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Spherical t-designs and Unitary t-designs

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Contents

Chapter 1

Spherical t-designs

The concept of a spherical t-design was first proposed by P. Deslarte in 1976 [8], and subsequently studied extensively by Y. Hong in 1982. Later, one of the fundamental theorems in the study of spherical t-designs (1.4.1) was proved by P. Seymour and T. Zaslavsky in 1984 [15]. While there has been sporadic research on the subject since then, interest in t-designs was greatly renewed due to their usefulness in the study of quantum information, which is discussed further in the section on quantum information (2).

This report will first discuss the history, and basic mathematical concepts of spherical t-designs, followed by some recent results on the subject. Then a review of basic quantum information will be given, to provide context for discussions on unitary t-designs, leading up to the groundbreaking results of [9].

1.1 Preliminaries

We begin with the definition of a spherical t-design on S^{d-1}, as well as an intuition in $\mathbb{R}^3$, to provide context for discussions on $\mathbb{R}^2$, $\mathbb{R}^3$, and the more general case of $\mathbb{R}^d$.

Definition 1.1.1 *A finite set X of points on the unit sphere S^{d-1} in $\mathbb{R}^d$ is called a spherical t-design if for every d-variable polynomial f_t of total degree at most t, we have*

$$\frac{1}{|X|}\sum_{x\in X} f_t(x) = \int_{S(\mathbb{R}^d)} f_t(u)d\mu(u).$$

Mathematically, a spherical t-design is a finite set of points on the sphere which, from an averaging point of view, is indistinguishable from the entire sphere. If one expects averaging indistinguishability, then a natural choice would be to use vertices of a polyhedron with plenty of symmetry, examples of which will be shown in the following sections. In addition, a finite set of points cannot behave indistinguishably with respect to averaging on all functions. Therefore it is natural to assume boundedness of degree in 1.1.1.

Intuitively, one can think of a spherical design as an object created to behave the same as the unit sphere, while containing less information. As a traditional sphere is a 3-dimensional shape, it is easiest to imagine a spherical t-design in $\mathbb{R}^3$. If you put a tetrahedron inside of a sphere, you lose a lot of the sphere, but you only have to place 4 vertices onto the surface of the sphere. A cube will retain more of the sphere, but you have twice as many vertices you must place. A dodecahedron begins to resemble a sphere, but now you have 20 vertices to align with the sphere. The mathematical value of this concept is that a sphere has infinite points, whereas these designs can be represented with a finite number of vertices. Thus, if you had a polynomial, and you wanted to take its average over the sphere, it would be much easier to use the set of points that are the vertices, rather than attempting to use the entire sphere. The problem arises in attempting to construct these designs. The higher the degree of the polynomial, the more vertices you require for the approximation to be accurate. This is the importance of the "t" in "t-design". By 1.1.1, at some degree t, using the points in X, the sum will cease to equal the integration. This motivates the construction of t-designs for higher values of t.

1.2 Explicit Spherical t-designs in $\mathbb{R}^2$ and $\mathbb{R}^3$

The simplest constructions of spherical t-designs can be generated based on the following theorem by Bannai:

Theorem 1.2.1 (See [3].) *A regular N-gon on $S^1 \subset \mathbb{R}^2$ is a t-design for $1 \leq t \leq N-1$.*

Bannai's theorem is a partial converse to the original theorem proved by Hong:

Theorem 1.2.2 (See [10].) *Let X be a spherical t-design in $\mathbb{R}^2$ with $|X| = n$. Then, when $t+1 \leq n \leq 2t+1$, X is a regular n-gon.*

One might find it interesting that Bannai, who will be referenced often in this report, introduced Hong to spherical designs [10]. These results will not be discussed in this paper, but it is important to note that not all t-designs on S^1 are regular N-gons. Hong also gives other classifications of t-designs. However, from this theorem we can easily create t-designs for many values of t. One additional fact that immediately follows from these theorems, especially highlighted by Bannai's version is this:

Corollary 1.2.3 *Any t-design is a (t−1)-design.*

Example 1.2.4 (Various Spherical Designs on S^1) *An equilateral triangle is a 1-design and a 2-design. A square is a t-design for $t \in \{1,2,3\}$. A regular heptagon is a t-design for $t \in \{1,2,3,4,5,6\}$.*

Example 1.2.5 (Graphical Representations of Spherical Designs on S^1)

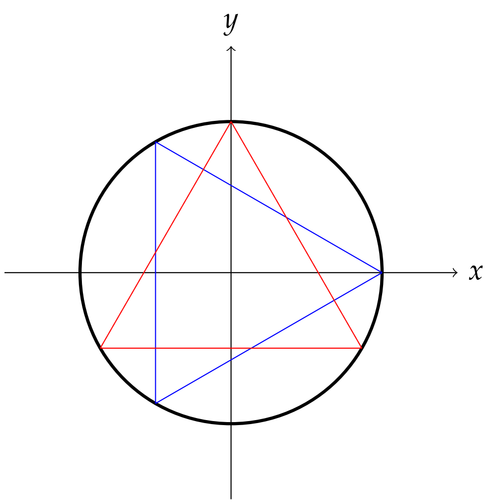

Figure 1: In blue, the spherical 2-design $\{(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$.
In red, the spherical 2-design $\{(0,1), (-\frac{\sqrt{3}}{2}, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, -\frac{1}{2})\}$.

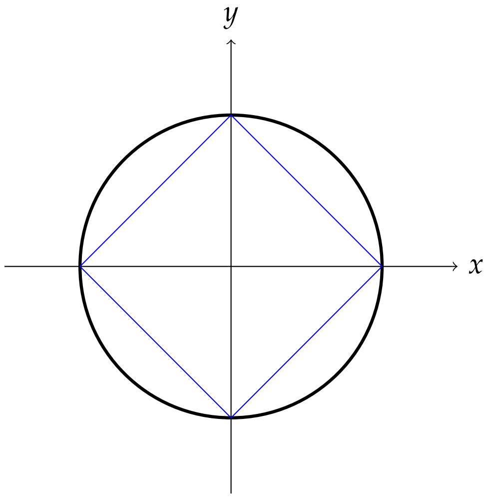

Figure 2: The spherical 3-design $\{(0,1),(1,0),(0,-1),(-1,0)\}$.

In $\mathbb{R}^2$, spherical designs are constructed on the unit circle, which means one can use trigonometry to easily construct their t-designs. Any set of three points $2\pi \div 3 = \frac{2\pi}{3}$ apart forms an equilateral triangle, and any four points separated by $2\pi \div 4 = \frac{\pi}{2}$ form a square.

In $\mathbb{R}^3$, and for higher dimensions, no explicit formula has been found which gives t-designs for all t, and so explicit constructions of spherical designs is still an active area of research. There are six widely known t-designs in $\mathbb{R}^3$:

Remark 1.2.6 (See [7].) *Any set of points in $\mathbb{R}^3$ with the center of mass at the origin is a 1-design.*

Remark 1.2.7 (See [3].) *A regular tetrahedron is a 2-design. A cube and a regular octahedron are both 3-designs. A regular dodecahedron and a regular icosahedron are both 5-designs.*

Example 1.2.8

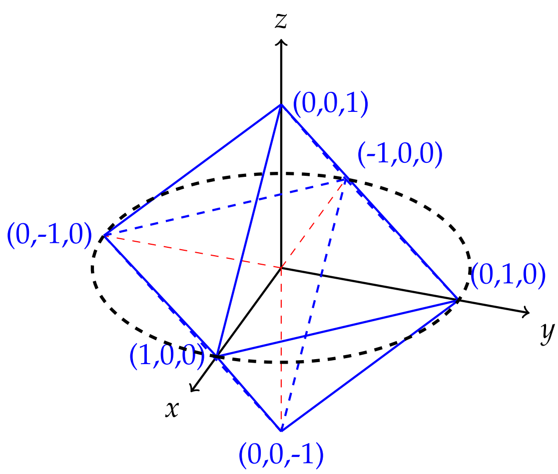

Figure 3: The spherical 3-design $\{(1,0,0),(-1,0,0),(0,1,0),(0,-1,0),(0,0,-1),(0,0,1)\}$.

1.3 Calculations with Spherical t-Designs on $\mathbb{R}^2$ and $\mathbb{R}^3$

For this section we will use one example of a 2-design in $\mathbb{R}^2$, one 3-design in $\mathbb{R}^2$, and one 3-design in $\mathbb{R}^3$, and demonstrate how using each provides the same result as integration. The examples are as follows,

(1) $X_1 = \{(1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$

(2) $X_2 = \{(0,1), (1,0), (0,-1), (-1,0)\}$

(3) $X_3 = \{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,-1), (0,0,1)\}$

$$Let\ f(x,y) = x^2.$$

$$\frac{1}{|X|}\sum_{x\in X} f_t(x) = \frac{1}{3}(f(1,0) + f(-\frac{1}{2}, \frac{\sqrt{3}}{2}) + f(-\frac{1}{2}, -\frac{\sqrt{3}}{2})) \tag{1.1}$$

$$= \frac{1}{3}(1 + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} cos^2\theta d\theta \ = \ \frac{1}{2\pi}\int_0^{2\pi} \frac{1+cos(2\theta)}{2} d\theta \tag{1.2}$$

$$= \frac{1}{2\pi}((\frac{2\pi}{2} + 0) - (0+0)) \ = \ \frac{\pi}{2\pi} \ = \ \frac{1}{2}$$

$$Let\ f(x,y) = x + y^3.$$

$$\frac{1}{|X|}\sum_{x\in X} f_t(x) = \frac{1}{4}(f(0,1) + f(1,0) + f(0,-1) + f(-1,0)) \tag{1.3}$$

$$\frac{1}{4}(1 + 1 - 1 - 1) = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi} cos\theta + sin^3\theta d\theta = \frac{1}{2\pi}\int_0^{2\pi} sin^3\theta d\theta \tag{1.4}$$

$$\frac{1}{2\pi}(\frac{1}{3}cos^3\theta - cos\theta)|_0^{2\pi} = \frac{1}{2\pi}(\frac{1}{3} - 1 - \frac{1}{3} + 1) = 0$$

$$Let\ f(x,y,z) = x^2 + y^2 + 2z^2.$$

$$\frac{1}{|X|}\sum_{x\in X} f_t(x) = \frac{1}{6}(f(1,0,0) + f(-1,0,0) + f(0,1,0) + f(0,-1,0) + f(0,0,1) + f(0,0,-1)) \tag{1.5}$$

$$= \frac{1}{6}(1+1+1+1+2+2) = \frac{4}{3}$$

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{5sin\phi + sin3\phi}{4} d\phi d\theta = \frac{1}{4\pi}\int_0^{2\pi} \frac{8}{3} d\theta \tag{1.6}$$

$$= \frac{16\pi}{12\pi} \ = \ \frac{4}{3}$$

1.4 Higher Dimensional Spherical t-designs

One of the fundamental theorems of spherical t-designs is the following:

Theorem 1.4.1 (See [15].) *For all $\mathbb{R}^n$, there exists a t-design for all t.*

However, currently there are only three known constructions of t-designs which apply for any $\mathbb{R}^n$.

Theorem 1.4.2 (See [3].) *Let $n \in \mathbb{Z}^+$. In $\mathbb{R}^n$ there exists a 2-design of $n+1$ points, a 3-design of 2^n points, and a 3-design of $2n$ points.*

Referring back to the previous section, in $\mathbb{R}^2$, these correspond to the triangle, and the square. In $\mathbb{R}^3$ that is the tetrahedron, the octahedron, and the cube. However, despite the lack of general rules, many specific constructions have been found that could be used to inspire research into general constructions, one category of these constructions will be discussed here.

Remark 1.4.3 (See [3].) *Lattices and their shells are a good source to look for spherical t-designs.*

The following are two examples of explicit constructions of spherical designs that have been found using lattices.

Example 1.4.4 (See [5].) *The Leech lattice is an 11-design on $S^{23} \subset \mathbb{R}^{24}$, containing 196560 points.*

The Leech lattice will be discussed again in the next section, as it belongs to a special class of spherical designs.

Theorem 1.4.5 (See [16].) *Let $m \in \mathbb{Z}^+$. Any shell of an extremal even unimodular lattice is a spherical t-design on S^{8m-1} for $t = 11 - 4(m\ mod 3)$.*

This theorem only applies under very specific conditions, but perhaps could be generalized further to either provide more designs, or be less restrictive.

1.5 Further Work on Spherical Designs

As seen in 1.2.7, a 3-design in $\mathbb{R}^3$ can be constructed with 8 points, or instead it could be constructed with only 6 points. Likewise, one can construct a 5-design with 12 or 20 points. This gives rise to a second area of research in spherical designs. Once a design has been discovered, the question is whether or not it is the most efficient design for that t. The reason for constructing t-designs is that they are easier to work with than spheres, and so the natural goal would be to find the most efficient design for every t in every dimension n, where efficiency is measured in the number of points that make up the design.

A lower bound has been proven for the minimum size of t-designs, for both even and odd values of t:

Theorem 1.5.1 (See [12].)

(1) *Let X be a spherical 2e-design on S^{n-1}, then $|X| \geq \binom{n+e-1}{e} + \binom{n+e-2}{e-1}$.*

(2) *Let X be a spherical (2e+1)-design on S^{n-1}, then $|X| \geq 2\binom{n+e-1}{e}$.*

Definition 1.5.2 *(Tight Design) A spherical t-design is considered tight if it achieves the lower bound that corresponds to* $t = 2e$ *or* $t = 2e + 1$.

An additional tangential definition is required for the study of tight designs.

Definition 1.5.3 *(Degree, see [3]) The degree s of a set X is defined in this context as*
$s = |\{u \cdot v | u, v \in X, u \neq v\}|$.

Plainly, it is the size of the set generated by taking the dot product of every vector in X with every other vector in X.

Example 1.5.4 *The degree of the 3-design on* S^2,
$X = \{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}$,
is $2 = |\{-1, 0\}|$.

With degree defined, we can introduce new methods of determining if a design is tight.

Theorem 1.5.5 (See [12].)

(1) *Let X be a spherical 2e-design. X is tight if and only if X is of degree e.*

(2) *Let X be a spherical (2e+1)-design. X is tight if and only if X is of degree e+1 and is antipodal.*

One can see by this theorem that 1.5.4 is tight.

Certain facts have been proved for tight spherical t-designs:

Theorem 1.5.6 (See [4].) *On* S^{n-1}, *for* $n \geq 3$, *if X is a tight spherical t-design, then either* $t \in \{1, 2, 3, 4, 5, 7\}$, *or X is the Leech lattice [1.4.4].*

In other words, outside of S^1, tight spherical t-designs only exist for $t \in \{1, 2, 3, 4, 5, 7\}$, with one exceptional example when $t = 11$. The implication of this theorem is that the lower bound on the size of spherical t-designs from 1.5.1 could be refined further for when t is not part of the set of values in 1.5.6. It would be both interesting and significant if there was a second greatest lower bound that could be proven for all other cases, but it is unlikely that there would be one greatest lower bound that all other t could achieve. A sub-problem of this is the interesting case of $t = 11$. There is a very strong dimensional constraint of $n = 24$, as the Leech lattice on S^{23} is the exceptional tight t-design for $t = 11$. This deserves a better understanding at a later time.

Theorem 1.5.7 (See [5].) *Let X be a tight spherical t-design on* S^{n-1}.

- *If* $t = 1$, *then X is two antipodal points.*
- *If* $t = 2$, *then X is a regular simplex of n+1 points.*
- *If* $t = 3$, *then X is a cross polytope,* $\{\pm e_i | 1 \leq i \leq n\}$, *where the* e_i*'s form an orthonormal basis of* $\mathbb{R}^n$.

The classification for when t is 4, 5 and 7, remains an open problem.

A second area of research with spherical t-designs concerns the applications of orthogonal transformations.

Definition 1.5.8 *(Rigid Design, see [2]) Let $X = \{u_1, u_2, ..., u_N\}$ be a spherical t-design. X is rigid if for any $\epsilon \in \mathbb{R}^+$, there does not exist a spherical t-design $X' = \{u'_1, u'_2, ..., u'_N\}$ that satisfies both the following.*

(1) $||u_i - u'_i|| < \epsilon$ for $i = 1, 2, ..., N$.

(2) There is no orthogonal transformation σ satisfying $X' = X^\sigma$.

Remark 1.5.9 (See [3].) *Tight t-designs are rigid.*

There are only two known rigid designs that are not tight, thus it is an open problem to create a rule that defines all rigid designs that are not tight.

Remark 1.5.10 (See [14], [6], [11].) *The following are the only two known rigid designs that are not tight.*

(1) (n+2)-point sets on S^{n-1} when n is even.

(2) 120 points of a 600-cell on S^3.

Finally, Bannai gives the following conjecture:

Conjecture 1.5.11 (See [3].) *For each given pair of n and t, there are only finitely many rigid spherical t-designs on S^{n-1}, up to orthogonal transformations.*

Chapter 2

Unitary t-designs

In the field of quantum information, it is very common to want to randomize over the entire unitary group. However, the unitary group is infinite, which is computationally difficult. Just as spherical t-designs are used to simplify computation on spheres which contain infinite points, unitary t-designs are used to approximate the infinite unitary group to simplify computation. The definition for spherical t-designs is valid for unitary t-designs, however, the proper way to write it in the context of unitary matrices is as follows:

Definition 2.0.1 *(Originally used in [13]) Let X be a finite subset of U(d). The following are equivalent.*

(1) X is a unitary t-design.

(2) $\frac{1}{|X|}\sum_{U\in X} U^{\otimes t}\otimes (U^{\dagger})^{\otimes t} = \int_{U(d)} U^{\otimes t}\otimes (U^{\dagger})^{\otimes t} dU$

See [1] for a detailed list of the applications of unitary t-designs.

2.1 Review of Unitary Matrices and Quantum Information

Definition 2.1.1 *(Unitary Matrix) Let* $U \in M_2(\mathbb{C})$. *U is unitary if* $U^* = U^{-1}$.

Remark 2.1.2 *In the context of unitary t-designs, the conventional notation is* $U^{\dagger} = U^{-1}$.

Example 2.1.3 *Let* $U = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. $U^{-1} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ *and* $U^{\dagger} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ *thus* $U^{-1} = U^{\dagger}$, *and so U is unitary.*

Definition 2.1.4 *(Unitary Group)* $U(n)$ *is the group of n x n unitary matrices, precisely* $U(n) := \{U \in M_n(\mathbb{C}) | U^{-1} = U^{\dagger}\}$.

Definition 2.1.5 *(Special Unitary Group)* $SU(n)$ *is the group of n x n unitary matrices with determinant one, precisely* $SU(n) := \{U \in U(n) | det(U) = 1\}$.

Quantum states, or just states, are the quantum equivalent of probability vectors. The entries are called amplitudes, and rather than adding up to one, it is the squares of the amplitudes that must sum to one. They are most commonly presented using the following notation.

Definition 2.1.6 *(Dirac Notation, named for its inventor Paul Dirac)*

- $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $|+\rangle := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
- $|-\rangle := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

When a state is measured, it is measured with respect to one of two commonly used bases.

Definition 2.1.7 *(Standard Basis)* $\{|0\rangle, |1\rangle\}$.

Definition 2.1.8 *(Hadamard Basis)* $\{|+\rangle, |-\rangle\}$.

Definition 2.1.9 *(Density Operator) Let $\{(p_k, |\phi_k\rangle) | \sum_{k=1}^{n} p_k = 1\}$ represent a probability distribution of states, where the ϕ_k's are unit vectors. A density operator ρ is a matrix with the following form.*

$$\rho := \textstyle\sum_k p_k |\phi_k\rangle \langle\phi_k|.$$

There are two types of density operators.

Definition 2.1.10 *(Pure State) A pure state is a density operator for which the corresponding probability distribution has $n = 1$. That is, $\rho = |\phi\rangle \langle\phi|$ for some unit vector ϕ.*

Example 2.1.11 *(Example of a pure state)*
$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ *is a pure state corresponding to the probability distribution $\{1, |+\rangle\}$.*

Definition 2.1.12 *(Mixed State) A mixed state is a density operator for which the corresponding probability distribution has $n > 1$.*

Example 2.1.13 *(Example of a mixed state)*
$\rho = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$ *is a mixed state corresponding to the probability distribution $\{(\frac{1}{4}, |0\rangle), (\frac{3}{4}, |1\rangle)\}$.*

Finally, there is a special class of mixed states.

Definition 2.1.14 *(Maximally Mixed State) A density operator ρ (2.1.9) is called a maximally mixed state if the vectors $\phi_1, ..., \phi_n$ are orthonormal.*

Example 2.1.15 *(Example of a maximally mixed state)*
$\frac{1}{2}I$ is a maximally mixed state because it represents infinitely many probability distributions, including:

(1) $\{\frac{1}{2}|0\rangle, \frac{1}{2}|1\rangle\}$

(2) $\{\frac{1}{2}|+\rangle, \frac{1}{2}|-\rangle\}$

One can apply unitary matrices to probability distributions to generate new probability distributions. Density operators make this easier, as the full calculation on the distribution can be reduced to a calculation on the operator.

Remark 2.1.16 *Let $U\rho$ represent the distribution $\{(p_k, U|\phi_k\rangle)\}$. $\rho' = U\rho$ can be calculated by $\rho' = U\rho U^\dagger$.*

In practice, this is used for the application of quantum gates to a quantum state. Say ρ is the state of a quantum system, and U represents a quantum gate. ρ' represents ρ after passing through U.

Example 2.1.17 *(Example application of unitary matrix to density operator) Let $U = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and let $\rho = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$.*

$$\rho' = UpU^\dagger \tag{2.1}$$
$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\rho' = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$$

2.2 The Relationship Between SU(2) and S^3

S^3 is isomorphic to $SU(2)$. Thus, a spherical t-design on S^3 can be represented as a unitary t-design on $SU(2)$.

Remark 2.2.1 *Let $v = (w, x, y, z) \in \mathbb{R}^4$. v can be written as a matrix in SU(2) in the following manner: $\begin{bmatrix} w + xi & zi - y \\ y + zi & w - xi \end{bmatrix}$.*

Example 2.2.2 *(A spherical 3-design on $\mathbb{R}^4$ written as a unitary 3-design on SU(2))*

(1) $\{\pm(1,0,0,0), \pm(0,1,0,0), \pm(0,0,1,0), \pm(0,0,0,1)\}$

(2) $\{\pm I, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

2.3 An Explicit Inductive Construction of Unitary t-Designs

In this last section we briefly touch on the results of [9], a recent paper by Bannai, Nakata, Okuda and Zhao. This paper contains many deep mathematical ideas which deserves a more thorough study than the brief overview covered here. It was chosen because previously only constructions for unitary 1-designs and 2-designs were well known, with very few for higher t's but this paper uses induction to create explicit unitary t-designs for all t and n. First we will review some important definitions used in the proof, then discuss how those definitions can be used to create designs, and then finally one specific result from the paper will be discussed.

Definition 2.3.1 *(Multi-Set) A set X is called a multi-set, if the elements of X can have multiplicities higher than one.*

Example 2.3.2 *(Example of a Multi-Set) $\{1, 1, 2, 3\}$ is a multi-set. The element 1 has multiplicity two.*

Definition 2.3.3 *(Direct Product of N Copies of a Group) Let G be some group, and* $N \in \mathbb{Z}_n$ *for some* $n \geq 1$*. The direct product of N copies of G,* G^N*, is defined as taking the direct product of G with itself, N times.*

Definition 2.3.4 *(Symmetric Group on* G^N*, [9]) Let* G^N *be the direct product of N copies of G.* $\mathfrak{S}_N$ *is the symmetric group on* G^N *with the action defined as the permutation of co-ordinates.*

Remark 2.3.5 (See [9].) *A* $\mathfrak{S}_N$ *orbit in* G^N *is a multi-set on G with N points. For some* $x = (x_1, ..., x_N) \in G^N$*, the* $\mathfrak{S}_N$ *orbit of x is* $X = \{x_1, ..., x_N\}_{mult}$*.*

Several operations are defined on these multi-sets. For the following several definitions, let $X = \{x_1, ..., x_N\}_{mult}$ on G.

Definition 2.3.6 *(Cardinality)* $|X| := N$*.*

Definition 2.3.7 *(Sum)* $\sum_{x \in X} \rho(x) := \sum_{i=1}^{N} \rho(x_i)$*.*

Definition 2.3.8 *(Dot Product/Minkowski Product) Let* $Y = \{y_1, ..., y_M\}_{mult}$*.*
$X \cdot Y := \{x_i y_j | 1 \leq i \leq N, 1 \leq j \leq M\}_{mult}$*.*

One should note that this is defined as the dot product due to the chosen notation. Computationally, it is the Minkowski product of sets.

Definition 2.3.9 *(Finite Dimensional Representation, [9]) Let G be a compact Hausdorff group.* (ρ, V) *is a finite dimensional complex G-representation if V is a finite dimensional complex vector space, and* $\rho : G \longrightarrow GL_{\mathbb{C}}(V)$ *is a group homomorphism for which* $G \times V \longrightarrow V$ *by* $(g, v) \longrightarrow gv$ *is continuous.*

Remark 2.3.10 *We call* (ρ, V) *a trivial representation if* $\forall g \in G$ *the relation* $\rho(g) = id_V$ *holds.*

Definition 2.3.11 *(G-stable Subspace) Let W be a subspace of V. W is G-stable if* $\forall g \in G$ *and* $\forall w \in W$*,* $\rho(g)w \in W$*.*

If the vector space of a representation has a G-stable subspace, then a sub-representation can be constructed.

Definition 2.3.12 *(Subrepresentation) Let* (ρ, V) *be a finite dimensional complex G-representation, and W a G-stable subspace of V.* (τ, W) *is a subrepresentation of* (ρ, V)*, where* $\tau : G \longrightarrow GL_{\mathbb{C}}$*,* $g \longrightarrow \tau(g) := (\rho(g))|_W$*.*

Definition 2.3.13 *(Irreducible) Let* (ρ, V) *be a finite dimensional complex G-representation.* (ρ, V) *is irreducible if* $V \neq 0$*, and the only G-stable subspace of V are 0 and V.*

For the next definition, note that (τ, W) is not considered a subrepresentation of (ρ, V).

Definition 2.3.14 *(G-intertwining, [9])*
A linear map $\theta : V \longrightarrow W$ *is G-intertwining if* $\forall g \in G$ *the relation* $\theta\rho(g) = \tau(g)\theta$ *holds.*

One can then say that (τ, W) and (ρ, V) are equivalent if θ is bijective.

Definition 2.3.15 *(Unitary Representation) Let (ρ, V) be a finite dimensional G-representation. (ρ, V) is unitary if V has a Hermitian inner product, and for any $g \in G$, $\rho(g)$ preserves the inner product.*

All of the previous definitions have been building up to this next definition, which is very important to the final results of the paper.

Definition 2.3.16 *(Unitary Dual of a Group, [9]) Let G be a compact Hausdorff group. The unitary dual $\widehat{G}$ of G is the set of representatives of the equivalence classes of the irreducible finite-dimensional unitary representations of G.*

Remark 2.3.17 (See [9].) *Two facts are well-known.*

(1) For any finite-dimensional G-representation, there exists an inner product such that the representation is unitary.

(2) Any two irreducible unitary G-representations are equivalent if and only if they are unitary equivalent.

Therefore, the unitary dual can simply be defined as the set of representatives for the equivalence classes of finite-dimensional irreducible G-representations.

In addition to the unitary dual, the other important definition for the paper is how to construct designs using compact groups.

Definition 2.3.18 *(ρ-design, [9]) Let G be a compact Hausdorff group with the Haar measure, and (ρ, V) a finite-dimensional G-representation. The finite multi-set X on G is a ρ-design if*

$$\frac{1}{|X|} \sum_{x \in X} \rho(x) = \int_G \rho(g) dg$$

in the vector space $End_{\mathbb{C}}(V)$.

It is important to note that in $\mathbb{R}^n$, t-design referred to the ability of the design to function for some polynomial of degree t. ρ-design only refers to the representation involved. With ρ-designs, further designs can be constructed.

Definition 2.3.19 *(Λ-design, [9]) Let Λ be a collection of finite-dimensional G-representations. X is a Λ-design if X is a ρ-design for all $\rho \in \Lambda$.*

Remark 2.3.20 (See [9].) *Several further designs can be constructed with ρ-designs and λ-designs.*

- *Let X be a ρ-design on G, and take some $g \in G$. Xg and gX are also ρ-designs on G.*
- *Let X be a Λ-design on G, and take some $g \in G$. Xg and gX are also Λ-designs on G.*
- *Let τ be a subrepresentation of ρ. Any ρ-design is a τ-design.*
- *Let X be both a ρ-design on G and a ϕ-design on G. Then X is also a $(\rho \oplus \phi)$-design on G.*

These definitions of designs on compact groups are used to prove the following two propositions.

Proposition 2.3.21 (See [9].) *Let X be a ρ-design on G, and let Y and Z be two non-empty finite multi-sets on G. Then $Y \cdot X \cdot Z$ is a ρ-design on G.*

Proposition 2.3.22 (See [9].) *The following are equivalent.*

(1) X is a ρ-design on G.

(2) X^{-1} is a ρ-design on G.

Bannai and his collaborators use representation theory of compact Lie groups to give an inductive algorithm for construction unitary t-designs, and as a result they obtain some explicit examples in $U(2)$ and $U(4)$. The base case for the algorithm in $U(4)$ is given below.

Lemma 2.3.23 (See [9].) *Let ω be a primitive fifth root of unity. $X = \{1, \omega, \omega^2, \omega^3, \omega^4\}$ is a unitary 4-design on $U(4)$.*

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