 x_3$$

Expanded and simplified, we find that

$$\begin{aligned} g(x_1 x_2 x_3) &= x_1 x_2^3 x_3 + 2r^2 x_1 x_2 x_3 + 3r x_1 x_2^2 x_3 + r x_1 x_2 x_3^2 + 5r x_1 x_2 x_3 \\ &\quad + 3 x_1 x_2^2 x_3 + 2 x_1 x_2 x_3 \\ &= x^{1,3,1} + (\text{terms of degree} < 5) \end{aligned}$$

hence (i) is satisfied.

Now for (ii), we have $n = 3$ so $\rho = r\delta = r(2, 1, 0) = (2r, r, 0)$. We need to write all compositions γ such that $|\gamma| < |\eta| = 5$.

If $|\gamma| = 0$, then $\gamma = (0, 0, 0)$
If $|\gamma| = 1$, then $\gamma = (1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$
If $|\gamma| = 2$, then $\gamma = (1, 1, 0), (1, 0, 1), (0, 1, 1), (2, 0, 0), (0, 2, 0)$, or $(0, 0, 2)$
If $|\gamma| = 3$, then $\gamma = (1, 1, 1), (3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 0, 2), (0, 1, 2)$, or $(0, 2, 1)$
If $|\gamma| = 4$, then $\gamma = (2, 1, 1), (1, 2, 1), (1, 1, 2), (2, 2, 0), (2, 0, 2), (0, 2, 2), (3, 1, 0), (3, 0, 1), (1, 3, 0), (1, 0, 3), (0, 1, 3), (0, 3, 1), (4, 0, 0), (0, 4, 0)$, or $(0, 0, 4)$

Notice that the compositions $(0, 0, 0)$, $(1, 0, 0)$, $(2, 0, 0)$, $(3, 0, 0)$, $(4, 0, 0)$, $(1, 1, 0), (1, 1, 1), (2, 1, 0), (2, 1, 1), (2, 2, 0)$, and $(3, 1, 0)$ are already sorted - they are partitions and so the shortest permutation to sort them is the identity permutation. Hence for these γ, $w_\gamma(\rho) = (2r, r, 0)$ and $-\overline{\gamma} = -(\gamma + (2r, r, 0))$. This means for any γ with last entry 0, we have that $g(-(\gamma + (2r, r, 0)) = 0$ since every term in g is multiplied by x_3.

Let's prove (ii) for $\gamma = (1, 1, 1)$ where $w_\gamma(\rho) = \rho$ given that
$g(x_1, x_2, x_3) = x_1x_2x_3(x_2 + 2 + 2r)(x_2 + 1 + r) + x_1x_2x_3r(x_3 + 1)$

$$\begin{aligned}
g(-((1,1,1) + (2r, r, 0)) &= g(-2r - 1, -r - 1, -1) \\
&= x_1x_2x_3(x_2 + 2 + 2r)((-r - 1) + (r + 1)) \\
&+ x_1x_2x_3r(-1 + 1) \\
&= x_1x_2x_3(x_2 + 2 + 2r)(0) + x_1x_2x_3r(0) \\
&= 0
\end{aligned}$$

Let's prove (ii) for $\gamma = (1, 2, 1)$ where $w_\gamma(\rho) \neq \rho$. We note that (1,2,1) is not a partition and the shortest permutation that sorts this composition into a partition is one that swaps the first and second entries. We apply this permutation to ρ.

$$-\overline{\gamma} = -(\gamma + w_\gamma(\rho)) = -((1, 2, 1) + (r, 2r, 0))$$

Then

$$\begin{aligned}
g(-((1,1,1) + (r, 2r, 0)) &= g((-r - 1, -2r - 2, -1)) \\
&= x_1x_2x_3(-2r - 2 + 2 + 2r)(x_2 + 1 + r) \\
&+ x_1x_2x_3r(-1 + 1) \\
&= x_1x_2x_3(0)(x_2 + 1 + r) + x_1x_2x_3r(0) \\
&= 0
\end{aligned}$$

Indeed, it can be shown that (ii) is satisfied for all compositions γ and thus the bar monomial $x^{\underline{1,3,1}}$ is the unique polynomial

$$g(x_1x_2x_3) = x_1x_2x_3(x_2 + 2 + 2r)(x_2 + 1 + r) + r(x_3 + 1)x_1x_2x_3$$

The bar game gives us a tool to identify a polynomial corresponding to a given composition η, and Theorem 6.0.1 allows us to confirm this polynomial is the unique bar monomial $x^{\underline{\eta}}$.

7 Summary and future directions

The proof of Theorem 6.0.1 uses recursions, permutations, downwards induction and iteration. Once it is established that bar monomials are uniquely defined by the sum of the weights of bar games, it follows that bar monomials have coefficients that lie in $\mathbb{N}[\alpha]$ since these weights are given by polynomials with non-negative integer coefficients.

Nonsymmetric interpolation polynomials

$$F_\eta^{r\delta} = \sum_{|\gamma|\leq|\eta|} \alpha^{|\gamma|-|\eta|} b_{\eta,\gamma}(\alpha) x^\gamma$$

can be expressed as

$$F_\eta^{r\delta} = \sum_{|\zeta|=|\eta|} b_{\eta,\zeta}(\alpha) x^{\underline{\zeta}}$$

where the nonsymmetric Jack polynomial has positive coefficients.

$$F_\eta^{(\alpha)} = \sum_{|\zeta|=|\eta|} b_{\eta,\zeta}(\alpha) x^\zeta, \qquad b_{\eta,\zeta}(\alpha) \in \mathbb{N}[\alpha]$$

Since both Jack polynomials and bar monomials have positive coefficients, the same must be true for $F_\eta^{r\delta}$. Using a similar argument, the normalized symmetric case $J_\lambda^{r\delta}$ and regular symmetric case $P_\lambda^{r\delta}$ of interpolation polynomials are shown to have positive coefficients.

One interesting topic for future research would be to investigate a natural extension of this positivity property to a family of super-symmetric polynomials called *Sergeev-Veselov* polynomials [5]. They are characterized by being symmetric in $m+n$ variables $(x_1, ..., x_n, y_1, ..., y_m)$ which are also symmetric in $(x_1, ..., x_n)$ and $(y_1, ..., y_m)$ separately. They are a one-parameter family of symmetric polynomials and have the property that by setting $x_1 = y_1 = t$, the parameter t disappears.

8 References

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