

ON EQUALITY OF RANKS OF LOCAL COMPONENTS OF AUTOMORPHIC REPRESENTATIONS

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ABSTRACT. We prove that the local components of an automorphic representation of an adelic semisimple group have equal rank in the sense of [31]. Our theorem is an analogue of the results previously obtained by Howe [16], Li [21], Dvorsky–Sahi [9], and Kobayashi–Savin [19]. Unlike previous works which are based on explicit matrix realizations and existence of parabolic subgroups with abelian unipotent radicals, our proof works uniformly for all of the (classical as well as exceptional) groups under consideration. Our result is an extension of the statement known for several semisimple groups (see [12], [30]) that if at least one local component of an automorphic representation is a minimal representation, then all of its local components are minimal.

1. INTRODUCTION

The notion of rank for a unitary representation of a semisimple group over a local field of characteristic zero is a powerful tool for studying *singular* (also known as *small*) unitary representations. The first such notion of rank, nowadays usually called *N-rank*, was introduced by Roger Howe [17] for the metaplectic group Mp_{2n} . In a nutshell, Howe’s idea is to consider orbits of the action of the Levi factor of the Siegel parabolic on its unipotent radical, and to associate them to unitary representations. Howe used his *N-rank* to construct singular unitary representations [18], to study the connection between singular representations and automorphic forms with degenerate Fourier coefficients [16], and to obtain explicit pointwise bounds for matrix coefficients of general irreducible unitary representations [17].

Following Howe’s work, a variety of notions of rank similar to Howe’s *N-rank* were defined, e.g., by Scaramuzzi [32] for GL_n , by Li [20], [21] for classical groups, and by Dvorsky and Sahi [9] for semisimple groups associated to Jordan algebras. The underlying idea of all of these works is similar to Howe’s original method, namely to consider the restriction of a unitary representation to an abelian unipotent radical.

In [31], the second author defined a new notion of rank that was applicable uniformly to nearly all semisimple groups over local fields of characteristic zero, and proved the *purity* theorem, which is one of the important steps in applications of all of the aforementioned notions of rank. The new idea that was introduced in [31] was to define rank based on Kirillov’s orbit method.

In this article, our goal is to prove that an analogue of a result of Howe [16] on the rank of local components of automorphic representations of Mp_{2n} also holds in the context of the rank defined by the second author in [31].

We now proceed to the statement of our main theorem. Let $\mathbf{P} := \{2, 3, 5, \dots, \infty\}$ denote the set of places of \mathbb{Q} . For every place $\nu \in \mathbf{P}$, we denote the corresponding completion of \mathbb{Q} by \mathbb{Q}_ν (note that $\mathbb{Q}_\infty = \mathbb{R}$). Let \mathbf{G} be an algebraic group that is defined over \mathbb{Q} . Throughout this paper, we assume that \mathbf{G} satisfies the following conditions:

- (i) \mathbf{G} is connected and absolutely almost simple.
- (ii) The absolute root system of \mathbf{G} is not A_1 , and its highest root is defined over \mathbb{Q} .
- (iii) The weak approximation property holds for \mathbf{G} with respect to every place $\nu \in \mathcal{P}$. In other words, $\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{Q}_\nu)$ for every place $\nu \in \mathcal{P}$.

Let $\mathbb{A} := \prod'_{\nu \in \mathcal{P}} \mathbb{Q}_\nu$ denote the ring of adèles (where \prod' indicates the restricted product). We consider a finite topological central covering

$$(1) \quad 1 \rightarrow F \xrightarrow{i} G_{\mathbb{A}} \xrightarrow{p} \mathbf{G}(\mathbb{A}) \rightarrow 1,$$

which is split over $\mathbf{G}(\mathbb{Q})$. We fix a splitting section for $\mathbf{G}(\mathbb{Q})$, and we denote the image of $\mathbf{G}(\mathbb{Q})$ under this section by $G_{\mathbb{Q}} \subseteq G_{\mathbb{A}}$. Now let (π, \mathcal{H}) be an irreducible unitary representation of $G_{\mathbb{A}}$. For every place $\nu \in \{2, 3, 5, \dots, \infty\}$ of \mathbb{Q} , we denote the local component of π at place ν by π_ν (for a precise definition of local components for covering groups, see Remark 4.4). Then π_ν is an irreducible unitary representation of a finite topological central extension G_ν of $\mathbf{G}(\mathbb{Q}_\nu)$. Let $\text{rk}(\pi_\nu)$ denote the rank of π_ν according to [31] (see Definition 3.4 below). Our main theorem is the following (see Theorem 9.5 for a restatement and proof).

Theorem. *Let $G_{\mathbb{A}}$ be a finite topological central extension of $\mathbf{G}(\mathbb{A})$, with \mathbf{G} as above. Assume that $G_{\mathbb{A}}$ splits over $\mathbf{G}(\mathbb{Q})$, and let $G_{\mathbb{Q}} \subseteq G_{\mathbb{A}}$ be the subgroup defined above. Let (π, \mathcal{H}) be an irreducible unitary representation of $G_{\mathbb{A}}$ which occurs as a subrepresentation of $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$. Then $\text{rk}(\pi_\nu)$ is independent of $\nu \in \mathcal{P}$.*

For further information on basic properties of the central extension (1), see for example [26, Sec. I.1.1]. Condition (iii) on \mathbf{G} is known to hold for many general classes of groups, e.g., simply connected or split groups.

We briefly comment on the relation between our paper and the recent article of Kobayashi and Savin [19]. In [19, Thm 1.1], Kobayashi and Savin prove a similar statement for groups that possess a maximal parabolic subgroup which has an abelian unipotent radical and which is conjugate to its opposite. Using the methods and results of [31, Sec. 6] on the relation between the N -rank and the rank introduced by the second author, one should be able to deduce the result of Kobayashi and Savin from our theorem for unitary automorphic representations. In addition, our method of proof is different from the one used by Kobayashi and Savin, because our techniques rely heavily on Kirillov's orbit method for unipotent groups.

We will now explain the key ideas of the proof of our main theorem. The proof is inspired by the original method of Howe, which is to detect the rank of a unitary representation by means of operators that come from elements of the convolution algebra of Schwartz (or locally constant compactly supported) functions on a unipotent group $U_\nu \subseteq G_\nu$. As shown by Howe, this is relatively easy if U_ν is abelian, as one only needs to choose a function whose Fourier transform has a suitable support (see the proof of [16, Lem. 2.4] and [21, Lem. 3.2]). However, in our case U_ν is non-abelian, and finding an element of the Schwartz algebra of U_ν which separates a point outside a closed subset of the unitary dual of U_ν requires a nontrivial argument (see Section 7 below). Of course this can be done by an element of the C^* -algebra of U_ν , but there is no reason to expect that such an element should come from the Schwartz algebra. It is proved in [22] that the primitive ideal of every irreducible unitary representation of a nilpotent Lie group has a dense intersection with the algebra of Schwartz functions, but this is not enough for our argument, and we need a generalization of this statement for annihilators of many closed subsets of the unitary dual of U_ν . We do not know if such

a statement is true. In the archimedean case, we get around this technical issue by using classical results of Dixmier [6] and Hulanicki (see [23] for a detailed discussion) about functional calculus of nilpotent Lie groups [6]. In the non-archimedean case, we address the analogous technical issue in Sections 5–7 using an interesting result of S. Gelfand and Kazhdan [13] about closed subsets of the unitary dual of a p -adic unipotent group.

Another point of diversion of our proof from Howe’s method is that, if $U_{\mathbb{A}}$ is non-abelian then it is not a Type I group [27, Sec. 7], and therefore uniqueness of direct integral decomposition does not hold for its unitary representations. We circumvent this issue by reducing the problem to the decomposition of $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$. By a result of Moore [27, Thm 11], the latter representation decomposes as a multiplicity-free direct sum of irreducible unitary representations that can be constructed using Kirillov’s orbit method (see Section 8). Of course we cannot use restriction to $U_{\mathbb{A}}$ as a $U_{\mathbb{A}}$ -intertwining map from $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ to $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$, but as shown in the proof of Theorem 9.5, we can carefully use the restriction of smooth vectors.

Even though our theorem is stated only for groups over the adèle ring of \mathbb{Q} , we believe that it remains true for groups over the adèle ring of an algebraic number field. The reason why we have hesitated to state our theorem in the latter more general context is that the results of Moore in [27] on the decomposition of $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$ are only stated for the adèle ring of \mathbb{Q} . It is quite likely that Moore’s result holds in the general case, and therefore the proof of our theorem applies *mutatis mutandis* to groups over the adèle ring of any algebraic number field.

Finally, we remark that it would be interesting to see if our main theorem can be used to obtain interesting information about the wavefront sets of the local components of a small automorphic representation of an exceptional group \mathbf{G} (see [14] and [15] for results in this direction). We intend to come back to this problem in the near future.

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2. THE RANK PARABOLIC

This section is devoted to the notation and structure theory that we will need to be able to recall the notion of rank and the purity theorem proved in [31]. The definition of rank, which will be reviewed in Section 3 below, is based on the existence of *Heisenberg parabolic subgroups*, which we will define first.

Recall that \mathbf{G} is an algebraic group defined over \mathbb{Q} , which satisfies conditions (i)–(iii) of Section 1. Fix a maximally \mathbb{Q} -split Cartan subgroup $\mathbf{H} \subseteq \mathbf{G}$. Let \mathfrak{h} denote the Lie algebra of \mathbf{H} , and let $\Phi \subseteq \mathfrak{h}^*$ denote the absolute root system of \mathbf{G} corresponding to \mathbf{H} . Choose a positive system Φ^+ in Φ , and let Δ denote the corresponding base of Φ . We denote the highest root of Φ by β_{Φ} . Thus, condition (ii) of Section 1 states that

$$(2) \quad \Phi \neq A_1 \text{ and } \beta_{\Phi} \text{ is defined over } \mathbb{Q}.$$

Let (\cdot, \cdot) denote the canonical symmetric bilinear form induced on \mathfrak{h}^* by the Killing form, and set

$$\Sigma_\Phi := \{\alpha \in \Delta : (\alpha, \beta_\Phi) \neq 0\}.$$

Then $|\Sigma_\Phi| = 2$ if Φ is of type A_n , $n > 1$, and $|\Sigma_\Phi| = 1$ otherwise [34]. Furthermore, the unipotent radical $\mathbf{U}_{\Delta-\Sigma_\Phi}$ of the \mathbb{Q} -parabolic subgroup $\mathbf{P}_{\Delta-\Sigma_\Phi} = \mathbf{L}_{\Delta-\Sigma_\Phi} \ltimes \mathbf{U}_{\Delta-\Sigma_\Phi}$ of \mathbf{G} is a Heisenberg group with center $\mathbf{Z}_{\Delta-\Sigma_\Phi} = \mathbf{U}_{\beta_\Phi}$, where \mathbf{U}_{β_Φ} is the unipotent subgroup of \mathbf{G} corresponding to β_Φ (see [34] or [12] for details). We call $\mathbf{P}_{\Delta-\Sigma_\Phi}$ the *Heisenberg parabolic subgroup* associated to Φ .

Next, starting with $\Phi_1 := \Phi$, we define a chain of irreducible root systems

$$\Phi = \Phi_1 \supseteq \cdots \supseteq \Phi_r \supseteq \Phi_{r+1} = \emptyset,$$

by the following inductive construction. Assume that Φ_n has been defined for some $n \geq 1$, and set $\Phi'_n := \{\alpha \in \Phi_n : (\alpha, \beta_{\Phi_n}) = 0\}$. Observe that Φ'_n is the root system of the Levi factor of the Heisenberg parabolic subgroup associated to Φ_n . By examination of all Dynkin diagrams, it follows that Φ'_n is a Cartesian product of at most three irreducible subsystems. Furthermore, at most one of these subsystems satisfies the assumption (2). We set Φ_{n+1} equal to the subsystem that satisfies (2) if it exists (in this case it will always be unique), and set $\Phi_{n+1} := \emptyset$ otherwise. If $\Phi_{n+1} \neq \emptyset$, then we set $\Phi_{n+1}^+ = \Phi_{n+1} \cap \Phi^+$. The inductive construction stops as soon as $\Phi_{n+1} = \emptyset$.

Set $\Gamma := \bigcup_{i=1}^r \Sigma_{\Phi_i}$. From now on let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ denote the standard \mathbb{Q} -parabolic subgroup of \mathbf{G} corresponding to $\Delta - \Gamma$. Note that

$$(3) \quad \mathbf{U} \simeq \mathbf{U}_r \ltimes (\mathbf{U}_{r-1} \ltimes (\cdots \ltimes \mathbf{U}_1) \cdots),$$

where \mathbf{U}_d for $1 \leq d \leq r$ denotes the unipotent radical of the Heisenberg parabolic subgroup associated to Φ_d . In particular, \mathbf{U} is a tower of semidirect products of Heisenberg groups. From now on, we denote the length of the tower of semidirect products (3) by $r(\mathbf{G})$.

Example 2.1. Assume that $\mathbf{G} = \mathrm{SL}_n$. If $n = 2k + 1$, $k \geq 1$, then \mathbf{P} is the Borel subgroup, whereas if $n = 2k$, $k > 1$, then \mathbf{P} is the parabolic subgroup corresponding to the middle node of the Dynkin diagram.

The following table indicates the values of $r(\mathbf{G})$ when the algebraic group \mathbf{G} is \mathbb{Q} -split.

\mathbf{G}	$r(\mathbf{G})$
A_n	$\lfloor \frac{n}{2} \rfloor$
B_n	$\lfloor \frac{n}{2} \rfloor$
C_n	$n - 1$
D_n	$\lfloor \frac{n-1}{2} \rfloor$
E_n	$\lfloor \frac{n}{2} \rfloor$
F_4	3
G_2	1

3. THE ν -RANK OF A UNITARY REPRESENTATION

For every place $\nu \in \mathbf{P}$, we identify $\mathbf{G}(\mathbb{Q}_\nu)$ with a subgroup of $\mathbf{G}(\mathbb{A})$ via the canonical embedding $\mathbb{Q}_\nu \hookrightarrow \mathbb{A}$. The exact sequence (1) restricts to a finite topological central extension

$$(4) \quad 1 \rightarrow F_\nu \xrightarrow{i_\nu} G_\nu \xrightarrow{p_\nu} \mathbf{G}(\mathbb{Q}_\nu) \rightarrow 1,$$

where $G_\nu := \mathfrak{p}^{-1}(\mathbf{G}(\mathbb{Q}_\nu))$ and $F_\nu := i^{-1}(G_\nu)$. As shown for example in [8, Lem. II.11] or [26, Appendix I], the above central extension splits over $\mathbf{U}(\mathbb{Q}_\nu)$, and there is a unique splitting section

$$(5) \quad \mathfrak{s}_\nu : \mathbf{U}(\mathbb{Q}_\nu) \rightarrow G_\nu.$$

Set $U_\nu := \mathfrak{s}_\nu(\mathbf{U}(\mathbb{Q}_\nu))$.

Recall that $r = r(\mathbf{G})$. For $1 \leq n \leq r$, let \mathbf{Z}_n denote the center of \mathbf{U}_n . Set $U_{n,\nu} := \mathfrak{s}_\nu(\mathbf{U}_n(\mathbb{Q}_\nu))$ and $Z_{n,\nu} := \mathfrak{s}_\nu(\mathbf{Z}_n(\mathbb{Q}_\nu))$. From (3) it follows that

$$(6) \quad U_\nu \simeq U_{r,\nu} \times (U_{r-1,\nu} \times (\cdots \times U_{1,\nu}) \cdots).$$

Furthermore, $U_{n,\nu}$ is a Heisenberg group, that is, $W_{n,\nu} := U_{n,\nu}/Z_{n,\nu}$ is a symplectic \mathbb{Q}_ν -vector space, so that the symplectic group $\mathrm{Sp}(W_{n,\nu})$ acts on $U_{n,\nu}$ by automorphisms which fix $Z_{n,\nu}$ pointwise. As shown in [31, Sec. 5], for every $1 \leq n \leq r-1$, the action by conjugation of

$$U_\nu^{(n)} := U_{r,\nu} \times (U_{r-1,\nu} \times (\cdots \times U_{n+1,\nu}) \cdots)$$

on $U_{n,\nu}$ factors through $\mathrm{Sp}(W_{n,\nu})$. Furthermore, again by [8, Lem. II.11], the metaplectic central extension

$$(7) \quad 1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathrm{Mp}(W_{n,\nu}) \rightarrow \mathrm{Sp}(W_{n,\nu}) \rightarrow 1$$

splits over $U_\nu^{(n)}$. By the Stone–von Neumann Theorem, for every nontrivial unitary character $\chi_{n,\nu}$ of $Z_{n,\nu}$ there exists a unique irreducible unitary representation $\rho_{\chi_{n,\nu}}$ of $U_{n,\nu}$ with central character $\chi_{n,\nu}$. Next we extend $\rho_{\chi_{n,\nu}}$ to a unitary representation of U_ν . To this end, first we extend $\rho_{\chi_{n,\nu}}$ to $U_\nu^{(n)} \times U_{n,\nu}$ by means of the restriction of the oscillator representation of $\mathrm{Mp}(W_{n,\nu}) \times U_{n,\nu}$ (this can be done because the exact sequence (7) is split over $U_\nu^{(n)}$). Subsequently, we extend $\rho_{\chi_{n,\nu}}$ to U_ν via the canonical quotient map

$$U_\nu \rightarrow U_\nu / (U_{n-1,\nu} \times (U_{n-2,\nu} \times (\cdots \times U_{1,\nu}) \cdots)) \cong U_\nu^{(n)} \times U_{n,\nu}.$$

As shown in [31, Cor. 4.2.3], for $1 \leq d \leq r$, the tensor product $\rho_{\chi_{1,\nu}} \otimes \cdots \otimes \rho_{\chi_{d,\nu}}$ is an irreducible unitary representation of U_ν .

Definition 3.1. For every $1 \leq d \leq r(\mathbf{G})$, the irreducible unitary representations $\rho_{\chi_{1,\nu}} \otimes \cdots \otimes \rho_{\chi_{d,\nu}}$ of U_ν that are constructed above are called *rankable* of rank d . The set of unitary equivalence classes of rankable representations of U_ν of rank d will be denoted by $\widehat{U}_\nu(d)$.

Let \widehat{U}_ν denote the unitary dual of U_ν . Recall that from the well known results of Kirillov in the archimedean case and [27, Thm 3] in the non-archimedean case, the orbit method gives a bijection between \widehat{U}_ν and the coadjoint orbits of U_ν .

Lemma 3.2. *For every $1 \leq d \leq r(\mathbf{G})$, the coadjoint orbit corresponding to any rankable representation of U_ν of rank d is an analytic manifold of dimension $\dim(U_{1,\nu}) + \cdots + \dim(U_{d,\nu}) - d$.*

Proof. This is basically a restatement of [31, Cor. 4.2.3]. For the definition of an analytic manifold over a local field, see [33, Sec. II.3.2]. Unfortunately, in [31], the details of the proof of the fact that coadjoint orbits are analytic manifolds were not given. The latter statement is a consequence of the following general observation. Let \mathbf{U} be a unipotent \mathbb{Q}_ν -algebraic group, and let $\mathbf{U} \times \mathbf{V} \rightarrow \mathbf{V}$ be a \mathbb{Q}_ν -action of \mathbf{U} on an affine space \mathbf{V} . Let V_ν and U_ν denote the sets of \mathbb{Q}_ν -points of \mathbf{U} and \mathbf{V} . Let $x \in V_\nu$, and let $\overline{\mathcal{O}}_x \subseteq \mathbf{V}$ denote the \mathbf{U} -orbit of x . Set $\mathcal{O}_x := \{u \cdot x : u \in U_\nu\}$. From [27, Lem. 7.1] it follows that $\mathcal{O}_x = \overline{\mathcal{O}}_x \cap V_\nu$, and from [1, Prop. 4.10] it follows that $\overline{\mathcal{O}}_x$ is a Zariski-closed subset

of \mathbf{V} . Consequently, \mathcal{O}_x is a Zariski-closed subset of V_ν , hence also a closed subset of the analytic \mathbb{Q}_ν -manifold V_ν . Finally, from [33, II.4.5, Thm 3] and [33, Sec. II.4.5, Thm 4] it follows that \mathcal{O}_x is also an analytic \mathbb{Q}_ν -submanifold of V_ν . \square

Now let π_ν be a unitary representation of G_ν . We can express the restriction of π_ν to U_ν in an essentially unique way as a direct integral

$$(8) \quad \pi_\nu|_{U_\nu} = \int_{\widehat{U}_\nu}^{\oplus} n_\sigma \sigma d\mu(\sigma) \quad \text{where } n_\sigma \in \mathbb{N} \cup \{\infty\}.$$

As a consequence of [31, Thm 5.3.1], we have the following theorem (see Remark 3.5 below).

Theorem 3.3. *Let π_ν be a nontrivial irreducible unitary representation of G_ν . Then there exists a unique integer $1 \leq d = d(\pi_\nu) \leq r(\mathbf{G})$ such that $\mu(\widehat{U}_\nu - \widehat{U}_\nu(d)) = 0$, where μ is the Borel measure in the decomposition (8).*

Definition 3.4. The integer $d(\pi_\nu)$ that is associated to the unitary representation π_ν in Theorem 3.3 is called the ν -rank of π_ν .

Remark 3.5. The proof of Theorem 3.3 needs some clarification. The tower of semidirect products of Heisenberg groups that appears in [31] is defined by successive consideration of highest roots which are defined over the local field \mathbb{Q}_ν . Obviously, a highest root that is defined over \mathbb{Q}_ν is not necessarily defined over \mathbb{Q} . Therefore the tower of semidirect products in [31] might be longer than the one introduced in (6). More precisely, it will be a unipotent group of the form

$$U^b := U_{r',\nu} \times (\dots \times U_{r,\nu} \times (U_{r-1,\nu} \times (\dots \times U_{1,\nu}) \dots)),$$

where $r' \geq r$. However, Theorem 3.3 follows from [31, Thm 5.3.1] and the fact that a rankable representation ρ of U^b (in the sense defined in [31]) of rank d restricts to a rankable representation of U_ν of rank $\min\{d, r\}$.

Remark 3.6. In [31, Sec. 1], it is assumed that the residual characteristic of the local field is odd. This assumption is superfluous and indeed it is never used in the proofs of [31].

Remark 3.7. The relation between the notion of ν -rank defined in Definition 3.4 and the rank in the sense of Howe [17], Li [20], and Scaramuzzi [32] was investigated in detail in [31, Sec. 6].

4. SMOOTH FORMS OF MODERATE GROWTH

For every place $\nu \in \mathbf{P}$, we set $\mathbf{G}^\nu := \prod'_{\eta \in \mathbf{P} - \{\nu\}} \mathbf{G}(\mathbb{Q}_\eta)$, and we identify \mathbf{G}^ν with a subgroup of $\mathbf{G}(\mathbb{A})$ in a natural way. Furthermore, we set

$$(9) \quad G^\nu := \mathfrak{p}^{-1}(\mathbf{G}^\nu).$$

According to [26, Appendix I], the central extension (1) splits over $\mathbf{U}(\mathbb{A})$, and the splitting section $\mathfrak{s} : \mathbf{U}(\mathbb{A}) \rightarrow G_\mathbb{A}$ is unique. Set $P_\nu := \mathfrak{p}^{-1}(\mathbf{P}(\mathbb{Q}_\nu))$, where we identify $\mathbf{P}(\mathbb{Q}_\nu)$ with a subgroup of $\mathbf{G}(\mathbb{Q}_\nu) \subseteq \mathbf{G}(\mathbb{A})$. Uniqueness of the sections \mathfrak{s} and \mathfrak{s}_ν , defined in (5), implies $\mathfrak{s}|_{\mathbf{U}(\mathbb{Q}_\nu)} = \mathfrak{s}_\nu$, hence

$$(10) \quad U_\nu = \mathfrak{s}(\mathbf{U}(\mathbb{Q}_\nu)) \quad \text{for every } \nu \in \mathbf{P},$$

and U_ν is normalized by P_ν . Set $\mathbb{A}_{\text{fin}} := \prod'_{\nu \in \mathbf{P} - \{\infty\}} \mathbb{Q}_\nu$, so that $\mathbb{A} \cong \mathbb{R} \times \mathbb{A}_{\text{fin}}$. Furthermore, set

$$U_\mathbb{A} := \mathfrak{s}(\mathbf{U}(\mathbb{A})), \quad U_\mathbb{Q} := G_\mathbb{Q} \cap U_\mathbb{A}, \quad \text{and } G_{\mathbb{A}_{\text{fin}}} := \mathfrak{p}^{-1}(\mathbf{G}(\mathbb{A}_{\text{fin}})).$$

From now on, we fix a norm $\|\cdot\|$ on $G_{\mathbb{R}}$ as follows. We choose a representation $\iota : G_{\mathbb{R}} \rightarrow \mathrm{SL}_n(\mathbb{R})$ for some $n > 1$ which descends to a faithful representation of $\mathbf{G}(\mathbb{R})$ whose image is closed in $\mathrm{Mat}_{n \times n}(\mathbb{R})$, and define

$$\|g\| := \left(\sum_{1 \leq i, j \leq n} |x_{i,j}|^2 \right)^{\frac{1}{2}} \quad \text{where } \iota(g) = [x_{i,j}]_{1 \leq i, j \leq n}.$$

The canonical injection $U_{\mathbb{Q}} \backslash U_{\mathbb{A}} \hookrightarrow G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ is a homeomorphism of $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ onto a closed subset of $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$. The homogeneous spaces $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ and $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ have finite invariant measures (in fact $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ is compact). As usual, $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ denotes the space of complex valued square integrable functions on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$. The representation of $G_{\mathbb{A}}$ on $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ by right translation will be denoted by $\mathbf{R}(\cdot)$, that is,

$$(\mathbf{R}(g)f)(x) := f(xg) \text{ for } f \in L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}), x \in G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \text{ and } g \in G_{\mathbb{A}}.$$

An irreducible unitary representation of $G_{\mathbb{A}}$ is called an *automorphic representation* if it occurs as a subrepresentation of $(\mathbf{R}, L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}))$.

Remark 4.1. From now on, we identify functions $f : G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$ with left $G_{\mathbb{Q}}$ -invariant functions $f : G_{\mathbb{A}} \rightarrow \mathbb{C}$ in the obvious way. Any two left $G_{\mathbb{Q}}$ -invariant measurable maps $G_{\mathbb{A}} \rightarrow \mathbb{C}$ that descend to maps $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$ which are almost everywhere equal to f , are also equal everywhere except on a subset of $G_{\mathbb{A}}$ of Haar measure zero (this follows from [24, Lem. 1.3]).

Definition 4.2. A continuous function $f : G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$ is called *smooth* if it satisfies the following two conditions.

- (i) There exists a compact open subgroup $K \subseteq G_{\mathbb{A}_{\mathrm{fin}}}$ such that $f(gk) = f(g)$ for $g \in G_{\mathbb{A}}$ and $k \in K$.
- (ii) For every $x \in G_{\mathbb{A}}$, the map $G_{\mathbb{R}} \rightarrow \mathbb{C}$, $y \mapsto f(xy)$ is in $C^\infty(G_{\mathbb{R}})$.

A continuous function $f : G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$ is said to be of *moderate growth* if it is smooth and satisfies

$$(11) \quad |f(xy)| \leq c_{x,f} \|y\|^{m_f} \text{ for every } x \in G_{\mathbb{A}} \text{ and every } y \in G_{\mathbb{R}},$$

where $m_f \in \mathbb{R}^+$ depends only on f , and $c_{x,f} \in \mathbb{R}^+$ depends only on x and f .

Let $C_c^\infty(G_{\mathbb{A}})$ denote the space of smooth compactly supported functions on $G_{\mathbb{A}}$ [26, Lem. I.2.5].

Remark 4.3. Let (π, \mathcal{H}) be a unitary representation of $G_{\mathbb{A}}$. Fix a Haar measure dg on $G_{\mathbb{A}}$ and set

$$\mathcal{H}^\circ := \left\{ \pi(\phi)v : v \in \mathcal{H} \text{ and } \phi \in C_c^\infty(G_{\mathbb{A}}) \right\}, \text{ where } \pi(\phi)v := \int_{G_{\mathbb{A}}} \phi(g)\pi(g)v dg.$$

Note that \mathcal{H}° is a $G_{\mathbb{A}}$ -invariant dense subspace of \mathcal{H} . We call \mathcal{H}° the *Gårding space* of (π, \mathcal{H}) . If (π, \mathcal{H}) is a subrepresentation of $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$, then from [26, Lem. I.2.5] it follows that every element of \mathcal{H}° can be represented by a unique smooth and moderate growth map $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$.

Remark 4.4. Consider an irreducible unitary representation (π, \mathcal{H}) of $G_{\mathbb{A}}$. If $G_{\mathbb{A}} = \mathbf{G}(\mathbb{A})$, then as is well known, we can express π as a restricted tensor product $\otimes'_{\pi \in \mathbf{P}} \pi_\nu$, where each π_ν is an irreducible unitary representation of $G_\nu = \mathbf{G}(\mathbb{Q}_\nu)$. The π_ν are called the *local components* of π . If $G_{\mathbb{A}} \neq \mathbf{G}(\mathbb{A})$, then $G_{\mathbb{A}}$ is not a restricted product of the local factors G_ν , and therefore the above definition of local components is not totally valid. There are various ways to fix this issue by generalizing the notion of local components (and possibly the restricted tensor product) to representations of $G_{\mathbb{A}}$.

The easiest way, which is sufficient for our goals, is as follows. For every $\nu_\circ \in \mathbf{P}$, the group $G_{\mathbb{A}}$ is an almost direct product of the groups G_{ν_\circ} and G^{ν_\circ} which are defined in Section 3 and (9). Thus we can consider π as a representation of $G_{\nu_\circ} \times G^{\nu_\circ}$. Since the group G_{ν_\circ} is Type I, by [25, Thm 1.8] we can decompose π into a tensor product $\pi_{\nu_\circ} \otimes \pi'_{\nu_\circ}$ of irreducible unitary representations of respective factors. We call π_{ν_\circ} the *local component* of π at ν_\circ .

With slightly more work (see [26, Sec. I.1.2]) one can show that indeed $G_{\mathbb{A}}$ is isomorphic to a quotient of a restricted product $\prod'_{\nu \in \mathbf{P}} G_\nu$. We can then inflate a representation of $G_{\mathbb{A}}$ to one of $\prod'_{\nu \in \mathbf{P}} G_\nu$, and use the restricted tensor product decomposition with respect to the latter group.

5. FUNCTIONAL CALCULUS ON NILPOTENT LIE GROUPS

Throughout this section N will be a simply connected nilpotent Lie group. Fix a Haar measure dn on N . For any $p \geq 1$, we denote the Banach space of complex-valued p -integrable functions on N by $L^p(N)$. For every $f_1, f_2 \in L^1(N)$, set $f_1 * f_2(a) := \int_N f_1(n) f_2(n^{-1}a) dn$. The conjugate-linear involution $f \mapsto f^\dagger$ of $L^1(N)$ is defined by $f^\dagger(n) := \overline{f(n^{-1})}$ for every $n \in N$. For every $f \in L^1(N)$, set

$$f^{*n} := \underbrace{f * \cdots * f}_{n \text{ times}} \quad \text{and} \quad e^{*f} := \sum_{n=0}^{\infty} \frac{f^{*n}}{n!} \in L^1(N).$$

Since N is simply connected, the exponential map is a diffeomorphism from $\mathfrak{n} := \text{Lie}(N)$ onto N .

Definition 5.1. The *Schwartz algebra* of N , denoted by $\mathcal{S}(N)$, is the space of functions $f : N \rightarrow \mathbb{C}$ such that $f \circ \exp$ is a Schwartz function on \mathfrak{n} in the sense of [35, Sec. 25].

It is well known (for example, see [11, Sec. 6.2]) that $\mathcal{S}(N)$ is a subalgebra of the convolution algebra $L^1(N)$.

Given any $\phi \in C_c^\infty(\mathbb{R})$, we set $\widehat{\phi}(t) := \int_{-\infty}^{\infty} e^{-ist} \phi(s) ds$ for every $t \in \mathbb{R}$. Furthermore, for every bounded self-adjoint operator A on a Hilbert space, we define $\phi(A)$ by functional calculus, as in [29, Chap. VII].

Proposition 5.2. *Let $f \in C_c(N)$, and let $\phi \in C_c^\infty(\mathbb{R})$. Assume that $f = f^\dagger$ and $\phi(0) = 0$. Then the following statements hold.*

- (i) *The integral $\phi\{f\} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\phi}(t) e^{*itf} dt$ converges absolutely in $L^1(N)$, and $\phi\{f\} \in \mathcal{S}(N)$.*
- (ii) *For every unitary representation (π, \mathcal{H}) of N , we have $\pi(\phi\{f\}) = \phi(\pi(f))$.*

Proof. Absolute convergence of the integral in (i) follows from [6, Lem. 7]. The fact that $f \in \mathcal{S}(N)$ follows from [23, Thm 2.6]. (It was pointed out to us by Professor Jean Ludwig that the latter statement was first proved by A. Hulanicki.) Part (ii) follows from [6, Lem. 7]. \square

6. THE UNITARY DUAL OF A UNIPOTENT p -ADIC GROUP

Let \mathbf{N} be a unipotent algebraic group defined over \mathbb{Q} , and let N be the group of \mathbb{Q}_ν -points of \mathbf{N} for some place $\nu \in \mathbf{P} - \{\infty\}$. As before, we denote the unitary dual of N by \widehat{N} . Members of \widehat{N} are equivalence classes of irreducible unitary representations of N . We now recall the definition of the Fell topology of \widehat{N} . For any $(\pi, \mathcal{H}) \in \widehat{N}$, $k \in \mathbb{N}$, $v_1, \dots, v_k, w_1, \dots, w_k \in \mathcal{H}$, $\varepsilon > 0$, and $\Omega \subseteq N$ compact, we define

$$(12) \quad \mathcal{U}(\pi, \Omega, \varepsilon; v_1, \dots, v_k; w_1, \dots, w_k)$$

to be the set of all $(\sigma, \mathcal{K}) \in \widehat{N}$ for which there exist $v'_1, \dots, v'_k, w'_1, \dots, w'_k \in \mathcal{K}$ such that

$$(13) \quad |\langle \pi(g)v_i, w_j \rangle - \langle \sigma(g)v'_i, w'_j \rangle| < \varepsilon \text{ for every } g \in \Omega \text{ and every } 1 \leq i, j \leq k.$$

The sets defined in (12) constitute a base for the Fell topology.

In [13], Gelfand and Kazhdan define a smooth version of the Fell topology on \widehat{N} . This topology, which will refer to by the *Gelfand–Kazhdan topology*, is also the one that is used in [2]. The main goal of this section is to prove that the Fell topology and the Gelfand–Kazhdan topology of \widehat{N} are indeed the same.

In order to define the Gelfand–Kazhdan topology, we need the notion of the *smooth dual* of N . Let (σ, V) be a representation of N on a complex vector space V . Recall that a vector $v \in V$ is called *smooth* if its stabilizer contains a compact open subgroup of N . Here and thereafter V^K denotes the subspace of K -fixed vectors of in V . The representation (σ, V) is called *smooth* if every $v \in V$ is smooth. We say (σ, V) is *admissible* if $\dim(V^K) < \infty$ for every compact open subgroup $K \subseteq N$. A smooth representation (σ, V) is called *pre-unitary* if it is equipped with an N -invariant positive definite Hermitian form. The set of equivalence classes of (algebraically) irreducible smooth representations of N is called the *smooth dual* of N .

Let (σ, V) be an irreducible smooth representation of N . From [36] it follows that (σ, V) is admissible and pre-unitary. Let $(\hat{\sigma}, \hat{V})$ denote the unitary representation of N corresponding to (σ, V) , so that \hat{V} is the Hilbert space completion of V .

Proposition 6.1. *The assignment $(\sigma, V) \mapsto (\hat{\sigma}, \hat{V})$ results in a bijective correspondence between the smooth dual and the unitary dual of N .*

Proof. Step 1. Let (σ, V) be an irreducible smooth representation of N , and let \hat{V}^∞ denote the subspace of smooth vectors of the unitary representation $(\hat{\sigma}, \hat{V})$. First we prove that $\hat{V}^\infty = V$. Clearly $V \subseteq \hat{V}^\infty$. To prove the reverse inclusion, let $v \in \hat{V}^\infty$ and choose a compact open subgroup $K \subseteq N$ such that K lies in the stabilizer of v . We can write $V = V^K \oplus V(K)$, where $V(K)$ is the kernel of the projection

$$P_K : V \rightarrow V, \quad P_K w := \int_K \sigma(n)w dn.$$

Here dn denotes the Haar measure of N which satisfies $\int_K dn = 1$. Let $\langle \cdot, \cdot \rangle$ denote the inner product of \hat{V} . If $v \notin V^K$ then after replacing v by $v - v_0$, where v_0 is the orthogonal projection of v on V^K , we can assume that $\langle v, V^K \rangle = 0$. By invariance of the inner product of \hat{V} , we obtain $\langle v, V(K) \rangle = 0$. It follows that $\langle v, V \rangle = 0$, which is a contradiction since V is dense in \hat{V} .

Step 2. We show that $(\sigma, V) \mapsto (\hat{\sigma}, \hat{V})$ is well-defined, that is, if (σ, V) is an irreducible smooth representation, then $(\hat{\sigma}, \hat{V})$ is an irreducible unitary representation. Suppose, on the contrary, that $\hat{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ where \mathcal{V}_1 and \mathcal{V}_2 are non-zero closed N -invariant subspaces of the Hilbert space \hat{V} . Then $V = \hat{V}^\infty = \mathcal{V}_1^\infty \oplus \mathcal{V}_2^\infty$. Since the subspace of smooth vectors of a unitary representation is dense, both \mathcal{V}_1^∞ and \mathcal{V}_2^∞ are non-zero. It follows that (σ, V) is reducible, which is a contradiction.

Step 3. We show that the assignment $(\sigma, V) \mapsto (\hat{\sigma}, \hat{V})$ is surjective. Fix an irreducible unitary representation (σ, \mathcal{V}) of N . It is now enough to show that the smooth representation of N on $V := \mathcal{V}^\infty$ (the space of smooth vectors of \mathcal{V}) is (algebraically) irreducible. Assume that $W \subsetneq V$ is an N -invariant subspace. As in Step 1, for every compact open subgroup $K \subseteq N$ we can write $V = V^K \oplus V(K)$. Note that $W = W^K \oplus (W \cap V(K))$. Since $W \neq V$, we can choose K such that $W^K \subsetneq V^K$. By the remark made at the end of [36, Sec. 5], V is admissible. Thus $\dim(V^K) < \infty$,

and therefore we can choose $w \in V^K$ such that $\langle w, W^K \rangle = 0$. As in Step 1, we have $\langle w, V(K) \rangle = 0$, so that $\langle w, W \rangle = 0$. It follows that the closure of W in \mathcal{V} is an N -invariant subspace which does not contain w . From irreducibility of (σ, \mathcal{V}) it follows that $W = \{0\}$. This completes the proof of irreducibility of V .

Step 4. We show that the assignment $(\sigma, V) \mapsto (\hat{\sigma}, \hat{V})$ is an injection. To this end, we need to show that if (σ_1, V_1) and (σ_2, V_2) are algebraically equivalent smooth representations, then the unitary representations $(\hat{\sigma}_1, \hat{V}_1)$ and $(\hat{\sigma}_2, \hat{V}_2)$ are unitarily equivalent. Let $T : V_1 \rightarrow V_2$ be a linear map such that $T\sigma_1(n) = \sigma_2(n)T$ for every $n \in N$. Then the inner product $\langle v, w \rangle' := \langle Tv, Tw \rangle$ on V_1 is also N -invariant. By [4, Prop. 2.1.15], up to scalar there exists at most one invariant inner product on V_1 . It follows that up to a scalar, T is an isometry. Therefore T extends to an intertwining operator $\hat{V}_1 \rightarrow \hat{V}_2$ by continuity. \square

We are now ready to give the definition of the Gelfand–Kazhdan topology of \hat{N} . Because of Proposition 6.1, it is enough to define this topology on the smooth dual of N . The Gelfand–Kazhdan topology is generated by the base of open sets

$$\mathcal{U} := \mathcal{U}(\pi, \Omega, \varepsilon; v_1, \dots, v_k; \lambda_1, \dots, \lambda_k),$$

where (π, V) is an irreducible smooth representation of N , $k \in \mathbb{N}$, $\varepsilon > 0$, $\Omega \subseteq N$ is compact, $v_1, \dots, v_k \in V$, and $\lambda_1, \dots, \lambda_k \in V^*$ (the algebraic dual of the vector space V). The definition of \mathcal{U} is similar to the definition of the base sets of the Fell topology, with relation (13) replaced by

$$(14) \quad |\lambda_i(\pi(g)v_j) - \lambda'_i(\pi(g)v'_j)| < \varepsilon \text{ for every } g \in \Omega \text{ and every } 1 \leq i, j \leq k.$$

Remark 6.2. Let (σ, V) be an irreducible smooth representation of N . Let $v_1, \dots, v_k \in V$, and let $\lambda \in V^*$. Fix a compact subset $\Omega \subseteq N$. Since N is a union of compact open subgroups, there exists a compact open subgroup $L \subseteq N$ such that $\Omega \subseteq L$. It follows that the vector space

$$W := \text{Span}_{\mathbb{C}} \{ \pi(g)v_i : g \in \Omega \text{ and } 1 \leq i \leq k \}$$

is finite dimensional. Recall that (π, V) is pre-unitary [36]. If $\langle \cdot, \cdot \rangle$ denotes the inner product of V , then it follows immediately that there exists a vector $v \in V$ such that $\lambda(\pi(g)v_i) = \langle \pi(g)v_i, v \rangle$ for every $g \in \Omega$ and every $1 \leq i \leq k$.

Proposition 6.3. *The Fell topology and the Gelfand–Kazhdan topology on \hat{N} are identical.*

Proof. The proof is straightforward and left to the reader. It follows from Remark 6.2 and the fact that in any unitary representation, an arbitrary matrix coefficient can be approximated uniformly by matrix coefficients of smooth vectors. \square

Recall that U_ν denotes the group defined in (6).

Corollary 6.4. *For every $\nu \in \mathbb{P}$, the unitary dual \hat{U}_ν , equipped with the Fell topology, is homeomorphic to the quotient space $\mathfrak{u}_\nu^*/\text{Ad}^*(U_\nu)$.*

Proof. For $\nu = \infty$, this is proved in [3]. For $\nu \in \mathbb{P} - \{\infty\}$, it is shown in [2, Thm 3.1] that $\mathfrak{u}_\nu^*/\text{Ad}^*(U_\nu)$ is homeomorphic to \hat{U}_ν equipped with the Gelfand–Kazhdan topology. The corollary now follows from Proposition 6.3. \square

7. EXISTENCE OF SEPARATING SCHWARTZ FUNCTIONS

For every $\nu \in \mathbf{P}$, recall that U_ν is isomorphic to the group of \mathbb{Q}_ν -points of the unipotent algebraic group \mathbf{U} defined in Section 2. From [5, Chap. IV] it follows that U_∞ is a simply connected nilpotent Lie group. As in Definition 5.1, we denote the algebra of Schwartz functions on U_∞ by $\mathcal{S}(U_\infty)$. For $\nu \in \mathbf{P} - \{\infty\}$, we set $\mathcal{S}(U_\nu) := C_c^\infty(U_\nu)$, where $C_c^\infty(U_\nu)$ is the convolution algebra of compactly supported locally constant complex-valued functions on U_ν .

Given $\nu \in \mathbf{P}$ and a unitary representation π of U_ν , we set

$$(15) \quad \pi(\psi) := \int_{U_\nu} \psi(n)\pi(n)dn \text{ for every } \psi \in \mathcal{S}(U_\nu),$$

where dn is a Haar measure on U_ν .

Proposition 7.1. *Fix $\nu \in \mathbf{P}$. Let $S \subseteq \widehat{U}_\nu$ be a closed subset with respect to the Fell topology, and let $\pi \in \widehat{U}_\nu - S$. Then there exists an element $\psi \in \mathcal{S}(U_\nu)$ such that $\pi(\psi) \neq 0$, but $\sigma(\psi) = 0$ for every $\sigma \in S$.*

Proof. We consider two separate cases.

Case 1: $\nu = \infty$. Let $C^*(U_\infty)$ denote the C^* -algebra of U_∞ . Since $C^*(U_\infty)$ is CCR (see [27, Thm 11] and the references therein), the Fell topology on \widehat{U}_∞ and the hull-kernel topology on the dual of $C^*(U_\infty)$ are identical [10, Sec. 7.2]. It follows that there exists an element $a \in C^*(U_\infty)$ such that $\|\pi(a)\| = 1$, whereas $\sigma(a) = 0$ for every $\sigma \in S$. Substituting a by aa^* if necessary, we can assume that $a = a^*$.

Let $C_c^\infty(U_\infty)$ denote the space of functions $U_\infty \rightarrow \mathbb{C}$ which are smooth and have compact support. Since $C_c^\infty(U_\infty)$ is a dense subspace of $C^*(U_\infty)$, we can choose $f \in C_c^\infty(U_\infty)$ satisfying $f = f^\dagger$ such that $\|\pi(f)\| = 1$, whereas $\|\sigma(f)\| < \frac{1}{4}$ for every $\sigma \in S$. Next we choose $\phi \in C_c^\infty(\mathbb{R})$ such that

$$\text{supp}(\phi) \subseteq [-2, 2], \quad \phi|_{[-\frac{1}{4}, \frac{1}{4}]} = 0, \quad \text{and} \quad \phi|_{[-\frac{5}{4}, -\frac{3}{4}]} = \phi|_{[\frac{3}{4}, \frac{5}{4}]} = 1.$$

Now set $\psi := \phi\{f\}$. By Proposition 5.2(i), we have $\psi \in \mathcal{S}(U_\infty)$. Set $A := \pi(f)$ and note that A is self-adjoint because $f = f^\dagger$. It follows that the spectral radius of A is equal to $\|A\| = 1$. Since $\phi(\pm 1) = 1$, we obtain $\|\phi(A)\| = \sup\{|\phi(x)| : x \in \text{Spec}(A)\} > 0$. Consequently, Proposition 5.2(ii) implies that $\pi(\psi) = \phi(A) \neq 0$. Similarly, for every $\sigma \in S$, the spectral radius of $\sigma(f)$ is equal to $\|\sigma(f)\| < \frac{1}{4}$. It follows that ϕ vanishes on the spectrum of $\sigma(f)$, and therefore $\sigma(\psi) = \phi(\sigma(f)) = 0$.

Case 2: $\nu \in \mathbf{P} - \{\infty\}$. By Proposition 6.1 and Proposition 6.3, we can assume that (π, V) is an irreducible smooth representation of N . The strategy of the proof is to use the results of [13]. Choose a compact open subgroup $K_0 \subseteq U_\nu$ such that $V^{K_0} \neq \{0\}$, and fix a sequence

$$K_0 \subseteq K_1 \subseteq \dots \subseteq K_n \subseteq \dots$$

of compact open subgroups of U_ν such that $U_\nu = \bigcup_{n=0}^\infty K_n$. Let \mathcal{H}_{K_0} denote the convolution algebra of K_0 -bi-invariant compactly supported complex-valued functions on U_ν , and let $\mathcal{H}_{K_0}^{K_n}$ denote the subalgebra of \mathcal{H}_{K_0} which consists of functions whose support lies in K_n . By [13, Prop. 4], the algebra $\mathcal{H}_{K_0}^{K_n}$ is isomorphic to the commutant of the image of the group algebra of K_n in the induced representation $\text{Ind}_{K_0}^{K_n} 1$, and therefore it is a finite dimensional semisimple associative algebra. Since $\mathcal{H}_{K_0} \subseteq \mathcal{S}(U_\nu)$, for every smooth representation σ of U_ν and every $\phi \in \mathcal{H}_{K_0}$ we define $\sigma(\phi)$ as in (15).

According to [27, Thm 4], the group U_ν is CCR, and therefore every point in \widehat{U}_ν is closed [10, Sec. 7.2]. Now let $\widehat{U}_\nu^{K_0}$ denote the subset of \widehat{U}_ν consisting of irreducible representations (σ, W) such that

$W^{K_0} \neq \{0\}$. We equip $\widehat{U}_\nu^{K_0}$ with the topology induced by the Fell topology of \widehat{U}_ν . Then $S \cap \widehat{U}_\nu^{K_0}$ and $\{\pi\}$ are closed subsets of $\widehat{U}_\nu^{K_0}$, and therefore by [13, Thm 6] and [13, Prop. 18] there exists an $n \in \mathbb{N}$ such that for every $(\sigma, W) \in S \cap \widehat{U}_\nu^{K_0}$, the $\mathcal{H}_{K_0}^{K_n}$ -modules W^{K_0} and V^{K_0} are disjoint. By Artin–Wedderburn theory,

$$(16) \quad \mathcal{H}_{K_0}^{K_n} \cong M_{d_1 \times d_1}(\mathbb{C}) \times \cdots \times M_{d_m \times d_m}(\mathbb{C})$$

for some integers $d_1, \dots, d_m \geq 1$, where $M_{d \times d}(\mathbb{C})$ is the associative algebra of $d \times d$ matrices with complex entries. The irreducible modules of $\mathcal{H}_{K_0}^{K_n}$ are the standard modules \mathbb{C}^{d_i} , $1 \leq i \leq m$, of the ideals $M_{d_i \times d_i}(\mathbb{C})$. From (16) and disjointness of V^{K_0} and W^{K_0} it follows that there exists an idempotent $\psi \in \mathcal{H}_{K_0}^{K_n}$ such that $\pi(\psi)V^{K_0} \neq \{0\}$, whereas $\sigma(\psi)W^{K_0} = \{0\}$ for every $(\sigma, W) \in S \cap \widehat{U}_\nu^{K_0}$. It follows that $\pi(\psi) \neq 0$, whereas

$$\sigma(\psi)W = \sigma(\psi * \psi)W = \sigma(\psi)^2W \subseteq \sigma(\psi)W^{K_0} = \{0\}.$$

But also when $W^{K_0} = \{0\}$, we have $\sigma(\phi)W \in W^{K_0} = \{0\}$ for every $\phi \in \mathcal{H}_{K_0}$, and in particular $\sigma(\psi) = 0$. \square

8. KIRILLOV THEORY FOR $U_{\mathbb{A}}$

As shown in [27], the group $U_{\mathbb{A}}$ is *not* of Type I because it is non-abelian. However, it is shown in [27, Thm 11] that the decomposition of the representation of $U_{\mathbb{A}}$ by right translation on $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$ can be described by means of Kirillov theory. We now recall Moore’s result from [27, Thm 11]. Let $\mathfrak{u}_{\mathbb{Q}}$ denote the Lie algebra of $U_{\mathbb{Q}}$, and let $\mathfrak{u}_{\mathbb{Q}}^*$ denote the dual of $\mathfrak{u}_{\mathbb{Q}}$. Fix a place $\nu \in \mathbb{P}$. Every $\mu \in \mathfrak{u}_{\mathbb{Q}}^*$ can be extended in a unique way to a linear functional $\mu_\nu \in \mathfrak{u}_\nu^* := \text{Lie}(U_\nu)$. Let ρ_{μ_ν} denote the irreducible unitary representation of U_ν that corresponds to the coadjoint orbit associated to μ_ν . Now let $(\mathbf{R}', L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}}))$ denote the representation of $U_{\mathbb{A}}$ on $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$ by right translation. It is shown in [27, Thm 11] that $(\mathbf{R}', L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}}))$ decomposes as a multiplicity-free direct sum of unitary representations

$$(17) \quad \rho_\mu := \otimes_{\nu \in \mathbb{P}} \rho_{\mu_\nu} \text{ for all } \mu \in \mathfrak{u}_{\mathbb{Q}}^*.$$

In the next section we will need the following lemma.

Lemma 8.1. *Let $\mu \in \mathfrak{u}_{\mathbb{Q}}^*$, and for every place $\nu \in \mathbb{P}$ let $\mathcal{O}_{\mu_\nu} \subseteq \mathfrak{u}_\nu^*$ denote the U_ν -orbit of μ_ν , where $\mu_\nu \in \mathfrak{u}_\nu^*$ is the canonical extension of μ . Then \mathcal{O}_{μ_ν} is an analytic \mathbb{Q}_ν -submanifold of \mathfrak{u}_ν , and $\dim(\mathcal{O}_{\mu_\nu})$ is independent of the place ν .*

Proof. The action of \mathbf{U} on \mathfrak{u}^* is algebraic and defined over \mathbb{Q} . Since $\mu \in \mathfrak{u}_{\mathbb{Q}}^*$, the stabilizer of μ is an algebraic group $\mathbf{S} \subseteq \mathbf{U}$ that is defined over \mathbb{Q} . Fix $\nu \in \mathbb{P}$. From the proof of Lemma 3.2 it follows that \mathcal{O}_{μ_ν} is an analytic \mathbb{Q}_ν -submanifold of \mathfrak{u}_ν^* . Let S_ν be the stabilizer of μ_ν in U_ν . Thus S_ν is the set of \mathbb{Q}_ν -points of \mathbf{S} , and hence it is an analytic \mathbb{Q}_ν -manifold of dimension $\dim(\mathbf{S})$ (see [28, Sec. 3.1]). On the other hand, by [33, Sec. II.4.5] we have $\dim(S_\nu) + \dim(\mathcal{O}_{\mu_\nu}) = \dim(U_\nu) = \dim(\mathbf{U})$. Consequently, $\dim(\mathcal{O}_{\mu_\nu}) = \dim(\mathbf{U}) - \dim(\mathbf{S})$ is independent of $\nu \in \mathbb{P}$. \square

9. RANK FOR GLOBAL REPRESENTATIONS

Recall the definition of $r := r(\mathbf{G})$ from Section 2. For every $\nu \in \mathbb{P}$ and $0 \leq d \leq r(\mathbf{G})$, let $\widehat{U}_\nu[d] \subseteq \widehat{U}_\nu$ denote the set consisting of irreducible unitary representations that correspond to

coadjoint orbits of dimension at most $\dim(U_{1,\nu}) + \cdots + \dim(U_{d,\nu}) - d$. Note that $\widehat{U}_\nu(d) \subseteq \widehat{U}_\nu[d]$, where $\widehat{U}_\nu(d)$ is defined in Definition 3.1.

Lemma 9.1. $\widehat{U}_\nu[d]$ is a closed subset of \widehat{U}_ν for every $0 \leq d \leq r(\mathbf{G})$ and every $\nu \in \mathbf{P}$.

Proof. Let $U_\nu(\lambda)$ denote the stabilizer of $\lambda \in \mathfrak{u}_\nu^*$ in U_ν . By Corollary 6.4, \widehat{U}_ν is homeomorphic to $\mathfrak{u}_\nu^*/\text{Ad}^*(U_\nu)$. Therefore it suffices to prove that for every $n \in \mathbb{N}$, the set

$$T_n := \{\lambda \in \mathfrak{u}_\nu^* : \dim(U_\nu(\lambda)) < n\}$$

is an open subset of \mathfrak{u}_ν^* . For every $\lambda \in \mathfrak{u}_\nu$, let $h_\lambda : U_\nu \rightarrow \mathfrak{u}_\nu^*$ be defined by $h_\lambda(g) := \text{Ad}^*(g)\lambda$. Then $\dim(U_\nu(\lambda)) = \dim \mathfrak{u}_\nu - \text{rank}(dh_\lambda(\mathbf{1}))$, where dh_λ is the differential of h_λ . Since the map $\lambda \mapsto \text{rank}(dh_\lambda(\mathbf{1}))$ is a lower semi-continuous function of λ , the complement of T_n is open. \square

For every $\nu \in \mathbf{P}$, set

$$J_{d,\nu} := \left\{ \phi \in \mathcal{S}(U_\nu) : \sigma(\phi) = 0 \text{ for every } \sigma \in \widehat{U}_\nu[d] \right\}.$$

Lemma 9.2. Fix $\nu \in \mathbf{P}$. Let (σ, \mathcal{H}) be a unitary representation of U_ν , and let

$$\sigma = \int_{\widehat{U}_\nu}^{\oplus} n_\tau \tau d\mu(\tau)$$

be the direct integral decomposition of σ . For $1 \leq d \leq r(\mathbf{G})$, the following statements are equivalent.

- (i) $\text{supp}(\mu) \subseteq \widehat{U}_\nu[d]$.
- (ii) $\sigma(\phi) = 0$ for every $\phi \in J_{d,\nu}$.

Proof. (i) \Rightarrow (ii): From [37, Sec. 14.9.2] and the definition of $J_{\nu,d}$ it follows that

$$\sigma(\phi) = \int_{\widehat{U}_\nu[d]}^{\oplus} n_\tau \tau(\phi) d\mu(\tau) = 0.$$

(ii) \Rightarrow (i): We prove the contrapositive, that is, if (i) is false then (ii) is false. Suppose that the support of μ does not lie inside $\widehat{U}_\nu[d]$. Since $\widehat{U}_\nu[d]$ is a closed subset of \widehat{U}_ν , we obtain $\mu(\widehat{U}_\nu - \widehat{U}_\nu[d]) > 0$. It follows that there exists some $(\tau_\circ, \mathcal{H}_{\tau_\circ}) \in \widehat{U}_\nu - \widehat{U}_\nu[d]$ such that $\mu(\mathcal{U}) > 0$ for every open neighborhood \mathcal{U} of $(\tau_\circ, \mathcal{H}_{\tau_\circ}) \in \widehat{U}_\nu - \widehat{U}_\nu[d]$ (because otherwise, since \widehat{U}_ν is second countable [7, Prop. 3.3.4], we will find a covering of $\widehat{U}_\nu - \widehat{U}_\nu[d]$ by countably many null open sets). By Proposition 7.1, there exists an element $\psi \in J_{d,\nu}$ such that $\tau_\circ(\psi) \neq 0$. Set $\psi^\dagger(n) := \overline{\psi(n^{-1})}$ for $n \in U_\nu$. Without loss of generality we can assume that $\psi = \psi^\dagger$, because otherwise we can replace ψ by either $i(\psi - \psi^\dagger)$ or $\psi + \psi^\dagger$. After scaling ψ by a real number, we can also assume that $\|\tau_\circ(\psi)\| = 1$. Since $\tau_\circ(\psi)$ is self-adjoint, we can choose $v \in \mathcal{H}_{\tau_\circ}$ such that $\|v\| = 1$ and $|\langle \tau_\circ(\psi)v, v \rangle| > \frac{3}{4}$.

Fix $\varepsilon > 0$ such that $\varepsilon(2 + \varepsilon + \|\psi\|_{L^1(U_\nu)}) < \frac{3}{4}$, and choose a compact subset $\Omega \subseteq U_\nu$ such that $\mathbf{1} \in \Omega$ and $\|\psi - \chi_\Omega \psi\|_{L^1(U_\nu)} < \varepsilon$, where $\mathbf{1}$ denotes the neutral element of U_ν and χ_Ω denotes the characteristic function of Ω . Set $\mathcal{U} := \mathcal{U}(\tau_\circ, \Omega, \varepsilon; v)$, defined as in (12). For every $(\tau, \mathcal{H}_\tau) \in \mathcal{U}$, there exists a vector $w \in \mathcal{H}_\tau$ such that

$$\sup \{ |\langle \tau_\circ(n)v, v \rangle - \langle \tau(n)w, w \rangle| : n \in \Omega \} < \varepsilon.$$

In particular, setting $g = \mathbf{1}$ we obtain $\|w\|^2 < 1 + \varepsilon$. Next set

$$a := |\langle \tau_\circ(\psi)v, v \rangle - \langle \tau_\circ(\chi_\Omega \psi)v, v \rangle| \text{ and } b := |\langle \tau(\psi)w, w \rangle - \langle \tau(\chi_\Omega \psi)w, w \rangle|.$$

By the choice of Ω , we have $a \leq \|\psi - \chi_\Omega \psi\|_{L^1(U_\nu)} < \varepsilon$ and $b \leq (1 + \varepsilon)\|\psi - \chi_\Omega \psi\|_{L^1(U_\nu)} < \varepsilon(1 + \varepsilon)$. Now set $c := |\langle \tau_\circ(\chi_\Omega \psi)v, v \rangle - \langle \tau(\chi_\Omega \psi)w, w \rangle|$. Then

$$c \leq \int_{\Omega} |\psi(n)| \cdot |\langle \tau_\circ(n)v, v \rangle - \langle \tau(n)w, w \rangle| dn < \varepsilon \|\psi\|_{L^1(U_\nu)}$$

By the above estimates and the triangle inequality we obtain $|\langle \tau_\circ(\psi)v, v \rangle - \langle \tau(\psi)w, w \rangle| \leq a + b + c < \frac{3}{4}$. Since $|\langle \tau_\circ(\psi)v, v \rangle| > \frac{3}{4}$, we obtain $\langle \tau(\psi)w, w \rangle \neq 0$, and in particular $\tau(\psi) \neq 0$. Finally, since $\sigma(\psi) = \int_{\widehat{U}_\nu} n_\tau \tau(\psi) d\mu(\tau)$, we obtain $\sigma(\psi) \neq 0$. \square

Lemma 9.3. *Fix a place $\nu \in \mathbf{P}$. Let $f \in L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$ be of moderate growth, and let $\psi \in \mathcal{S}(U_\nu)$. Then the map*

$$G_{\mathbb{Q}} \setminus G_{\mathbb{A}} \rightarrow \mathbb{C}, \quad x \mapsto \int_{U_\nu} \psi(n) f(xn) dn$$

is continuous and in $L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$.

Proof. Set $\Phi_1(x) := \int_{U_\nu} \psi(n) f(xn) dn$ and $\Phi_2(x) := \int_{U_\nu} |\psi(n) f(xn)| dn$ for every $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$. Recall that by Definition 4.2, a moderate growth element of $L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$ is a smooth map $G_{\mathbb{Q}} \setminus G_{\mathbb{A}} \rightarrow \mathbb{C}$.

Step 1. We show that $\Phi_2(x) < \infty$ for every $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$. If $\nu \in \mathbf{P} - \{\infty\}$, then the statement is obvious since ψ is compactly supported. Next assume that $\nu = \infty$, and fix $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$. Let $\|\cdot\|_{\mathfrak{u}_\infty}$ be a norm on \mathfrak{u}_∞ . By (11) we can assume that there exist $c_1, m_1 > 0$ such that

$$|f(x \exp(y))| \leq c_1 (\|y\|_{\mathfrak{u}_\infty} + 1)^{m_1} \text{ for every } y \in \mathfrak{u}_\infty.$$

Since $\psi \in \mathcal{S}(U_\infty)$, there exists a constant $c_2 > 0$ such that

$$\psi(\exp(y)) \leq c_2 (\|y\|_{\mathfrak{u}_\infty} + 1)^{-m_1 - 2 \dim(\mathfrak{u}_\infty)} \text{ for every } y \in \mathfrak{u}_\infty.$$

Since the Haar measure on U_ν is the pushforward of the Lebesgue measure of $\mathfrak{u}_\nu := \text{Lie}(U_\nu)$ via the exponential map, we obtain $\Phi_2(x) < \infty$.

Step 2. From Step 1 it follows that the integral defining $\Phi_1(x)$ is convergent for every $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$. In this step we assume that $\nu \in \mathbf{P} - \{\infty\}$, and we prove that Φ_1 is a continuous map. (The case $\nu = \infty$ will be addressed in Step 3 below.) Fix $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$. Our goal is to prove continuity of Φ_1 at x . Set $\Omega := \text{supp}(\psi)$. For every $y \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$,

$$(18) \quad \Phi_1(x) - \Phi_1(y) = \int_{\Omega} \psi(n) (f(xn) - f(yn)) dn.$$

Fix any $\varepsilon > 0$. By continuity of the map $\Omega \times (G_{\mathbb{Q}} \setminus G_{\mathbb{A}}) \rightarrow \mathbb{C}$ defined as $(n, x) \mapsto f(xn)$, and by compactness of Ω , there exists an open neighborhood $\mathcal{U} \subseteq G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$ of x such that

$$(19) \quad |f(xn) - f(yn)| < \frac{\varepsilon}{\|\psi\|_{L^1(U_\nu)}} \text{ for every } y \in \mathcal{U} \text{ and every } n \in \Omega.$$

From (19) and (18) it follows that $|\Phi_1(x) - \Phi_1(y)| < \varepsilon$ for every $y \in \mathcal{U}$. Since $\varepsilon > 0$ is arbitrary, the latter inequality proves continuity of Φ_1 at an arbitrary point $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$.

Step 3. In this step we assume that $\nu = \infty$, and we prove that the map Φ_1 is continuous. Fix $x \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$. Our goal is to prove continuity of Φ_1 at x . Suppose that $\Omega \subseteq U_\infty$ is an arbitrary relatively compact open set. Then for every $y \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$,

$$(20) \quad \Phi_1(x) - \Phi_1(y) = \int_{\Omega} \psi(n) (f(xn) - f(yn)) dn + \int_{U_\infty - \Omega} \psi(n) (f(xn) - f(yn)) dn.$$

Choose any $\varepsilon > 0$. Our goal is to obtain upper estimates for the above integrals on Ω and $U_\infty - \Omega$ for a suitably chosen Ω .

Every $y \in G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$ can be written as $y = xz_{\text{fin}}z_\infty$ where $z_{\text{fin}} \in G_{\mathbb{A}_{\text{fin}}}$ and $z_\infty \in G_\infty$. If y is chosen sufficiently close to x , then z_{fin} and z_∞ will be sufficiently close to the neutral elements of $G_{\mathbb{A}_{\text{fin}}}$ and $G_{\mathbb{R}}$, respectively. Therefore smoothness of $f : G_{\mathbb{Q}} \setminus G_{\mathbb{A}} \rightarrow \mathbb{C}$ and the growth bound (11) imply that there exists a sufficiently small neighborhood $\mathcal{U}_1 \subseteq G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$ of x such that for every $y \in \mathcal{U}_1$ and every $n \in U_\infty$,

$$(21) \quad \begin{aligned} |f(xn) - f(yn)| &\leq |f(xn)| + |f(xz_{\text{fin}}z_\infty n)| = |f(xn)| + |f(xz_\infty n z_{\text{fin}})| \\ &= |f(xn)| + |f(xz_\infty n)| \leq c_{x,f} \|n\|^{m_f} + c_{x,f} \|z_\infty n\|^{m_f} \leq c_3 \|n\|^{m_f}, \end{aligned}$$

where $c_3 > 0$ is a constant. Fix a norm $\|\cdot\|_{\mathfrak{u}_\infty}$ on \mathfrak{u}_∞ . From (21) it follows that there exist $c_4, m_4 > 0$ such that

$$(22) \quad |f(x \exp(u)) - f(y \exp(u))| \leq c_4 (\|u\|_{\mathfrak{u}_\infty} + 1)^{m_4} \text{ for every } u \in \mathfrak{u}_\infty \text{ and every } y \in \mathcal{U}_1.$$

Since $\psi \in \mathcal{S}(U_\nu)$, we can choose an Ω suitably large such that

$$(23) \quad |\psi(\exp(u))| \leq \frac{\varepsilon}{c_4} (\|u\|_{\mathfrak{u}_\infty} + 1)^{-m_4 - 2 \dim(\mathfrak{u}_\infty)} \text{ for every } u \in \mathfrak{u}_\infty \text{ such that } \exp(u) \in U_\infty - \Omega.$$

Set $c_5 := \int_{U_\nu} (\|u\|_{\mathfrak{u}_\infty} + 1)^{-2 \dim(\mathfrak{u}_\infty)} du$. Since the Haar measure of U_∞ is the pushforward of the Lebesgue measure of \mathfrak{u}_∞ , the latter integral is convergent. From (22) and (23) it follows that

$$(24) \quad \left| \int_{U_\infty - \Omega} \psi(n)(f(xn) - f(yn)) dn \right| < c_5 \varepsilon \text{ for every } y \in \mathcal{U}_1.$$

With an argument similar to the case $\nu \neq \infty$ in Step 2, we can show that there exists a sufficiently small neighborhood $\mathcal{U}_2 \subseteq G_{\mathbb{Q}} \setminus G_{\mathbb{A}}$ of x (which depends on Ω) such that for every $y \in \mathcal{U}_2$ we have

$$(25) \quad \left| \int_{\Omega} \psi(n)(f(xn) - f(yn)) dn \right| < \varepsilon.$$

From (24) and (25) it follows that for every $y \in \mathcal{U}_1 \cap \mathcal{U}_2$ we have

$$\left| \int_{U_\infty} \psi(n)(f(xn) - f(yn)) dn \right| < (c_5 + 1) \varepsilon.$$

The latter inequality implies continuity of Φ_1 at x .

Step 4. We prove that $\Phi_1 \in L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$. It is enough to prove that $\Phi_2 \in L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$. Measurability of Φ_2 follows from Fubini's Theorem. Next fix any $h \in L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$. By Fubini's Theorem and the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_{G_{\mathbb{Q}} \setminus G_{\mathbb{A}}} |\Phi_2(x)h(x)| dx &= \int_{U_\nu} \left(\int_{G_{\mathbb{Q}} \setminus G_{\mathbb{A}}} |\psi(n)f(xn)h(x)| dx \right) dn \\ &\leq \int_{U_\nu} |\psi(n)| \cdot \|f\|_{L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})} \cdot \|h\|_{L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})} dn \\ &\leq \|\psi\|_{L^1(U_\nu)} \cdot \|f\|_{L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})} \cdot \|h\|_{L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})}. \end{aligned}$$

Thus the map $h \mapsto \int_{L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})} \Phi_2(x)h(x) dx$ is a bounded linear functional on $L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$, hence by the Riesz representation theorem we obtain $\Phi_2 \in L^2(G_{\mathbb{Q}} \setminus G_{\mathbb{A}})$. \square

Lemma 9.4 below probably follows from standard results in the literature. We include a complete proof because we did not find a suitable reference. The tricky point is to use Fubini's Theorem carefully to justify that one can change the order of certain integrals.

Before stating Lemma 9.4, we remind the reader that by Definition 4.2, a moderate growth element $f \in L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ is assumed to be a smooth map $f : G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$. In particular, the restriction $f|_{U_{\mathbb{Q}} \backslash U_{\mathbb{A}}}$ is well-defined and continuous. Recall that $\mathcal{S}(U_{\nu})$ denotes the Schwartz space of U_{ν} .

Lemma 9.4. *Fix a place $\nu \in \mathbf{P}$. Let $f \in L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ be of moderate growth. Set $R_{U_{\nu}} := R|_{U_{\nu}}$, where R denotes the representation of $G_{\mathbb{A}}$ on $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ by right translation, and let $\psi \in \mathcal{S}(U_{\nu})$. Then*

- (i) $(R_{U_{\nu}}(\psi)f)(x) = \int_{U_{\nu}} \psi(n)f(xn)dn$ for almost every $x \in G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$.
- (ii) Let R' denote the representation of $U_{\mathbb{A}}$ on $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$ by right translation, and set $R'_{U_{\nu}} := R'|_{U_{\nu}}$. Then $(R'_{U_{\nu}}(\psi)f)|_{U_{\mathbb{Q}} \backslash U_{\mathbb{A}}}(x) = \int_{U_{\nu}} \psi(n)f(xn)dn$ for almost every $x \in U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$.

Proof. We will only give the proof of (i). The proof of (ii) is similar and indeed somewhat easier, since $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ is compact. Set $\Phi_1(x) := \int_{U_{\nu}} \psi(n)f(xn)dn$ for every $x \in G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$.

Fix any $h \in L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$. By Lemma 9.3 we are allowed to use Fubini's Theorem to write

$$\begin{aligned} \langle R_{U_{\nu}}(\psi)f, h \rangle &= \int_{U_{\nu}} \psi(n) \langle R_{U_{\nu}}(n)f, h \rangle dn = \int_{U_{\nu}} \int_{G_{\mathbb{Q}} \backslash G_{\mathbb{A}}} \psi(n)f(xn)\overline{h(x)} dx dn \\ &= \int_{G_{\mathbb{Q}} \backslash G_{\mathbb{A}}} \int_{U_{\nu}} \psi(n)f(xn)\overline{h(x)} dn dx = \int_{G_{\mathbb{Q}} \backslash G_{\mathbb{A}}} \Phi_1(x)\overline{h(x)} dx = \langle \Phi_1, h \rangle. \end{aligned}$$

Since $h \in L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ is arbitrary, from the above calculation it follows that $R_{U_{\nu}}(\psi)f = \Phi_1$ as elements of $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$. \square

Next we state and prove our main theorem (from the introduction). Recall that $(R, L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}))$ denotes the unitary representation of $G_{\mathbb{A}}$ on $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ by right translation. Furthermore, recall the definition of ν -rank of an irreducible unitary representation of G_{ν} given in Definition 3.4, where $\nu \in \mathbf{P}$.

Theorem 9.5. *Let (π, \mathcal{H}) be an irreducible unitary representation of $G_{\mathbb{A}}$ which occurs as a sub-representation of $(R, L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}))$, and let the π_{ν} , $\nu \in \mathbf{P}$, denote the local components of π , as in Remark 4.4. Then the ν -rank of π_{ν} is independent of ν .*

Proof. Let \mathcal{H}° denote the Gårding space of (π, \mathcal{H}) defined in Remark 4.3. For every $\nu \in \mathbf{P}$, let d_{ν} denote the ν -rank of π_{ν} . If $d_{\nu} = r(\mathbf{G})$ for every $\nu \in \mathbf{P}$, then there is nothing to prove. Next assume that $d_{\nu} < r(\mathbf{G})$ for some $\nu \in \mathbf{P}$, and choose $\nu \in \mathbf{P}$ such that d_{ν} has the smallest possible value. It suffices to prove that $d_{\nu_1} \leq d_{\nu}$ for every other $\nu_1 \in \mathbf{P}$.

Set $R_{U_{\nu}} := R|_{U_{\nu}}$. Lemma 9.2 implies that $R_{U_{\nu}}(\phi)f = 0$ for every $\phi \in J_{d_{\nu}, \nu}$ and every $f \in \mathcal{H}$. Consider the vector space W of complex-valued functions on $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ defined as

$$W := \left\{ f|_{U_{\mathbb{Q}} \backslash U_{\mathbb{A}}} : f \in \mathcal{H}^{\circ} \right\}.$$

Note that by Remark 4.3, elements of \mathcal{H}° are represented by continuous maps $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$, and therefore their restriction to $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ is well-defined. Since $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$ is compact, elements of W are bounded functions on $U_{\mathbb{Q}} \backslash U_{\mathbb{A}}$, and in particular $W \subseteq L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$. Let $\mathcal{H} \subseteq L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$

denote the closure of W inside $L^2(U_{\mathbb{Q}} \backslash U_{\mathbb{A}})$. Since \mathcal{H}° is $G_{\mathbb{A}}$ -invariant, the space W is $U_{\mathbb{A}}$ -invariant. From $U_{\mathbb{A}}$ -invariance of W it follows that \mathcal{K} is also $U_{\mathbb{A}}$ -invariant. Consequently, we obtain a unitary representation (σ, \mathcal{K}) of $U_{\mathbb{A}}$ on \mathcal{K} obtained from the restriction of R' (see Lemma 9.4(ii)).

Step 1. Let $R'_{U_{\nu_1}}$ be as in Lemma 9.4(ii). In this step we prove that

$$(26) \quad R'_{U_{\nu_1}}(\phi_1)w = 0 \text{ for every } \phi_1 \in J_{d_\nu, \nu_1} \text{ and } w \in \mathcal{K}.$$

Fix $\phi \in J_{d_\nu, \nu}$. For any $f \in \mathcal{H}^\circ$, Lemma 9.4(i) and continuity of the map $x \mapsto \int_{U_\nu} \phi(n)f(xn)dn$ (see Lemma 9.3) imply that $\int_{U_\nu} \phi(n)f(xn)dn = 0$ for every $x \in G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$. Thus from Lemma 9.4(ii) it follows that

$$(27) \quad R'_{U_\nu}(\phi)w = 0 \text{ for every } w \in W.$$

From Moore's result mentioned in Section 8 it follows that $\sigma = \bigoplus_{\mu \in S} \rho_\mu$, where $S \subseteq \mathfrak{u}_{\mathbb{Q}}^*$ and the representations $\rho_\mu = \otimes_{\nu \in P} \rho_{\mu_\nu}$ are defined in (17). Now fix $\mu \in S$. Since $\rho_\mu|_{U_\nu}$ is a direct sum of countably many copies of ρ_{μ_ν} , from (27) and Lemma 9.2 it follows that $\rho_{\mu_\nu} \in \widehat{U}_\nu[d_\nu]$. Consequently, by Lemma 8.1 we obtain

$$\rho_{\mu_{\nu_1}} \in \widehat{U}_{\nu_1}[d_\nu] \text{ for every } \nu_1 \in P.$$

Therefore Lemma 9.2 implies (26).

Step 2. Let G^{ν_1} be defined as in Section (9). For every $g \in G^{\nu_1}$, the map

$$\mathbf{U}(\mathbb{Q}_{\nu_1}) \rightarrow G_{\mathbb{A}}, \quad n \mapsto g s_{\nu_1}(n) g^{-1}$$

is a splitting section. From the uniqueness of this section (see Section 3) it follows that $ng = gn$ for every $n \in U_{\nu_1}$.

Step 3. By the weak approximation property, $\mathbf{G}(\mathbb{Q})\mathbf{G}^{\nu_1}$ is dense in $\mathbf{G}(\mathbb{A})$. It follows that $G_{\mathbb{Q}}G^{\nu_1}F$ is a dense subset of $G_{\mathbb{A}}$, where $F \subseteq G_{\mathbb{A}}$ denotes the kernel of the central extension (1).

Step 4. In this step we prove that

$$(28) \quad R_{U_{\nu_1}}(\phi_1)f = 0 \text{ for every } f \in \mathcal{H}^\circ \text{ and } \phi_1 \in J_{d_\nu, \nu_1}.$$

Recall that by Remark 4.1 and Remark 4.3, we can represent every element of \mathcal{H}° by a unique continuous map $G_{\mathbb{A}} \rightarrow \mathbb{C}$. Furthermore, Lemma 9.3 and $G_{\mathbb{A}}$ -invariance of \mathcal{H}° imply that the map $x \mapsto \int_{U_{\nu_1}} \phi_1(n)f(xng)dn$ is continuous for every $f \in \mathcal{H}^\circ$ and $g \in G_{\mathbb{A}}$. Thus by Lemma 9.4(ii) and Step 1 we obtain

$$(29) \quad \int_{U_{\nu_1}} \phi_1(n)f(xng)dn = 0 \text{ for every } \phi_1 \in J_{d_\nu, \nu_1}, f \in \mathcal{H}^\circ, g \in G_{\mathbb{A}}, \text{ and } x \in U_{\mathbb{Q}} \backslash U_{\mathbb{A}}.$$

Setting $x = U_{\mathbb{Q}}$ (the identity coset in $U_{\mathbb{A}}$) in (29), and using Step 2, we obtain that

$$(30) \quad \int_{U_{\nu_1}} \phi_1(n)f(gn)dn = 0 \text{ for every } \phi_1 \in J_{d_\nu, \nu_1}, g \in G^{\nu_1}, \text{ and } f \in \mathcal{H}^\circ.$$

Note that in (30) we consider f as a map $G_{\mathbb{A}} \rightarrow \mathbb{C}$ (see Remark 4.1).

By Lemma 9.4(i), continuity of the left hand side of (30) as a function of g (see Lemma 9.3), and Step 3, in order to complete the proof of (28) it suffices to prove that the vanishing condition (30) holds for every $g \in G_{\mathbb{Q}}G^{\nu_1}F$. From left $G_{\mathbb{Q}}$ -invariance of the left hand side of (30) it follows that

this vanishing condition also holds for every $g \in G_{\mathbb{Q}}G^{\nu_1}$. Finally, Schur's Lemma implies that for every $z \in F$, the action of $\pi(z)$ on \mathcal{H} is by a scalar $\gamma(z) \in \mathbb{C}$, and therefore for every $g \in G_{\mathbb{Q}}G^{\nu_1}$,

$$\int_{U_{\nu_1}} \phi_1(n)f(gzn)dn = \int_{U_{\nu_1}} \phi_1(n)f(gnz)dn = \gamma(z) \int_{U_{\nu_1}} \phi_1(n)f(gn)dn = 0.$$

Step 5. Since \mathcal{H}° is dense in \mathcal{H} , the assertion (28) of Step 4 holds for every $f \in \mathcal{H}$ as well. Now Lemma 9.2 and Theorem 3.3 imply that the ν_1 -rank of π_{ν_1} is at most d_{ν} . \square

Remark 9.6. For several groups \mathbf{G} it is known (see [12], [30]) that for every $\nu \in \mathbf{P}$, an irreducible unitary representation of G_{ν} is minimal if and only if its ν -rank is equal to one. For such \mathbf{G} , Theorem 9.5 implies that if at least one local component of an automorphic representation of $G_{\mathbb{A}}$ is a minimal representation, then all of its local components are minimal.

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