Differentiable vectors and unitary representations of Fréchet–Lie supergroups

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Abstract

A locally convex Lie group G has the Trotter property if, for every $x_1, x_2 \in \mathfrak{g}$,

$$\exp_G(t(x_1 + x_2)) = \lim_{n \to \infty} \left(\exp_G\left(\frac{t}{n}x_1\right) \exp_G\left(\frac{t}{n}x_2\right) \right)^n$$

holds uniformly on compact subsets of $\mathbb{R}$. All locally exponential Lie groups have this property, but also groups of automorphisms of principal bundles over compact smooth manifolds. A key result of the present article is that, if G has the Trotter property, $\pi \colon G \to \mathrm{GL}(V)$ is a continuous representation of G on a locally convex space, and $v \in V$ is a vector such that $\overline{\mathbf{d}\pi}(x)v := \frac{d}{dt}|_{t=0}\pi(\exp_G(tx))v$ exists for every $x \in \mathfrak{g}$, then the map $\mathfrak{g} \to V, x \mapsto \overline{\mathbf{d}\pi}(x)v$ is linear.

Using this result we conclude that, for a representation of a locally exponential Fréchet–Lie group G on a metrizable locally convex space, the space of $\mathcal{C}^k$-vectors coincides with the common domain of the k-fold products of the operators $\overline{\mathbf{d}\pi}(x)$. For unitary representations on Hilbert spaces, the assumption of local exponentiality can be weakened to the Trotter property.

As an application, we show that for smooth (resp., analytic) unitary representations of Fréchet–Lie supergroups $(G, \mathfrak{g})$ where G has the Trotter property, the common domain of the operators of $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ can always be extended to the space of smooth (resp., analytic) vectors for G.

Keywords: infinite dimensional Lie group, representation, differentiable vector, smooth vector, analytic vector, derived representation, Lie supergroup, Trotter property.
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Introduction

Let G be a Lie group modeled on a locally convex space. Assume that G has a (smooth) exponential function $\exp_G \colon \mathbf{L}(G) = \mathfrak{g} \to G$, so that every smooth one-parameter group of

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G is of the form $\gamma_x(t) := \exp_G(tx)$ for some $x \in \mathbf{L}(G)$. For every continuous representation $\pi \colon G \to \mathrm{GL}(V)$ of G on a locally convex space V, set

$$\mathcal{D}_x := \left\{ v \in V \colon \frac{d}{dt}\bigg|_{t=0} \pi(\exp_G(tx))v \ \ \text{exists} \right\}.$$

Thus, $\mathcal{D}_x$ is the domain of the infinitesimal generator

$$\overline{\mathsf{d}\pi}(x)v := \frac{d}{dt}\bigg|_{t=0} \pi(\exp_G(tx))v$$

of the one-parameter group $\pi(\exp_G(tx))$. Let $\mathcal{D}^1 := \bigcap_{x \in \mathfrak{g}} \mathcal{D}_x$ be the common domain of these operators. Clearly, each vector v with a continuously differentiable orbit map (a $\mathcal{C}^1$-vector) is contained in $\mathcal{D}^1$, but the converse problem turns out to be a tricky question. The main difficulty is to establish that, for $v \in \mathcal{D}^1$, the map

$$\omega_v \colon \mathfrak{g} \to V, \quad x \mapsto \overline{\mathsf{d}\pi}(x)v$$

is linear and, if this is the case, and ω_v is continuous, to show that this implies that v is a $\mathcal{C}^1$-vector. As we have seen in [Ne10], the latter problem can be solved rather easily if G is locally exponential, but in practice this assumption appears rather strong because it is not satisfied for groups of diffeomorphisms. With respect to the linearity of ω_v, we managed to show in [Ne10, Thm. 2.8] that ω_v is linear if G is a Banach–Lie group and V is a Banach space. To achieve this result, we had to build on quite involved results of Neklyudov ([Nek08]).

In the present note we obtain much more satisfactory solutions to both problems with rather direct proofs. The key to our new approach is the recent paper [BB11] by I. and D. Beltiţă, where they address the linearity problem for ω_v in the context of topological groups. In Section 2, we show that their approach can actually be carried much further to obtain our Theorem 2.7, which asserts that for a Lie group G with the Trotter property, and a continuous function $\varphi \colon G \to V$ (where V is a locally convex space), if the derivatives $D_x\varphi(g) := \frac{d}{dt}|_{t=0}\varphi(g\gamma_x(t))$ with respect to all one-parameter groups γ_x, $x \in \mathfrak{g}$, are continuous maps, then the map $x \mapsto D_x\varphi$ is linear.

That a Lie group G has the *Trotter property* means that, for $x_1, x_2 \in \mathfrak{g}$,

$$\exp_G(t(x_1 + x_2)) = \lim_{n \to \infty} \left(\exp_G\left(\frac{t}{n}x_1\right) \exp_G\left(\frac{t}{n}x_2\right) \right)^n$$

holds uniformly on compact subsets of $\mathbb{R}$. The main advantage of this property is that the class of Lie groups with this property contains all locally exponential Lie groups (hence all Banach–Lie groups), groups of automorphisms of principal bundles over compact smooth manifolds (in particular, diffeomorphism groups of compact smooth manifolds), and direct limit Lie groups (see Section 3).

Theorem 2.7 turns out to be just the right tool to deal with differentiable vectors in continuous representations of Lie groups. Combining it with the techniques developed in [Ne10], we show in Section 4 that, for a continuous representation of a locally exponential Fréchet–Lie group on a metrizable locally convex space, $\mathcal{D}^1$ coincides with the space of $\mathcal{C}^1$-vectors, and there is a similar characterization of $\mathcal{C}^k$-vectors (Theorem 4.10). This already generalizes the corresponding Banach results from [Ne10] considerably with substantially simpler proofs.

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In Section 5 we turn to the special class of unitary representations which are differentiable in the sense that the space of $\mathcal{C}^1$-vectors is dense. For these we can weaken the assumption of local exponentiality to the Trotter property. Namely, for any Fréchet–Lie group G with the Trotter property, we obtain the natural characterization of the space of $\mathcal{C}^k$-vectors as the common domain of the k-fold products of the operators $\overline{\mathrm{d}\pi}(x)$, $x \in \mathfrak{g}$ (Theorem 5.3).

In Section 6 we apply all this to unitary representations of Lie supergroups $(G, \mathfrak{g})$, which we consider as a pair consisting of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ and a Lie group G whose Lie algebra is the even part $\mathfrak{g}_{\overline{0}}$ of $\mathfrak{g}$ (see [CCTV06]). A crucial difficulty in dealing with unitary representations of Lie supergroups is the specification of the common domain of the operators corresponding to the odd part $\mathfrak{g}_{\overline{1}}$ (see [CCTV06] and [MNS11] for a detailed discussion). For the large class of Fréchet–Lie supergroups where G has the Trotter property, we roughly show that, if the representation of G is smooth, resp., analytic, the common domain of the operators of $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ can always be extended to the space of smooth, resp., analytic vectors for G. This generalizes the respective stability results for Banach–Lie supergroups from [MNS11] and for finite-dimensional Lie supergroups from [CCTV06]. We thus obtain a natural context for a global unitary representation theory for Lie supergroups modeled on Fréchet spaces, which applies in particular to diffeomorphism groups and gauge groups.

Contents

1 Locally convex Lie groups

In this section we briefly recall the basic concepts related to infinite-dimensional Lie groups. Throughout these notes all topological groups and vector spaces are assumed to be Hausdorff.

Definition 1.1. (a) Let E and F be locally convex spaces, $U \subseteq E$ open and $f \colon U \to F$ a map. Then the *derivative of f at x in the direction h* is defined as

$$\mathrm{d}f(x)(h) := (\partial_h f)(x) := \left.\frac{d}{dt}\right|_{t=0} f(x + th) = \lim_{t \to 0} \frac{1}{t}\big(f(x + th) - f(x)\big)$$

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whenever it exists. The function f is called *differentiable at x* if $\mathsf{d}f(x)(h)$ exists for all $h \in E$. It is called *continuously differentiable*, if it is differentiable at all points of U and

$$\mathsf{d}f\colon U \times E \to F, \quad (x, h) \mapsto \mathsf{d}f(x)(h)$$

is a continuous map. Note that this implies that the maps $\mathsf{d}f(x)$ are linear (cf. [GN12, Lemma 2.2.14]). The map f is called a $\mathcal{C}^k$-*map*, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$\mathsf{d}^j f(x)(h_1, \ldots, h_j) := (\partial_{h_j} \cdots \partial_{h_1} f)(x)$$

exist for all integers $1 \leq j \leq k$, $x \in U$ and $h_1, \ldots, h_j \in E$, and all maps $\mathsf{d}^j f\colon U \times E^j \to F$ are continuous. As usual, $\mathcal{C}^\infty$-maps are called *smooth*.

(b) If E and F are complex locally convex spaces, then f is called *complex analytic* if it is continuous and for each $x \in U$ there exists a 0-neighborhood V with $x + V \subseteq U$ and continuous homogeneous polynomials $\beta_k\colon E \to F$ of degree k such that for each $h \in V$ we have

$$f(x + h) = \sum_{k=0}^\infty \beta_k(h),$$

as a pointwise limit ([BS71]). The map f is called *holomorphic* if it is $\mathcal{C}^1$ and for each $x \in U$ the map $\mathsf{d}f(x)\colon E \to F$ is complex linear (cf. [Mil84, p. 1027]). If F is sequentially complete, then f is holomorphic if and only if it is complex analytic ([BS71, Ths. 3.1, 6.4]).

(c) If E and F are real locally convex spaces, then we call a map $f\colon U \to F$, $U \subseteq E$ open, *real analytic* or a $\mathcal{C}^\omega$-*map*, if for each point $x \in U$ there exists an open neighborhood $V \subseteq E_{\mathbb{C}}$ and a holomorphic map $f_{\mathbb{C}}\colon V \to F_{\mathbb{C}}$ with $f_{\mathbb{C}}|_{U \cap V} = f|_{U \cap V}$ (cf. [Mil84]). The advantage of this definition, which differs from the one in [BS71], is that it also works nicely for non-complete spaces. Any analytic map is smooth, and the corresponding chain rule holds without any condition on the underlying spaces, which is the key to the definition of analytic manifolds (see [Gl02] for details).

Once the concept of a smooth function between open subsets of locally convex spaces is established (cf. [Ne06], [GN12]), it is clear how to define a locally convex smooth manifold. A *(locally convex) Lie group G* is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element. Then each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(\mathbf{1}) = x$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x, y] = [x_l, y_l](\mathbf{1})$ for $x, y \in \mathfrak{g}$. We shall also use the functorial notation $\mathbf{L}(G) := (\mathfrak{g}, [\cdot, \cdot])$ for the Lie algebra of G. The adjoint action of G on $\mathfrak{g}$ is defined by $\mathrm{Ad}(g) := T_{\mathbf{1}}(c_g)$, where $c_g(x) = gxg^{-1}$ is the conjugation map. The adjoint action is smooth and each $\mathrm{Ad}(g)$ is a topological isomorphism of $\mathfrak{g}$. If $\mathfrak{g}$ is a Fréchet, resp., a Banach space, then G is called a *Fréchet-*, resp., a *Banach–Lie group*.

A smooth map $\exp_G\colon \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma'_x(0) = x$. The Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathfrak{g}$ mapped diffeomorphically by $\exp_G$ onto an open subset of G. If, in addition, G is analytic

and the exponential function is an analytic diffeomorphism in a 0-neighborhood, then G is called a *BCH–Lie group* (for Baker–Campbell–Hausdorff). The class of BCH–Lie groups contains in particular all Banach–Lie groups ([Ne06, Prop. IV.1.2]).

2 Linearity of differentials

The main result of this section is Theorem 2.7. It extends [BB11, Thm. 2.5] in the sense that it weakens the necessary assumptions considerably and thus makes it much simpler to apply. Its main application is that, for a continuous representation of the Lie group G (with the Trotter property) on a locally convex space V and an element $v \in V$ for which $\omega_v(x) := \frac{d}{dt}|_{t=0}\pi(\exp_G(tx))v$ exists for every $x \in \mathbf{L}(G)$, the map $\omega_v \colon \mathfrak{g} \to V$ is linear (Theorem 4.6). For continuous representations of Banach–Lie groups on Banach spaces this was known from [Ne10, Thm. 8.2], but the proof used quite involved results of Neklyudov ([Nek08]) which are now bypassed. Therefore our results also provide a considerable simplification of the arguments for Banach–Lie groups.

Let $S \subseteq \mathbb{R}$ and G be a topological group. We say that a sequence of functions $\gamma_n : S \to G$ is *uniformly convergent* to $\gamma : S \to G$ if for every **1**-neighborhood $U \subseteq G$ there exists an $N \geq 1$ such that $\gamma_n(s) \in \gamma(s)U$ for every $s \in S$ and every $n \geq N$. If the maps $\gamma_n : S \to G$ are continuous then γ is continuous as well. If, in addition, S is compact, then the definition is symmetric, that is, for every **1**-neighborhood $U \subseteq G$ there exists an $N \geq 1$ such that $\gamma_n(s) \in U\gamma(s)$ for every $s \in S$ and every $n \geq N$.

Definition 2.1. (cf. [BB11]) (a) Let G be a topological group and $\mathfrak{L}(G) := \mathrm{Hom}(\mathbb{R}, G)$ denote the set of continuous one-parameter groups. We endow this set with the topology of uniform convergence on compact subsets of $\mathbb{R}$. For $X, X_1, X_2 \in \mathfrak{L}(G)$ we say that $X = X_1 + X_2$ if

$$X(t) = \lim_{n \to \infty} \left(X_1\left(\frac{t}{n}\right) X_2\left(\frac{t}{n}\right) \right)^n \tag{1}$$

holds uniformly in t on compact subsets of $\mathbb{R}$.

(b) We say that a Lie group G has the *Trotter property* if it has a smooth exponential function and the one-parameter groups $\gamma_x(t) := \exp_G(tx)$, $x \in \mathbf{L}(G)$, satisfy $\gamma_{x_1+x_2} = \gamma_{x_1} + \gamma_{x_2}$ in the sense defined above, i.e.,

$$\exp_G(t(x_1 + x_2)) = \lim_{n \to \infty} \left(\exp_G\left(\frac{t}{n}x_1\right) \exp_G\left(\frac{t}{n}x_2\right) \right)^n \quad \text{for all } x_1, x_2 \in \mathbf{L}(G),$$

uniformly on compact subsets of $\mathbb{R}$.

Lemma 2.2. *Let G be a topological group and $\gamma_n \colon I = [0,1] \to G$ be a sequence of continuous curves converging uniformly to a limit curve γ. Then*

$$\gamma(I) \cup \bigcup_{n=1}^{\infty} \gamma_n(I)$$

is a compact subset of G.

Proof. Let $X := Y \times I$, where $Y := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then Y is compact, and therefore X is a compact subset of $\mathbb{R}^2$. The map $F \colon X \to G$ defined by

$$F(s,t) := \begin{cases} \gamma_n(t) & \text{for } s = \frac{1}{n} \\ \gamma(t) & \text{for } s = 0 \end{cases}$$

is continuous, and therefore its image is a compact subset of G. $\qquad\square$

Lemma 2.3. *Let G and H be topological groups, $K \subseteq G$ be a compact subset and $f \colon G \to H$ continuous. Then f is uniformly continuous on K in the following sense: For every $\mathbf{1}$-neighborhood U_H in H there exists a $\mathbf{1}$-neighborhood U_G in G such that*

$$(\forall x \in K)\ f(xU_G) \subseteq f(x)U_H.$$

Proof. This essentially follows from Remark 3.9(b) but we also give a direct proof. The map

$$F \colon K \times G \to H, \quad F(x,z) := f(x)^{-1} f(xz)$$

is continuous, so that the inverse image $F^{-1}(U_H^0)$ of the interior U_H^0 of U_H is an open subset of $K \times G$ containing $K \times \{\mathbf{1}\}$. Hence there exists an open subset $U_G \subseteq G$ with $K \times U_G \subseteq F^{-1}(U_H^0)$. $\qquad\square$

Definition 2.4. ([BB11, Def. 2.1]) Let G be a topological group and V be a locally convex space. For $\varphi \colon G \to V$, $X \in \mathfrak{L}(G)$, and $g \in G$, we write

$$(D_X \varphi)(g) = \lim_{t \to 0} \frac{\varphi(gX(t)) - \varphi(g)}{t} \tag{2}$$

whenever the limit on the right-hand side exists.

The next lemma is a more accurate version of [BB11, Prop. 2.2]. We denote the space of continuous maps from G into V by $\mathcal{C}(G, V)$.

Lemma 2.5. *Let G be a topological group, V be a locally convex space and $\varphi \in \mathcal{C}(G, V)$. Then the following assertions hold:*

(i) *Let $X \in \mathfrak{L}(G)$ and $n \geq 1$ be such that the function $D_X^n \varphi$ is defined and continuous. For every $g \in G$ and $t \in \mathbb{R}$ we then have*

$$\varphi(gX(t)) = \sum_{j=0}^{n} \frac{t^j}{j!} (D_X^j \varphi)(g) + t^n \chi_1^X(g, t)$$

where $\chi_1^X \colon G \times \mathbb{R} \to V$ is continuous and $\chi_1^X(g, 0) = 0$ for every $g \in G$.

(ii) *If $X_1, X_2 \in \mathfrak{L}(G)$ are such that $D_{X_1} \varphi$ and $D_{X_2} \varphi$ are defined and continuous, then*

$$\varphi(gX_1(t)X_2(t)) = \varphi(g) + t((D_{X_1} + D_{X_2})\varphi)(g) + t\chi_2(g, t) \quad \text{for all} \quad (g, t) \in G \times \mathbb{R},$$

where $\chi_2 \colon G \times \mathbb{R} \to V$ is a continuous function satisfying $\chi_2(g, 0) = 0$ for every $g \in G$.

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Proof. (i) The Taylor formula leads to the asserted identity with the remainder term

$$\chi_1^X(g,t) = \frac{1}{(n-1)!} \int_0^1 \left((1-s)^{n-1}(D_X^n\varphi)(gX(ts)) - \frac{1}{n}(D_X^n\varphi)(g) \right) ds$$

which is continuous on $G \times \mathbb{R}$ because the function $D_X^n\varphi$ is continuous and hence one can use Lemma 2.3 for compact sets of the form $\{g\} \times [0,1] \subseteq G \times \mathbb{R}$.

(ii) First we use (i) to obtain a continuous function $\chi_1^{X_2}$ on $G \times \mathbb{R}$ vanishing in $(g,0)$ and satisfying

$$\varphi(gX_2(t)) = \varphi(g) + t(D_{X_2}\varphi)(g) + t\chi_1^{X_2}(g,t) \quad \text{for} \quad (g,t) \in G \times \mathbb{R}.$$

This leads immediately to

$$\varphi(gX_1(t)X_2(t)) = \varphi(gX_1(t)) + t(D_{X_2}\varphi)(gX_1(t)) + t\chi_1^{X_2}(gX_1(t),t).$$

From (i) we also obtain a continuous function $\chi_1^{X_1}$ on $G \times \mathbb{R}$ vanishing in $(g,0)$ and satisfying

$$\varphi(gX_1(t)) = \varphi(g) + t(D_{X_1}\varphi)(g) + t\chi_1^{X_1}(g,t) \quad \text{for} \quad (g,t) \in G \times \mathbb{R}.$$

By plugging this formula into the previous one, we get

$$\varphi(gX_1(t)X_2(t)) = \varphi(g) + t(D_{X_1}\varphi)(g) + t(D_{X_2}\varphi)(g) + t\chi_2(g,t),$$

where

$$\chi_2(g,t) = (D_{X_2}\varphi)(gX_1(t)) - (D_{X_2}\varphi)(g) + \chi_1^{X_1}(g,t) + \chi_1^{X_2}(gX_1(t),t)$$

is a continuous function on $G \times \mathbb{R}$ vanishing in all pairs $(g,0)$. $\square$

Based on the preceding lemma, we obtain Lemma 2.6 below, which is a sharpening of [BB11, Lemma 2.3]. Here the main point is that Lemmas 2.2 and 2.3 permit us to draw stronger conclusions from the proof given in [BB11].

Lemma 2.6. *Let G be a topological group and V be a locally convex space. Let $X, X_1, X_2 \in \mathcal{L}(G)$ with $X = X_1 + X_2$ and $\varphi \in \mathcal{C}(G,V)$ be such that $D_{X_1}\varphi$ and $D_{X_2}\varphi$ exist and are continuous. Then $D_X\varphi$ also exists and satisfies*

$$D_X\varphi = D_{X_1}\varphi + D_{X_2}\varphi. \tag{3}$$

Proof. For $t \in \mathbb{R}$ we put $\gamma(t) = X_1(t)X_2(t)$. Fix $g_0 \in G$. We have to show that $D_X\varphi$ is defined in g_0 and satisfies

$$(D_X\varphi)(g_0) = (D_{X_1}\varphi)(g_0) + (D_{X_2}\varphi)(g_0). \tag{4}$$

To this end fix an arbitrary continuous seminorm $|\cdot|$ on V and let $\varepsilon > 0$ be arbitrary. Since the functions $D_{X_j}\varphi$ are continuous for $j = 1, 2$, there exists an open **1**-neighborhood U in G such that

$$(\forall g \in U) \quad |((D_{X_1} + D_{X_2})\varphi)(g_0 g) - ((D_{X_1} + D_{X_2})\varphi)(g_0)| < \frac{\varepsilon}{2}. \tag{5}$$

Let $\delta_1 > 0$ be such that $X(t) \in U$ for $-\delta_1 \leq t \leq \delta_1$. By using (1) with uniform convergence on the interval $[-\delta_1, \delta_1]$, we obtain an $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ and $-\delta_1 \leq t \leq \delta_1$, then

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$\gamma(t/n)^n \in U$. There also exists a $\delta_2 \in (0, \delta_1)$ such that for $n = 1, \dots, n_1$ and $-\delta_2 \le t \le \delta_2$ we have $\gamma(t/n)^n \in U$. Therefore

$$\gamma\left(\frac{t}{n}\right)^n \in U \text{ if } -\delta_2 \le t \le \delta_2 \text{ and } n \in \mathbb{N}.$$

If $1 \le k \le n$ and $|t| \le \delta_2$, then $|(k/n)t| \le |t| \le \delta_2$, so that we also obtain

$$\gamma\left(\frac{t}{n}\right)^k = \gamma\left(\frac{(k/n)t}{k}\right)^k \in U.$$

According to Lemma 2.2, the set

$$C := X([-\delta_2, \delta_2]) \cup \bigcup_{n \in \mathbb{N}} \left\{ \gamma\left(\frac{t}{n}\right)^n : -\delta_2 \le t \le \delta_2 \right\}$$

is compact. Therefore the continuity of χ_2 in Lemma 2.5(ii) implies (by Lemma 2.3) the existence of $\delta_3 \in (0, \delta_2)$ such that, for $x \in g_0 C$ and $0 < |t| \le \delta_3$, we have

$$|\chi_2(x,t)| = \left| \frac{1}{t} (\varphi(x\gamma(t)) - \varphi(x)) - ((D_{X_1} + D_{X_2})\varphi)(x) \right| < \frac{\varepsilon}{2}. \tag{6}$$

Note that we also have $\gamma(t/n)^k \in C$ for $0 \le k \le n$ and $|t| \le \delta_2$. This allows us to use (5) in order to show that if $0 < |t| \le \delta_3$ and $n \in \mathbb{N}$, then

$$\left| \frac{1}{t}\left(\varphi\left(g_0\gamma\left(\frac{t}{n}\right)^n\right) - \varphi(g_0)\right) - ((D_{X_1} + D_{X_2})\varphi)(g_0) \right|$$

$$\le \frac{1}{n} \sum_{k=1}^{n} \left| \frac{1}{t/n}\left(\varphi\left(g_0\gamma\left(\frac{t}{n}\right)^k\right) - \varphi\left(g_0\gamma\left(\frac{t}{n}\right)^{k-1}\right)\right) - ((D_{X_1} + D_{X_2})\varphi)\left(g_0\gamma\left(\frac{t}{n}\right)^{k-1}\right) \right|$$

$$+ \frac{1}{n} \sum_{k=1}^{n} \left| ((D_{X_1} + D_{X_2})\varphi)\left(g_0\gamma\left(\frac{t}{n}\right)^{k-1}\right) - ((D_{X_1} + D_{X_2})\varphi)(g_0) \right|$$

$$< \frac{1}{n} \sum_{k=1}^{n} \frac{\varepsilon}{2} + \frac{1}{n} \sum_{k=1}^{n} \frac{\varepsilon}{2} = \varepsilon.$$

Since φ is continuous, we have $\lim_{n \to \infty} \varphi(g_0\gamma(t/n)^n) = \varphi(g_0 X(t))$ in V. The above estimates thus lead for $0 < |t| \le \delta_3$ to

$$\left| \frac{1}{t}(\varphi(g_0 X(t)) - \varphi(g_0)) - ((D_{X_1} + D_{X_2})\varphi)(g_0) \right| \le \varepsilon.$$

Since $|\cdot|$ is an arbitrary continuous seminorm on the Hausdorff locally convex space V, it follows that $D_X \varphi$ exists in g_0 and that (4) is satisfied. $\qquad\square$

The following theorem extends [BB11, Thm 2.4] in the sense that it applies to all continuous functions $G \to V$, which facilitates the application of this theorem considerably.

Theorem 2.7. *Let G be a Lie group with the Trotter property. Then, for every locally convex space V and $\varphi \in \mathcal{C}(G, V)$ for which $D_x \varphi := D_{\gamma_x} \varphi$ exists and is continuous for every $x \in \mathbf{L}(G)$, the map*

$$\mathbf{L}(G) \to \mathcal{C}(G, V), \quad x \mapsto D_x \varphi$$

is linear.

3 Examples of groups with the Trotter property

An important point of the Trotter property is that almost all natural classes of infinite-dimensional Lie groups have this property, even if they are not locally exponential, such as groups of diffeomorphisms.

Lemma 3.1. *Let G and H be topological groups, $f : G \to H$ be a continuous map, and $\gamma_n : I = [0, 1] \to G$ be a sequence of continuous curves converging uniformly to a limit curve γ. Then the sequence $f \circ \gamma_n$ converges uniformly to $f \circ \gamma$.*

Proof. The set $K := \gamma(I)$ is compact and the statement follows from Lemma 2.3. $\qquad\Box$

Lemma 3.2. *Let G be a topological group and $\gamma : \mathbb{R} \to G$ be a continuous curve with $\gamma(0) = \mathbf{1}$. Set*

$$\gamma_n : \mathbb{R} \to G \ , \gamma_n(t) := \gamma\left(\frac{t}{n}\right)^n.$$

Assume that there exists an $\varepsilon > 0$ such that the sequence $(\gamma_n|_{[-\varepsilon,\varepsilon]})_{n=1}^\infty$ converges uniformly. Then $(\gamma_n)_{n=1}^\infty$ converges uniformly on compact subsets of $\mathbb{R}$.

Proof. It suffices to show that the sequence $(\gamma_n)_{n=1}^\infty$ converges uniformly on $[-2\varepsilon, 2\varepsilon]$, since one can iterate the argument. Define

$$\eta : [-\varepsilon, \varepsilon] \to G, \quad \eta(t) := \lim_{n\to\infty} \gamma_n(t)$$

and observe that

$$\gamma_{2n}(t) = \gamma_n\left(\frac{t}{2}\right)^2.$$

Since the square map $f : G \to G, g \mapsto g^2$ is continuous, from Lemma 3.1 it follows that the sequence $(\gamma_{2n})_{n=1}^\infty$ converges uniformly on $[-2\varepsilon, 2\varepsilon]$ to the curve $t \mapsto \eta\left(\frac{t}{2}\right)^2$. A minor modification of the above argument proves that $(\gamma_{2n+1})_{n=0}^\infty$ also converges uniformly on $[-2\varepsilon, 2\varepsilon]$ to the same curve. In fact,

$$\gamma_{2n+1}(t) = \gamma\left(\frac{t}{2n+1}\right)^{2n+1} = \gamma\left(\frac{t}{2n+1}\right)\gamma_n(h_n(t))^2, \quad \text{where} \quad h_n(t) := \frac{nt}{2n+1}. \tag{7}$$

Since $\gamma(0) = \mathbf{1}$, the curves $t \mapsto \gamma\left(\frac{t}{2n+1}\right)$ converge uniformly on $[-2\varepsilon, 2\varepsilon]$ to the constant function with value $\mathbf{1}$. The curves $(\gamma_n \circ h_n)_{n=0}^\infty$ converge uniformly on $[-2\varepsilon, 2\varepsilon]$ to $\eta\left(\frac{t}{2}\right)$. Since the map $\mu : G \times G \times G \to G$ defined by $\mu(x, y, z) := xyz$ is continuous, Lemma 3.1 implies that $(\gamma_{2n+1})_{n=0}^\infty$ converges uniformly on $[-2\varepsilon, 2\varepsilon]$ to $\eta\left(\frac{t}{2}\right)^2$. $\qquad\Box$

The following proposition provides a criterion for a Lie group to have the Trotter property. We shall see below that it is crucial to verify that certain groups of diffeomorphisms have the Trotter property.

Proposition 3.3. *Let G be a Lie group with a smooth exponential map. Assume that, for every $x_1, x_2 \in \mathbf{L}(G)$, there exists an $\varepsilon > 0$ such that*

$$\exp_G(t(x_1 + x_2)) = \lim_{n\to\infty} \left(\exp_G\left(\frac{t}{n}x_1\right) \exp_G\left(\frac{t}{n}x_2\right) \right)^n \tag{8}$$

holds uniformly on $[-\varepsilon, \varepsilon]$. Then G has the Trotter property.

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Proof. Setting $\gamma : \mathbb{R} \to G$, $\gamma(t) := \exp_G(tx_1) \exp_G(tx_2)$ in Lemma 3.2 implies that the right hand side of (8) is uniformly convergent on compact subsets of $\mathbb{R}$ to a map $\eta : \mathbb{R} \to G$. For every $t \in \mathbb{R}$ and $\gamma_n(t) := \gamma\left(\frac{t}{n}\right)^n$ we have

$$\eta(2t) = \lim_{n \to \infty} \gamma_{2n}(2t) = \lim_{n \to \infty} \gamma_n(t)^2 = \eta(t)^2.$$

Since $\eta(t) = \exp_G(t(x_1 + x_2))$ for $t \in [-\varepsilon, \varepsilon]$, it follows that $\eta(t) = \exp_G(t(x_1 + x_2))$ for every $t \in \mathbb{R}$. $\qquad\square$

Covering groups

Since the condition in Proposition 3.3 is local, we immediately obtain the following statement.

Corollary 3.4. *If $q_G \colon \widehat{G} \to G$ is a covering morphism of Lie groups, then G has the Trotter property if and only if $\widehat{G}$ has the Trotter property.*

Locally exponential groups

Proposition 3.5. *Every locally exponential Lie group has the Trotter property.*

Proof. Let G be a locally exponential Lie group and $x, y \in \mathfrak{g} = \mathbf{L}(G)$. Then

$$\gamma : \mathbb{R} \to G, \quad \gamma(t) := \exp_G(tx) \exp_G(ty)$$

is a smooth curve. Since G is locally exponential, there exists an $\varepsilon > 0$ and a smooth curve $\beta \colon [-\varepsilon, \varepsilon] \to \mathfrak{g}$ with $\gamma(t) = \exp_G(\beta(t))$ for $|t| \leq \varepsilon$. Then $x + y = \gamma'(0) = \beta'(0)$ follows from $T_0(\exp_G) = \mathrm{id}$. By Taylor's formula $\beta(t) = t\delta(t)$ for a continuous function δ on $[-\varepsilon, \varepsilon]$ satisfying $\delta(0) = \beta'(0)$. Then

$$\beta_n(t) := n\beta\left(\frac{t}{n}\right) = t\delta\left(\frac{t}{n}\right)$$

converges uniformly on every compact subset of $\mathbb{R}$ to the curve $\eta(t) := t(x + y)$. Considering $(\mathfrak{g}, +)$ as a topological group, Lemma 3.1 implies that the sequence $\gamma_n = \exp_G \circ \beta_n$ converges uniformly on compact subsets of $\mathbb{R}$ to $\exp_G \circ \eta$. $\qquad\square$

Example 3.6. Proposition 3.5 implies that mapping groups, and in particular loop groups, have the Trotter property. If M is a compact manifold and K is a locally exponential Lie group with the Lie algebra $\mathfrak{k}$, then it follows from [Ne06, Th. IV.1.12] that, for $r \in \mathbb{N}_0 \cup \{\infty\}$, the mapping group $\mathcal{C}^r(M, K)$ is a locally exponential Lie group with Lie algebra $\mathcal{C}^r(M, \mathfrak{k})$. Note also that central extensions of locally exponential Lie groups are locally exponential by [Ne06, Th. IV.2.11].

Diffeomorphism groups

Our next goal is to prove that groups of automorphisms of principal bundles on compact smooth manifolds have the Trotter property. To this end, we begin by a closer look at the topology of these groups.

If X, Y are topological spaces, then we write $\mathcal{C}(X, Y)$ for the space of continuous maps $X \to Y$, endowed with the compact open topology. Suppose that X is locally compact. Then the group $\mathrm{Homeo}(X)$, endowed with the topology inherited from the embedding

$$\mathrm{Homeo}(X) \to \mathcal{C}(X, X)^2, \quad \varphi \mapsto (\varphi, \varphi^{-1}),$$

is a topological group ([Str06, Cor. 9.15]). If, in addition, X is a $\mathcal{C}^k$-manifold for some $k \in \mathbb{N}$, then we endow the group $\mathrm{Diff}^k(M)$ of $\mathcal{C}^k$-diffeomorphisms with the group topology inherited from the embedding

$$T^k \colon \mathrm{Diff}^k(X) \to \mathrm{Homeo}(T^k(X)), \quad \varphi \mapsto T^k(\varphi).$$

This topology is called the *modified compact open $\mathcal{C}^k$-topology*. If X is a smooth manifold, then we endow the group $\mathrm{Diff}(X) = \mathrm{Diff}^\infty(X)$ with the group topology inherited from the embedding

$$\mathrm{Diff}(X) \to \prod_{k \in \mathbb{N}_0} \mathrm{Homeo}(T^k(X)), \quad \varphi \mapsto (T^k(\varphi))_{k \in \mathbb{N}_0}$$

where $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This topology is called the *modified smooth compact open topology*.

In general, this topology does not turn $\mathrm{Diff}(X)$ into a Lie group. A typical example is the disjoint union $X = \coprod_{n \in \mathbb{N}} \mathbb{S}^1$ of infinitely many circles (in this case the group $\mathrm{Diff}(X)$ contains the compact group $\mathbb{T}^\mathbb{N}$ as a topological subgroup, and this is not compatible with a manifold structure). However, there are many important cases where topological subgroups of $\mathrm{Diff}(X)$ actually carry natural Lie group structures.

Examples 3.7. The following groups carry Lie group structures compatible with their modified smooth compact open topology.

(a) If M is a compact manifold, then $\mathrm{Diff}(M)$ carries a natural Lie group structure ([Ha82], [Mil84], [Ne06]). From the smoothness of the action of the Lie group $\mathrm{Diff}(M)$ on M, one easily derives that the modified smooth compact open topology is coarser than the Lie topology, but the construction of charts for the manifold structure on $\mathrm{Diff}(M)$ (see [Ne06, Ex. II.3.14]) easily implies that the two topologies coincide.

(b) The group $\mathrm{Aut}(P) = \mathrm{Diff}(P)^K$ of automorphisms of a principal K-bundle $q \colon P \to M$ over a compact smooth manifold M. It is a Lie group extension of the Lie group $\mathrm{Diff}(M)$ by the gauge group $\mathrm{Gau}(P)$ ([Wo07], [ACMM89]).

(c) The group $\mathrm{Aut}(\mathbb{V})$ of automorphisms of a vector bundle over a compact smooth manifold can be identified with the automorphism group of the corresponding frame bundle $\mathrm{Fr}(\mathbb{V})$, which is a principal $\mathrm{GL}(V)$-bundle. Therefore (b) applies to $\mathrm{Aut}(\mathbb{V})$.

(d) The higher tangent bundles $T^k(M)$ of a compact smooth manifold carry natural bundle structures, which, for $k > 1$, are not vector bundles. They are *multilinear bundles* (cf. [Be08, Sect. 15]). This implies that they are also associated to a principal K-bundle $q_P \colon P \to M$, where K is a finite-dimensional Lie group of polynomial diffeomorphisms of the fiber, such that $\mathrm{Aut}(T^k(M)) \cong \mathrm{Aut}(P)$. As in (c), we thus obtain on $\mathrm{Aut}(T^k(M))$ a Lie group structure from (b).

Remark 3.8. Let $\mathcal{V}(M)$ denote the space of smooth vector fields on the smooth manifold M. If $X \in \mathcal{V}(M)$ is complete, we write $(\Phi_t^X)_{t \in \mathbb{R}}$ for the corresponding smooth flow.

Then the smoothness of the flow map $\Phi^X\colon \mathbb{R} \times M \to M$ and $\Phi^X_{-t} = (\Phi^X_t)^{-1}$ implies that $\Phi^X\colon \mathbb{R} \to \mathrm{Diff}(M)$ is a continuous one-parameter group with respect to the modified smooth compact open topology. Here we use that the induced flow $T^k(\Phi^X_t) = \Phi^{T^k(X)}_t$ on $T^k(M)$ is also smooth.

If M is compact, then $\mathcal{V}(M) = \mathbf{L}(\mathrm{Diff}(M))$ is the Lie algebra of the group of all diffeomorphisms of M and the exponential function is given by the time-1-flow $\exp(X) = \Phi^X_1$. Note that the compactness of M implies that every vector field on M is complete.

Similarly, the Lie algebra $\mathfrak{aut}(P)$ of the automorphism group $\mathrm{Aut}(P)$ of a principal K-bundle over M is the Lie algebra $\mathcal{V}(P)^K$ of K-invariant vector fields. All these vector fields are complete and $\exp(X) = \Phi^X_1$ defines the exponential function $\mathfrak{aut}(P) \to \mathrm{Aut}(P)$.

Remark 3.9. (a) For the following we recall from Theorem 1 in [Bou89, Ch. 2, §4.1] that every compact space X carries a unique uniform structure compatible with the topology.

(b) We also recall that, for a locally compact space X and a uniform space Y, on $\mathcal{C}(X,Y)$ the topology of uniform convergence on compact subsets coincides with the compact open topology (cf. [Bou74, Ch. 10]).

(c) Putting both pieces together, we see that, if X is a compact space and Y a compact subset of a topological space Z, then the compact open topology on the subspace $\mathcal{C}(X,Y) \subseteq \mathcal{C}(X,Z)$ coincides with the topology of uniform convergence. This means that, for a sequence $f_n\colon X \to Z$ of continuous map for which $\bigcup_{n\in\mathbb{N}} f_n(X)$ has compact closure Y, uniform convergence in $\mathcal{C}(X,Y)$ is equivalent to convergence in $\mathcal{C}(X,Z)$ in the compact open topology.

Lemma 3.10. *Let M be a smooth manifold and $X, Y \in \mathcal{V}(M)$ be complete vector fields for which $X + Y$ is also complete. Then for each compact subset $C \subseteq M$ there exists an $\varepsilon > 0$ such that*
$$\Phi^{X+Y}_t = \lim_{n\to\infty} \left(\Phi^X_{t/n} \circ \Phi^Y_{t/n} \right)^n \tag{9}$$
holds uniformly on $|t| \leq \varepsilon$, where both sides are considered as functions from $\mathbb{R}$ into $\mathcal{C}(C, M)$, endowed with the compact open topology.

Proof. From [Nel69, §4, Th. 1] it follows that, every point $m_0 \in M$ has a neighborhood V for which there exists an $\varepsilon_V > 0$ such that $\gamma_n(t) := \left(\Phi^X_{t/n} \circ \Phi^Y_{t/n} \right)^n$ converges to Φ^{X+Y}_t uniformly on V and for $|t| \leq \varepsilon_V$. Covering C with finitely many such neighborhoods $V_1, \ldots, V_N$, we put $\varepsilon := \min\{\varepsilon_{V_1}, \ldots, \varepsilon_{V_N}\}$. For $|t| \leq \varepsilon$ and on C, both sides of (9) attain values in some compact subset $D \subseteq M$. By Remark 3.9 the topology of uniform convergence on $\mathcal{C}(C, D)$ coincides with the compact open topology induced from $\mathcal{C}(C, M)$. $\qquad\square$

Theorem 3.11. *Let K be a finite-dimensional Lie group and $q\colon P \to M$ be a smooth K-principal bundle over the compact smooth manifold M. Then the Lie group $\mathrm{Aut}(P) = \mathrm{Diff}(P)^K$ of bundle automorphisms has the Trotter property.*

Proof. Let $X, Y \in \mathfrak{aut}(P) = \mathcal{V}(P)^K$. We need to show that
$$\Phi^{X+Y}_t = \lim_{n\to\infty} \left(\Phi^X_{t/n} \circ \Phi^Y_{t/n} \right)^n \tag{10}$$

holds uniformly on compact subsets of $\mathbb{R}$ with respect to the modified smooth compact open topology on $\mathrm{Aut}(P)$. We thus have to show that, for every $r \in \mathbb{N}_0$, we have

$$\Phi_t^{T^r(X)+T^r(Y)} = \lim_{n\to\infty} \left(\Phi_{t/n}^{T^r(X)} \circ \Phi_{t/n}^{T^r(Y)} \right)^n \tag{11}$$

uniformly on compact subsets of $\mathbb{R}$ in the topological group $\mathrm{Homeo}(T^r(P))$ with respect to the modified compact open topology. Here $T^r(X)\colon T^r(P) \to T^{r+1}(P)$ denotes the canonical lift of $X \in \mathfrak{aut}(P)$ to a smooth vector field on $T^r(P)$ which generates the flow $T^r(\Phi_t^X) = \Phi_t^{T^r(X)}$ on $T^r(P)$.

From Lemma 3.10 it follows that, for every compact subset $C \subseteq P$, there exists an $\varepsilon > 0$ such that

$$\Phi_t^{X+Y}|_C = \lim_{n\to\infty} \left(\Phi_{t/n}^X \circ \Phi_{t/n}^Y \right)^n |_C$$

holds uniformly on $[-\varepsilon, \varepsilon]$ in the space $\mathcal{C}(C, P)$, endowed with the compact open topology. Since M is compact, there exists a compact subset $C \subseteq P$ whose interior C^0 satisfies $q_P(C^0) = M$. We now have $P = C^0 \cdot K$, and for every compact subset $C' \subseteq P$ there exists a finite subset $F \subseteq K$ with $C' \subseteq C \cdot F$. This implies that

$$\Phi_t^{X+Y} = \lim_{n\to\infty} \left(\Phi_{t/n}^X \circ \Phi_{t/n}^Y \right)^n$$

actually holds uniformly on $[-\varepsilon, \varepsilon]$ in $\mathcal{C}(P, P)$.

For $r \in \mathbb{N}$, let $C^r \subseteq T^r(P)$ be a compact subset which is a neighborhood of C in $T^r(P)$ (recall that $C \subseteq P \subseteq T^r(P)$). According to Lemma 3.10, there exists an $\varepsilon_r > 0$ such that

$$\Phi_t^{T^r(X)+T^r(Y)}|_{C^r} = \lim_{n\to\infty} \left(\Phi_{t/n}^{T^r(X)} \circ \Phi_{t/n}^{T^r(Y)} \right)^n |_{C^r}$$

holds uniformly on $[-\varepsilon_r, \varepsilon_r]$ in the space $\mathcal{C}(C^r, T^r(P))$, endowed with the compact open topology. As above, we see that the same statement holds with $\widetilde{C} := (C^r)^0 \cdot K$ instead of C^r (here we use the canonical lift of the K-action on P to $T^k(P)$). This is an open subset of $T^r(P)$ containing the canonical image of P. As $T^r(P)$ is associated to a principal bundle with some structure group H over P (cf. Example 3.7(d)) and the maps $T^r(\varphi)$, $\varphi \in \mathrm{Diff}(P)$, are bundle automorphisms, it follows with a similar argument as above, applied to the flows on the corresponding principal H-bundle, that actually

$$\Phi_t^{T^r(X)+T^r(Y)} = \lim_{n\to\infty} \left(\Phi_{t/n}^{T^r(X)} \circ \Phi_{t/n}^{T^r(Y)} \right)^n \tag{12}$$

holds uniformly on $[-\varepsilon_r, \varepsilon_r]$ in the space $\mathcal{C}(T^r(P), T^r(P))$, endowed with the compact open topology.

Since

$$(\Phi_t^{T^r(X)+T^r(Y)})^{-1} = \Phi_{-t}^{T^r(X)+T^r(Y)} = \lim_{n\to\infty} \left(\Phi_{-t/n}^{T^r(Y)} \circ \Phi_{-t/n}^{T^r(X)} \right)^n,$$

we can also apply the preceding argument with X and Y exchanged to see that, for some $0 < \varepsilon_r' < \varepsilon_r$, we have uniform convergence in $[-\varepsilon_r', \varepsilon_r']$ of (12) with respect to the modified compact open topology on the group $\mathrm{Homeo}(T^r(P))$. Lemma 3.2 now implies that (12) holds uniformly on compact subsets of $\mathbb{R}$. $\qquad\square$

Corollary 3.12. *If M is a compact smooth manifold, then the Lie group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M has the Trotter property.*

Direct limits

Example 3.13. If G is the direct limit of a sequence (G_n) of finite-dimensional Lie groups and injective homomorphisms $G_n \hookrightarrow G_{n+1}$, then $\mathbf{L}(G)$ is isomorphic as a topological Lie algebra to the corresponding direct limit of finite-dimensional Lie algebras $\mathbf{L}(G_n)$ and G has the Trotter property (cf. [Gl05, Prop. 4.6]).

Semidirect products

Let V be a complete locally convex space, G a Lie group with a smooth exponential function and $\alpha\colon G \to \mathrm{GL}(V)$ be a homomorphism defining a smooth action of G on V, so that we can form the semidirect product Lie group $V \rtimes_\alpha G$. This Lie group has a smooth exponential function, given explicitly by

$$\exp_{V \rtimes_\alpha G}(v, x) = \big(\beta(x)v, \exp_G(x)\big) \quad \text{with} \quad \beta(x) = \int_0^1 \alpha(\exp_G(sx))\, ds$$

([Ne06, Ex. II.5.9]).

Proposition 3.14. *If G has the Trotter property, then so does $H := V \rtimes_\alpha G$.*

Proof. Let $X = (u, x), Y = (w, y) \in \mathbf{L}(H) \cong V \rtimes \mathfrak{g}$, $z := x + y$, and consider the smooth curves

$$\Gamma(t) := \exp_H(tX) \exp_H(tY) \quad \text{and} \quad \Gamma_n(t) := \Gamma(t/n)^n$$

in H. In terms of the semidirect product structure, we write $\Gamma(t) = (\delta(t), \gamma(t))$ where

$$\delta(t) = t\beta(tx)u + \alpha(\exp_G(tx))t\beta(ty)w \ \text{ and } \ \gamma(t) = \exp_G(tx)\exp_G(ty).$$

If G has the Trotter property, then $\gamma_n(t) = \gamma(t/n)^n$ converges uniformly on each compact interval in $\mathbb{R}$ to $\exp_G(tz)$, the second component of $\exp_H(t(X + Y))$. To see what happens in the first component, we note that

$$\gamma_n(t) = \Big(\delta(t/n), \gamma(t/n)\Big)^n = \Big(B_n(t)n\delta(t/n), \gamma(t/n)^n\Big),$$

where

$$B_n(t) = \frac{1}{n}\Big(\mathbf{1} + \alpha(\gamma(t/n)) + \cdots + \alpha(\gamma(t/n))^{n-1}\Big).$$

Next we observe that $\delta\colon \mathbb{R} \to V$ is a smooth curve with $\delta(0) = 0$ and $\delta'(0) = u + w$. Therefore $n\delta(t/n)$ converges uniformly on compact subsets of $\mathbb{R}$ to $t(u + w)$. It therefore suffices to show that, on compacts subsets $C_V \subseteq V$, we have

$$B_n(t) \to \beta(t(x + y)) = \beta(tz) = \int_0^1 \alpha(\exp_G(stz))\, ds.$$

Fix $T > 0$. Writing

$$B_n(t)v - \beta(tz)v = B_n(t)v - \int_0^1 \alpha(\exp_G(stz))\,ds$$

$$= \frac{1}{n}\Big(\sum_{j=0}^{n-1} \alpha(\gamma(t/n))^j v - \alpha\Big(\exp_G\Big(\frac{jt}{n}z \Big) \Big) v \Big)$$

$$+ \frac{1}{n} \sum_{j=0}^{n-1} \alpha\Big(\exp_G\Big(\frac{jt}{n}z \Big) \Big) v - \int_0^1 \alpha(\exp_G(stz))v\,ds,$$

we see that the second summand converges to 0 because it describes a Riemann sum approximation of the integral $\beta(tz)v$, which converges uniformly for $|t| \leq T$ and $v \in C_V$.

By Lemma 2.2, for every compact subset $C \subseteq V$, the set

$$\{\alpha(\gamma_n(t))v \colon v \in C, |t| \leq T, n \in \mathbb{N}\}$$

has compact closure $\widetilde{C}$ which contains also the elements $\alpha(\exp_G(tz))v$, $v \in C$, $|t| \leq T$.

Now let $p \colon V \to \mathbb{R}$ be a continuous seminorm and $C \subseteq V$ be a compact subset. Then for every $\varepsilon > 0$ there exists a **1**-neighborhood $U = U_{\varepsilon,C} \subseteq G$ such that $p(\alpha(g)c - c) \leq \varepsilon$ for $g \in U_{\varepsilon,C}$, $c \in \widetilde{C}$.

Since $\gamma_n(t) \to \exp_G(tz)$ uniformly on $[-T, T]$, for each **1**-neighborhood $U \subseteq G$ there exists an element $N_U \in \mathbb{N}$ such that

$$\gamma_n(t) \in U \exp_G(tz) \quad \text{for} \quad n \geq N_U, |t| \leq T.$$

For $0 \leq j < n$ we then have

$$\gamma(t/n)^j = \gamma(tj/nj)^j = \gamma_j(tj/n) \in U \exp_G\Big(\frac{tj}{n}z \Big) \quad \text{for} \quad j \geq N_U.$$

For $j \geq N_U$ we thus obtain for $v \in C$ the estimate

$$p\Big(\alpha(\gamma(t/n))^j v - \alpha(\exp_G\big(\frac{jt}{n}z \big))v \Big) \leq \varepsilon$$

and therefore

$$p\Big(\frac{1}{n}\Big(\sum_{j=0}^{n-1} \alpha(\gamma(t/n))^j v - \alpha\Big(\exp_G\Big(\frac{jt}{n}z \Big) \Big) v \Big) \leq \frac{N_U}{n} 2 \sup p(\widetilde{C}) + \frac{n - N_U}{n}\varepsilon < 2\varepsilon$$

if n is sufficiently large. This proves that $B_n(t)v \to \beta(tz)v$ uniformly for $|t| \leq T$ and $v \in C$ and hence that $H = V \rtimes_\alpha G$ has the Trotter property. $\square$

Regular Lie groups

For the definition and properties of C^0-regularity, which are used in the next theorem and its proof, see Appendix B.

Theorem 3.15. *Suppose that G is a C^0-regular Lie group with the Trotter property and that $q \colon \widehat{G} \to G$ is a central extension by a C^0-regular abelian Lie group Z with Lie algebra $\mathfrak{z}$ (f.i., $Z = \mathfrak{z}/\Gamma$ where $\mathfrak{z}$ is complete and $\Gamma \subseteq \mathfrak{z}$ is a discrete subgroup). Then $\widehat{G}$ also has the Trotter property.*

Proof. Since C^0-regularity is an extension property by Theorem B.7, the group $\widehat{G}$ is C^0-regular. We want to show that $\widehat{G}$ also has the Trotter property. So let $\widehat{x}, \widehat{y} \in \widehat{\mathfrak{g}}$, $x := \mathbf{L}(q)\widehat{x}$ and $y := \mathbf{L}(q)\widehat{y}$. We also put $z := x + y$ and $\widehat{z} := \widehat{x} + \widehat{y}$. We consider the smooth curves

$$\widehat{\gamma}(t) := \exp_{\widehat{G}}(t\widehat{x})\exp_{\widehat{G}}(t\widehat{y}) \quad \text{and} \quad \gamma(t) := \exp_G(tx)\exp_G(ty) = q(\widehat{\gamma}(t)).$$

To see that $\widehat{\gamma}_n(t) := \widehat{\gamma}(t/n)^n$ converges uniformly on compact subsets of $\mathbb{R}$ to the curve $\exp_{\widehat{G}}(t(\widehat{x} + \widehat{y}))$, it suffices to show that the corresponding sequence of (left) logarithmic derivatives $\delta(\widehat{\gamma}_n)$ converges uniformly on compact subsets to the constant curve $\widehat{x} + \widehat{y}$.

From the product rule $\delta(\alpha\beta) = \delta(\beta) + \mathrm{Ad}(\beta)^{-1}\delta(\alpha)$ it follows that

$$\delta(\widehat{\gamma}_n) = \frac{1}{n}\delta(\widehat{\gamma})_{t/n} + \mathrm{Ad}(\widehat{\gamma}(t/n))^{-1}\frac{1}{n}\delta(\widehat{\gamma})_{t/n} + \cdots + \mathrm{Ad}(\widehat{\gamma}(t/n))^{-n+1}\frac{1}{n}\delta(\widehat{\gamma})_{t/n}$$

$$= \frac{1}{n}\Big(\sum_{j=0}^{n-1}\mathrm{Ad}(\widehat{\gamma}(t/n))^{-j}\delta(\widehat{\gamma})_{t/n}\Big).$$

The adjoint action of $\widehat{G}$ on its Lie algebra $\widehat{\mathfrak{g}}$ factors through an action $\widehat{\mathrm{Ad}} \colon G \to \mathrm{Aut}(\widehat{\mathfrak{g}})$, so that

$$\delta(\widehat{\gamma}_n) = \frac{1}{n}\Big(\sum_{j=0}^{n-1}\widehat{\mathrm{Ad}}(\gamma(t/n))^{-j}\delta(\widehat{\gamma})_{t/n}\Big).$$

We also obtain from the product rule that

$$\delta(\widehat{\gamma})_t = \widehat{y} + \mathrm{Ad}(\exp_{\widehat{G}}(t\widehat{y}))^{-1}\widehat{x} = \widehat{y} + \widehat{\mathrm{Ad}}(\exp_G(-ty))\widehat{x},$$

so that

$$\delta(\widehat{\gamma})_{t/n} \to \widehat{x} + \widehat{y} = \widehat{z}$$

holds uniformly on compact intervals of $\mathbb{R}$.

As $\gamma_n(t) \to \exp_G(tz)$ uniformly on compact intervals in $\mathbb{R}$, we obtain as in the proof of Proposition 3.14 that

$$\lim_{n\to\infty}\frac{1}{n}\Big(\sum_{j=0}^{n-1}\widehat{\mathrm{Ad}}(\gamma(t/n))^{-j}\delta(\widehat{\gamma})_{t/n}\Big) = \lim_{n\to\infty}\frac{1}{n}\Big(\sum_{j=0}^{n-1}\widehat{\mathrm{Ad}}(\gamma(t/n))^{-j}\widehat{z}\Big)$$

$$= \lim_{n\to\infty}\frac{1}{n}\Big(\sum_{j=0}^{n-1}\widehat{\mathrm{Ad}}\big(\exp_G(\frac{-jt}{n}z)\big)\widehat{z}\Big) = \lim_{n\to\infty}\frac{1}{n}\Big(\sum_{j=0}^{n-1}\widehat{z}\Big) = \widehat{z}.$$

This means that $\delta(\widehat{\gamma}_n) \to \widehat{z}$ in the space $\mathcal{C}(\mathbb{R}, \widehat{\mathfrak{g}})$. Since $\widehat{G}$ is C^0-regular, this implies that

$$\widehat{\gamma}_n = \mathrm{Evol}_{\widehat{G}}(\delta(\widehat{\gamma}_n)) \to \mathrm{Evol}_{\widehat{G}}(\widehat{z}),$$

which is the Trotter–Formula. $\square$

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Theorem 3.16. (Glöckner; [Gl12b]; see [OMYK82] for the compact case) *For every finite-dimensional smooth manifold M, the group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms is C^0-regular.*

Combining Glöckner's theorem with Corollary 3.12 and Theorem 3.15, we obtain:

Corollary 3.17. *Every central extension of a diffeomorphism group $\mathrm{Diff}(M)$ of a compact smooth manifold by a finite-dimensional center has the Trotter property. In particular, the Virasoro group Vir has the Trotter property.*

4 $\mathcal{C}^k$-vectors

In this section, G denotes a Lie group with a smooth exponential function

$$\exp_G \colon \mathbf{L}(G) = \mathfrak{g} \to G.$$

Definition 4.1. Let (π, V) be a representation of the Lie group G (with a smooth exponential function) on the locally convex space V.

(a) We say that π is *continuous* if the action of G on V defined by $(g, v) \mapsto \pi(g)v$ is continuous.

(b) An element $v \in V$ is a $\mathcal{C}^k$-vector, $k \in \mathbb{N}_0 \cup \{\infty\}$, if the orbit map $\pi^v \colon G \to V, g \mapsto \pi(g)v$ is a $\mathcal{C}^k$-map. We write $V^k := V^k(\pi)$ for the linear subspace of $\mathcal{C}^k$-vectors and we say that the representation π is *smooth* if the space V^∞ of smooth vectors is dense. A vector $v \in V$ is called an *analytic* vector if the orbit map π^v is analytic. The space of analytic vectors is denoted by V^ω.

(c) For each $x \in \mathfrak{g}$, we write

$$\mathcal{D}_x := \left\{ v \in V \colon \frac{d}{dt}\Big|_{t=0} \pi(\exp_G(tx))v \ \ \text{exists} \right\}$$

for the domain of the infinitesimal generator

$$\overline{\mathrm{d}\pi}(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp_G(tx))v$$

of the one-parameter group $\pi(\exp_G(tx))$, $\mathcal{D}^1 := \bigcap_{x \in \mathfrak{g}} \mathcal{D}_x$ and $\omega_v(x) := \overline{\mathrm{d}\pi}(x)v$ for $v \in \mathcal{D}^1$. Each $\mathcal{D}_x$ and therefore also $\mathcal{D}^1$ are linear subspaces of V, but at this point we do not know whether ω_v is linear (cf. [Ne10, Thm. 8.2] for a positive answer for Banach–Lie groups).

(d) We define inductively

$$\mathcal{D}^n := \{v \in \mathcal{D}^1 \colon (\forall x \in \mathfrak{g}) \, \overline{\mathrm{d}\pi}(x)v \in \mathcal{D}^{n-1}\} \quad \text{for} \quad n > 1,$$

so that

$$\omega_v^n(x_1, \ldots, x_n) := \overline{\mathrm{d}\pi}(x_1) \cdots \overline{\mathrm{d}\pi}(x_n)v$$

is defined for $v \in \mathcal{D}^n$ and $x_1, \ldots, x_n \in \mathfrak{g}$. We further put $\mathcal{D}^\infty := \bigcap_{n \in \mathbb{N}} \mathcal{D}^n$.

Remark 4.2. (a) For every representation (π, V) we have

$$V^1(\pi) \subseteq \mathcal{D}^1 \quad \text{and} \quad V^k(\pi) \subseteq \mathcal{D}^k \quad \text{for} \quad k \in \mathbb{N}.$$

Note that ω_v^k is continuous and k-linear for every $v \in V^k(\pi)$.

(b) By definition, we have $\mathcal{D}^2 \subseteq \mathcal{D}^1$, so that we obtain by induction that $\mathcal{D}^{n+1} \subseteq \mathcal{D}^n$ for every $n \in \mathbb{N}$.

(c) If $v \in V^k(\pi)$ then from the continuity of the action $G \times V \to V$ it follows that the map $G \times \mathfrak{g}^k \to V$, $(g, x_1, \ldots, x_k) \mapsto \pi(g)\overline{\mathrm{d}\pi}(x_1) \cdots \overline{\mathrm{d}\pi}(x_k)v$ is continuous.

The following lemma ([Ne10, Lemma 3.3]) provides a criterion for $\mathcal{C}^1$-vectors.

Lemma 4.3. *Suppose that (π, V) is a continuous representation of the Lie group G on V. Then a vector $v \in \mathcal{D}^1$ is a $\mathcal{C}^1$-vector if and only if the following two conditions are satisfied:*

(i) *For every smooth curve $\gamma \colon [-\varepsilon, \varepsilon] \to G$, $\varepsilon > 0$, with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$, the derivative $\frac{d}{dt}|_{t=0}\pi(\gamma(t))v$ exists and equals $\overline{\mathrm{d}\pi}(x)v$.*

(ii) *$\omega_v \colon \mathfrak{g} \to V, x \mapsto \overline{\mathrm{d}\pi}(x)v$ is continuous.*

If G is locally exponential, then (i) *follows from* (ii).

Definition 4.4. A Lie group G is called *locally m-exponential* if there exist closed subspaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_m \subseteq \mathfrak{g} = \mathbf{L}(G)$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ is a topological direct sum and the map

$$M \colon \mathfrak{g} \to G, \quad M(x_1 + \cdots + x_m) := \exp_G(x_1) \cdots \exp_G(x_m)$$

is a local diffeomorphism in a neighborhood of $(0, \ldots, 0)$.

The following lemma extends the implication (ii) $\Rightarrow$ (i) of Lemma 4.3 to the larger class of locally m-exponential groups.

Lemma 4.5. *Suppose that (π, V) is a continuous representation of the locally m-exponential Lie group G on V. Let $v \in \mathcal{D}^1$ be such that $\omega_v \colon \mathfrak{g} \to V, x \mapsto \overline{\mathrm{d}\pi}(x)v$ is continuous and linear. Then, for every smooth curve $\gamma \colon [-\varepsilon, \varepsilon] \to G$ with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$, the derivative $\frac{d}{dt}|_{t=0}\pi(\gamma(t))v$ exists and equals $\overline{\mathrm{d}\pi}(x)v$. In particular, v is a $\mathcal{C}^1$-vector.*

Proof. Any smooth curve γ with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$ can be written for sufficiently small values of $t \in \mathbb{R}$ as $\exp_G(\eta_1(t)) \cdots \exp_G(\eta_m(t))$ with smooth curves $\eta_j \colon [-\varepsilon, \varepsilon] \to \mathfrak{g}$ satisfying $\eta_j(0) = 0$ and $\sum_j \eta_j'(0) = x$. From the proof of [Ne10, Lem. 3.3] we know that for every smooth curve $\eta \colon [-\varepsilon, \varepsilon] \to \mathfrak{g}$ satisfying $\eta(0) = 0$ we have

$$\frac{d}{ds}\bigg|_{s=0} \pi(\exp_G \eta(s))v = \omega_v(\eta'(0)).$$

For two smooth curves $\gamma_j \colon [-\varepsilon, \varepsilon] \to G$ with $\gamma_j(0) = \mathbf{1}$ we have

$$\pi(\gamma_1(t)\gamma_2(t))v - v = \pi(\gamma_1(t))\big(\pi(\gamma_2(t))v - v\big) + \big(\pi(\gamma_1(t))v - v\big),$$

and since G acts continuously on V, we obtain

$$\frac{d}{dt}\bigg|_{t=0} \pi(\gamma_1(t)\gamma_2(t))v = \frac{d}{dt}\bigg|_{t=0} \pi(\gamma_1(t))v + \frac{d}{dt}\bigg|_{t=0} \pi(\gamma_2(t))v.$$

By induction this leads to

$$\frac{d}{dt}\Big|_{t=0} \pi(\gamma(t))v = \sum_{j=1}^{m} \frac{d}{dt}\Big|_{t=0} \pi(\exp_G \eta_j(t))v = \sum_{j=1}^{m} \omega_v(\eta_j'(0)) = \omega_v\Big(\sum_{j=1}^{m} \eta_j'(0)\Big) = \omega_v(x). \ \square$$

As an immediate consequence of Theorem 2.7, we obtain:

Theorem 4.6. *Let (π, V) be a continuous representation of the Lie group G with the Trotter property on the locally convex space V. Then, for each $v \in \mathcal{D}^1$, the map*

$$\omega_v \colon \mathfrak{g} \to V, \quad v \mapsto \overline{\mathrm{d}\pi}(x)v$$

is linear.

Proof. For $x \in \mathfrak{g}$ and the continuous orbit map $\pi^v \colon G \to V$, $\pi^v(g) := \pi(g)v$ we have $D_x \pi^v = \pi^{\overline{\mathrm{d}\pi}(x)v}$, which exists and is continuous for $x \in \mathfrak{g}$ and $v \in \mathcal{D}^1$. Hence the assertion follows from Theorem 2.7. $\square$

Lemma 4.7. ([Ne10, Lemma 3.4]) *If G is locally exponential, then a vector $v \in V$ is a C^k-vector if and only if $v \in \mathcal{D}^k$ and the maps ω_v^n, $n \leq k$, are continuous and n-linear. In particular, v is a smooth vector if and only if $v \in \mathcal{D}^\infty$ and all the maps ω_v^n are continuous and n-linear.*

Lemma 4.8. *Let $V_0, V_1, \ldots, V_n$ be topological vector spaces and*

$$\omega \colon V_1 \times \cdots \times V_n \to V_0$$

be an n-linear map. If ω is continuous in some point $(v_1^0, \ldots, v_n^0)$, then ω is continuous.

Proof. For $n = 1$ this is obvious. If $n > 1$ then we can write

$$\omega(v_1 - v_1^0, \ldots, v_n - v_n^0) = \omega(v_1, \ldots, v_n) \pm \cdots + (-1)^n \omega(v_1^0, \ldots, v_n^0).$$

Each of the summands on the right hand side is a function of $(v_1, \ldots, v_n)$. Continuity of ω at $(v_1^0, \ldots, v_n^0)$ implies that all of these summands are continuous at $(v_1^0, \ldots, v_n^0)$. We conclude that ω is continuous at $(0, \ldots, 0)$, and it is well-known that this implies the continuity of ω. $\square$

Lemma 4.9. *Let (π, V) be a continuous representation of the Fréchet–Lie group G with the Trotter property on the metrizable locally convex space V. Then, for each $v \in \mathcal{D}^n$, the map*

$$\omega_v^n \colon \mathfrak{g}^n \to V, \quad (x_1, \ldots, x_n) \mapsto \overline{\mathrm{d}\pi}(x_1)\overline{\mathrm{d}\pi}(x_2) \cdots \overline{\mathrm{d}\pi}(x_n)v$$

is continuous and n-linear.

Proof. Theorem 4.6 implies that ω_v^n is n-linear. We argue by induction on $n \in \mathbb{N}_0$ that it is continuous. For $n = 0$ this is trivial. Now we assume $n > 0$ and that $\omega_w^{n-1} \colon \mathfrak{g}^{n-1} \to V$ is a continuous $(n-1)$-linear map for every $w \in \mathcal{D}^{n-1}$.

Hence, for $t > 0$, the maps $F_t \colon \mathfrak{g}^n \to V$, defined by

$$F_t(x_1, x_2, \ldots, x_n) := \frac{1}{t}\Big(\pi(\exp_G(tx_1))\omega_v^{n-1}(x_2, \ldots, x_n) - \omega_v^{n-1}(x_2, \ldots, x_n)\Big)$$

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are continuous and satisfy

$$\lim_{n \to \infty} F_{\frac{1}{n}}(x_1, x_2, \ldots, x_n) = \omega_v^n(x_1, \ldots, x_n).$$

Since V is metrizable and $\mathfrak{g}^n$ is a Baire space, it follows from [Bou74, Ch. IX, §5, Ex. 22(a)] that the set of discontinuity points of ω_v^n is of the first category, hence not all of $\mathfrak{g}^n$. We conclude that there exists a point in which ω_v^n is continuous, so that its continuity follows from Lemma 4.8. $\square$

Theorem 4.10. *Let (π, V) be a continuous representation of the locally exponential Fréchet–Lie group G on the metrizable locally convex space V. Then $\mathcal{D}^k$ coincides with the space of $\mathcal{C}^k$-vectors for any $k \in \mathbb{N} \cup \{\infty\}$.*

Proof. It only remains to combine Lemma 4.7 with Lemma 4.9. $\square$

Remark 4.11. The preceding theorem generalizes the corresponding Banach results from [Ne10] considerably with substantially simpler proofs.

5 Differentiable vectors for unitary representations

The main goal of this section is to prove Theorem 5.3 which, for unitary representations, extends Theorem 4.10 to groups with the Trotter property.

In this section G is a Lie group with a smooth exponential function. We shall see in this section, that we can weaken the assumption of local exponentiality in Theorem 4.10 for unitary representations.

Definition 5.1. Let $(\pi, \mathcal{H})$ be a unitary representation of the Lie group G. We say that π is:

(i) *differentiable* if the space $\mathcal{H}^1$ of $\mathcal{C}^1$-vectors is dense in $\mathcal{H}$.

(ii) *smooth* if the space $\mathcal{H}^\infty$ of smooth vectors is dense in $\mathcal{H}$.

(iii) *analytic* if the space $\mathcal{H}^\omega$ of analytic vectors is dense in $\mathcal{H}$.

The following lemma supplements the general Lemma 4.3. It implies in particular, that for unitary representations with a dense space of $\mathcal{C}^1$-vectors, the $\mathcal{C}^1$-vectors can be characterized in terms of one-parameter groups.

Lemma 5.2. *Let $(\pi, \mathcal{H})$ be a differentiable unitary representation of the Lie group G. If $v \in \mathcal{D}^1$ is such that the map $\omega_v \colon \mathfrak{g} \to \mathcal{H}, x \mapsto \overline{\mathrm{d}\pi}(x)v$ is continuous, then $v \in \mathcal{H}^1$.*

Proof. Let $\gamma \colon [-\varepsilon, \varepsilon] \to G$ be a smooth curve with $\gamma(0) = \mathbf{1}$ and $\gamma'(0) = x$, and

$$\xi : [-\varepsilon, \varepsilon] \to \mathfrak{g} , \ \xi(t) := \mathrm{d}\ell_{\gamma(t)^{-1}}(\gamma(t))(\gamma'(t))$$

be its left logarithmic derivative. For $w \in \mathcal{H}^1$ we then obtain the relation

$$\frac{d}{dt} \pi(\gamma(t)^{-1})w = -\overline{\mathrm{d}\pi}(\xi(t))\pi(\gamma(t)^{-1})w$$

and hence

$$\frac{d}{dt}\langle\pi(\gamma(t))v,w\rangle = -\langle v,\overline{\mathbf{d}\pi}(\xi(t))\pi(\gamma(t)^{-1})w\rangle = \langle\pi(\gamma(t))\overline{\mathbf{d}\pi}(\xi(t))v,w\rangle. \tag{13}$$

Set $\alpha(t) := \pi(\gamma(t))v$ and $\alpha_w(t) := \langle\alpha(t),w\rangle$. The curve $\beta(t) := \pi(\gamma(t))\overline{\mathbf{d}\pi}(\xi(t))v$ is also continuous because ω_v is continuous, and the action of G on $\mathcal{H}$ defined by π is continuous. By (13), for each $w \in \mathcal{H}^1$ the function α_w is differentiable and $\alpha'_w(t) = \beta_w(t) := \langle\beta(t),w\rangle$. Since β_w is continuous, it follows that

$$\langle\int_0^t \beta(\tau)\,d\tau,w\rangle = \int_0^t \beta_w(\tau)\,d\tau = \alpha_w(t) - \alpha_w(0) = \langle\alpha(t) - \alpha(0),w\rangle.$$

Since $\mathcal{H}^1$ is assumed to be dense in $\mathcal{H}$, we obtain $\alpha(t) = \alpha(0) + \int_0^t \beta(\tau)\,d\tau$. Now the continuity of β shows that α is $\mathcal{C}^1$ with $\alpha'(0) = \beta(0) = \overline{\mathbf{d}\pi}(x)v$. Finally Lemma 4.3 shows that v is a $\mathcal{C}^1$-vector. $\qquad\square$

Recall that $\mathcal{H}^n$ denotes the space of $\mathcal{C}^n$-vectors of $(\pi,\mathcal{H})$.

Theorem 5.3. *Let $(\pi,\mathcal{H})$ be a differentiable unitary representation of the Fréchet–Lie group G with the Trotter property. Then $\mathcal{D}^n = \mathcal{H}^n$ for every $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. Let $n \in \mathbb{N}$ and $v \in \mathcal{D}^n$. By Lemma 4.9, the map

$$\omega_v^n : \mathfrak{g}^n \to \mathcal{H}, \quad (x_1,\ldots,x_n) \mapsto \overline{\mathbf{d}\pi}(x_1)\cdots\overline{\mathbf{d}\pi}(x_n)v$$

is continuous and n-linear. For $n = 1$, Lemma 5.2 implies that v is a $\mathcal{C}^1$-vector. For $n > 1$, the inductive argument for the proof of [Ne10, Lem. 3.4] works without change. For the reader's convenience we give the details.

Let $n > 1$ and $v \in \mathcal{D}^n$. Recall that $\pi^v : G \to \mathcal{H}$ denotes the orbit map of v, that is, $\pi^v(g) := \pi(g)v$. Since $v \in \mathcal{D}^1 = \mathcal{H}^1$, by induction hypothesis the map

$$T(\pi^v) : T(G) \simeq G \times \mathfrak{g} \to \mathcal{H}\ ,\ (g,x) \mapsto T(\pi^v)(g,x)$$

is well defined. It remains to prove that the latter map is $\mathcal{C}^{n-1}$. Lemma 4.3 implies that $T(\pi^v)(g,x) = \pi(g)\overline{\mathbf{d}\pi}(x)v = \pi^w(g)$ where $w := \overline{\mathbf{d}\pi}(x)v \in \mathcal{D}^{n-1}$. By induction hypothesis, $w \in \mathcal{H}^{n-1}$. Thus, a direct calculation shows that the map $T(\pi^v)$ has directional derivatives of order $j \leq n-1$ and they are sums of terms of the form $\pi(g)\omega_v^j(x_1,\ldots,x_j)w$ for $j \leq n-1$. From Remark 4.2(c) it follows that $T(\pi^v)$ is a $\mathcal{C}^{n-1}$ map. $\qquad\square$

6 Unitary representations of Lie supergroups

We now apply Theorem 5.3 to unitary representations of Lie supergroups $(G,\mathfrak{g})$. We begin by recalling the definition of Lie supergroups and their unitary representations. See [CCTV06] and [MNS11] for further details.

By a *locally convex Lie superalgebra* we mean a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ over $\mathbb{R}$ or $\mathbb{C}$ with the following two properties.

(i) $\mathfrak{g}$ is a locally convex space and the Lie superbracket is continuous.

(ii) $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is a topological direct sum, i.e., the involution $\sigma \colon \mathfrak{g} \to \mathfrak{g}$ defined by $\sigma(x_0 + x_1) = x_0 - x_1$ for $x_0 \in \mathfrak{g}_{\overline{0}}$ and $x_1 \in \mathfrak{g}_{\overline{1}}$ is continuous.

The parity of a homogeneous element $x \in \mathfrak{g}$ is denoted by $p(x) \in \{\overline{0}, \overline{1}\}$.

Definition 6.1. A *Lie supergroup* is an ordered pair $(G, \mathfrak{g})$ with the following properties.

(i) G is a Lie group, modeled on a locally convex space.

(ii) $\mathfrak{g}$ is a locally convex Lie superalgebra over $\mathbb{R}$.

(iii) $\mathfrak{g}_{\overline{0}} = \mathbf{L}(G)$ is the Lie algebra of G.

(iv) There exists a homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ defining a smooth action $G \times \mathfrak{g} \to \mathfrak{g}$ by even automorphisms of $\mathfrak{g}$. For every $x \in \mathfrak{g}_{\overline{0}}$ and $y \in \mathfrak{g}$, we have $\mathsf{d}\,\mathrm{Ad}_y(\mathbf{1})(x) = [x, y]$ where $\mathrm{Ad}_y : G \to \mathfrak{g}$ is defined by $\mathrm{Ad}_y(g) := \mathrm{Ad}(g)y$.

We refer to the homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ of Definition 6.1(iv) as the *adjoint action* of G on $\mathfrak{g}$. If G is an analytic Lie group and the adjoint action of G on $\mathfrak{g}$ defines an analytic map $G \times \mathfrak{g} \to \mathfrak{g}$, then we call the Lie supergroup $(G, \mathfrak{g})$ *analytic*.

Definition 6.2. Let $(G, \mathfrak{g})$ be a Lie supergroup. A *smooth unitary representation* of $(G, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ satisfying the following properties.

(SR1) $(\pi, \mathcal{H})$ is a smooth unitary representation of G on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ such that, for every $g \in G$, the operator $\pi(g)$ preserves the $\mathbb{Z}_2$-grading.

(SR2) For $\mathcal{B} := \mathcal{H}^\infty$, $\rho^\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{B})$ is a representation of the Lie superalgebra $\mathfrak{g}$.

(SR3) $\rho^\pi(x) = \mathsf{d}\pi(x)\big|_{\mathcal{B}}$ for every $x \in \mathfrak{g}_{\overline{0}}$.

(SR4) $e^{-\frac{\pi i}{4}}\rho^\pi(x)$ is a symmetric operator for every $x \in \mathfrak{g}_{\overline{1}}$, i.e., $-i\rho^\pi(x) \subseteq \rho^\pi(x)^*$.

(SR5) Every element of the component group G/G° has a coset representative $g \in G$ such that $\pi(g)\rho^\pi(x)\pi(g)^{-1} = \rho^\pi(\mathrm{Ad}(g)x)$ for every $x \in \mathfrak{g}_{\overline{1}}$.

If $(G, \mathfrak{g})$ is an analytic Lie supergroup, then an *analytic unitary representation* of $(G, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$, where $(\pi, \mathcal{H})$ is an analytic unitary representation of G on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ such that, for every $g \in G$, the operator $\pi(g)$ preserves the $\mathbb{Z}_2$-grading and the other conditions above hold for $\mathcal{B} := \mathcal{H}^\omega$.

Remark 6.3. If G is connected, then obviously (SR5) holds trivially. This point is the main difference between Definition 6.2 above and the definition given in [CCTV06, Def. 2] for finite-dimensional Lie groups, where it is assumed that

$$\pi(g)\rho^\pi(x)\pi(g)^{-1} = \rho^\pi(\mathrm{Ad}(g)x) \quad \text{for every } x \in \mathfrak{g}_{\overline{1}} \text{ and every } g \in G, \tag{14}$$

while the infinitesimal action is supposed to satisfy a weaker condition. Indeed Proposition 6.13 below implies that for a (possibly disconnected) G equation (14) follows from Definition 6.2.

We will need a slightly more general notion than smooth and analytic unitary representations, which we introduce in the next definition (see [MNS11, Def. 4.2]).

Definition 6.4. Let $(G, \mathfrak{g})$ be a Lie supergroup. A *pre-representation* of $(G, \mathfrak{g})$ is a 4-tuple $(\pi, \mathcal{H}, \mathcal{B}, \rho^{\mathcal{B}})$ which satisfies the following properties.

(PR1) $(\pi, \mathcal{H})$ is a smooth unitary representation of G on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_{\overline{0}} \oplus \mathcal{H}_{\overline{1}}$. Moreover, $\pi(g)$ is an even operator for every $g \in G$.

(PR2) $\mathcal{B}$ is a dense $\mathbb{Z}_2$-graded subspace of $\mathcal{H}$ contained in $\mathcal{D}^1 := \bigcap_{x \in \mathfrak{g}_{\overline{0}}} \mathcal{D}(\overline{\mathrm{d}\pi}(x))$.

(PR3) $\rho^{\mathcal{B}} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{B})$ is a representation of the Lie superalgebra $\mathfrak{g}$.

(PR4) If $x \in \mathfrak{g}_{\overline{0}}$ then $\rho^{\mathcal{B}}(x) = \overline{\mathrm{d}\pi}(x)\big|_{\mathcal{B}}$ and $\rho^{\mathcal{B}}(x)$ is essentially skew-adjoint.

(PR5) If $x \in \mathfrak{g}_{\overline{1}}$ then $e^{-\frac{\pi i}{4}} \rho^{\mathcal{B}}(x)$ is a symmetric operator, i.e., $-i\rho^{\mathcal{B}}(x) \subseteq \rho^{\mathcal{B}}(x)^*$.

(PR6) For every element of the component group G/G°, there exists a coset representative $g \in G$ such that $\pi(g)^{-1}\mathcal{B} \subseteq \mathcal{B}$ and

$$\pi(g)\rho^{\mathcal{B}}(x)\pi(g)^{-1} = \rho^{\mathcal{B}}(\mathrm{Ad}(g)x) \text{ for every } x \in \mathfrak{g}_{\overline{1}}.$$

In the following we write

$$\mathcal{D}^\infty := \bigcap_{n \in \mathbb{N}} \mathcal{D}^n \quad \text{and} \quad \mathcal{D}^n := \bigcap_{x_1, \dots, x_n \in \mathfrak{g}_{\overline{0}}} \mathcal{D}(\overline{\mathrm{d}\pi}(x_1) \cdots \overline{\mathrm{d}\pi}(x_n)).$$

The set of smooth (resp., analytic) vectors of the unitary representation $(\pi, \mathcal{H})$ is denoted by $\mathcal{H}^\infty$ (resp., $\mathcal{H}^\omega$).

Remark 6.5. (i) Observe that in (PR3) there are no continuity assumptions on the map $\rho^{\mathcal{B}}$.
  (ii) (PR2/3) imply that $\mathcal{B} \subseteq \mathcal{D}^\infty$. In addition, Theorem 5.3 asserts that, if G is a Fréchet–Lie group with the Trotter property, then $\mathcal{H}^\infty = \mathcal{D}^\infty$, so that we obtain $\mathcal{B} \subseteq \mathcal{H}^\infty$.

The following lemma is stated in [MNS11, Lemma 4.4] for Banach–Lie supergroups, but is remains true with the same proof in the general context.

Lemma 6.6. *Let $(G, \mathfrak{g})$ be a Lie supergroup and $(\pi, \mathcal{H}, \mathcal{B}, \rho^{\mathcal{B}})$ be a pre-representation of $(G, \mathfrak{g})$. Then the following assertions hold.*

  (i) *For every $x \in \mathfrak{g}_{\overline{0}}$, we have $\overline{\rho^{\mathcal{B}}(x)} = \overline{\mathrm{d}\pi}(x)$. In particular $\mathcal{D}^\infty \subseteq \mathcal{D}^1 \subseteq \mathcal{D}(\overline{\rho^{\mathcal{B}}(x)})$.*

  (ii) *For every $x \in \mathfrak{g}_{\overline{1}}$, the operator $e^{-\frac{\pi i}{4}} \rho^{\mathcal{B}}(x)$ is essentially self-adjoint and $\overline{\rho^{\mathcal{B}}(x)}^2 = \frac{1}{2}\overline{\mathrm{d}\pi}([x, x])$. In particular $\mathcal{D}^\infty \subseteq \mathcal{D}^1 \subseteq \mathcal{D}(\overline{\rho^{\mathcal{B}}(x)})$.*

Definition 6.7. Let $(\pi, \mathcal{H}, \mathcal{B}, \rho^{\mathcal{B}})$ be a pre-representation of a Lie supergroup $(G, \mathfrak{g})$. For every $x = x_{\overline{0}} + x_{\overline{1}} \in \mathfrak{g}^{\mathbb{C}}$ we define a linear operator $\widetilde{\rho}^{\mathcal{B}}(x)$ on $\mathcal{H}$ with $\mathcal{D}(\widetilde{\rho}^{\mathcal{B}}(x)) = \mathcal{D}^\infty$ as follows. If $x_{\overline{0}} = a_{\overline{0}} + ib_{\overline{0}}$ and $x_{\overline{1}} = a_{\overline{1}} + ib_{\overline{1}}$, where $a_{\overline{0}}, b_{\overline{0}} \in \mathfrak{g}_{\overline{0}}$ and $a_{\overline{1}}, b_{\overline{1}} \in \mathfrak{g}_{\overline{1}}$, then we put

$$\widetilde{\rho}^{\mathcal{B}}(x)v = \overline{\rho^{\mathcal{B}}(a_{\overline{0}})}v + i\overline{\rho^{\mathcal{B}}(b_{\overline{0}})}v + \overline{\rho^{\mathcal{B}}(a_{\overline{1}})}v + i\overline{\rho^{\mathcal{B}}(b_{\overline{1}})}v \quad \text{for} \quad v \in \mathcal{D}^\infty.$$

The following auxiliary lemma is [MNS11, Lemma 2.5].

Lemma 6.8. *Let P_1 and P_2 be two symmetric operators on a complex Hilbert space $\mathcal{H}$ such that $\mathcal{D}(P_1) = \mathcal{D}(P_2)$. Let $\mathscr{L} \subseteq \mathcal{D}(P_1)$ be a dense linear subspace of $\mathcal{H}$ such that $P_1\big|_{\mathscr{L}} = P_2\big|_{\mathscr{L}}$. Assume that the latter operator is essentially self-adjoint. Then $P_1 = P_2$.*

Proposition 6.9. *Let $(G, \mathfrak{g})$ be a Lie supergroup and $(\pi, \mathcal{H}, \mathcal{B}, \rho^{\mathcal{B}})$ be a pre-representation of $(G, \mathfrak{g})$. Then the following assertions hold.*

(i) *$\widetilde{\rho}^{\mathcal{B}}(x)\mathcal{D}^\infty \subseteq \mathcal{D}^\infty$ for every $x \in \mathfrak{g}^{\mathbb{C}}$.*

(ii) *The map $\widetilde{\rho}^{\mathcal{B}} \colon \mathfrak{g}^{\mathbb{C}} \to \mathrm{End}(\mathcal{D}^\infty)$ is a homomorphism of Lie superalgebras, i.e., it is linear, and, if $x, y \in \mathfrak{g}^{\mathbb{C}}$ are homogeneous, then*

$$\widetilde{\rho}^{\mathcal{B}}([x,y]) = \widetilde{\rho}^{\mathcal{B}}(x)\widetilde{\rho}^{\mathcal{B}}(y) - (-1)^{p(x)p(y)}\widetilde{\rho}^{\mathcal{B}}(y)\widetilde{\rho}^{\mathcal{B}}(x).$$

(iii) *For each $v \in \mathcal{H}^\infty$, the map $\omega_v \colon \mathfrak{g} \to \mathcal{H}, x \mapsto \widetilde{\rho}^{\mathcal{B}}(x)v$ is linear and continuous.*

Proof. (i) By Lemma 6.6(i) and the definition of $\widetilde{\rho}^{\mathcal{B}}$ it suffices to prove the statement when $x \in \mathfrak{g}_{\overline{1}}$. Therefore it is enough to prove that

$$\overline{\rho^{\mathcal{B}}(x)}v \in \mathcal{D}^n \text{ for every } x \in \mathfrak{g}_{\overline{1}}, v \in \mathcal{D}^{n+1}, n \in \mathbb{N}. \tag{15}$$

Let $y \in \mathfrak{g}_{\overline{0}}$. For every $w \in \mathcal{B}$ and $v \in \mathcal{D}^2$, using Lemma 6.6, we can write

$$\langle \overline{\rho^{\mathcal{B}}(x)}v, \overline{\mathrm{d}\pi}(y)w \rangle = \langle \overline{\rho^{\mathcal{B}}(x)}v, \rho^{\mathcal{B}}(y)w \rangle = e^{\frac{\pi i}{2}}\langle v, \overline{\rho^{\mathcal{B}}(x)}\rho^{\mathcal{B}}(y)w \rangle$$

$$= e^{\frac{\pi i}{2}}\langle v, \rho^{\mathcal{B}}(y)\rho^{\mathcal{B}}(x)w + \rho^{\mathcal{B}}([x,y])w \rangle = e^{\frac{\pi i}{2}}\langle v, \rho^{\mathcal{B}}(y)\rho^{\mathcal{B}}(x)w \rangle + e^{\frac{\pi i}{2}}\langle v, \rho^{\mathcal{B}}([x,y])w \rangle$$

$$= \langle \overline{\rho^{\mathcal{B}}(x)\mathrm{d}\pi}(y)v, w \rangle + \langle \overline{\rho^{\mathcal{B}}([x,y])}v, w \rangle.$$

It follows that the complex linear functional $\mathcal{B} \to \mathbb{C}, w \mapsto \langle \overline{\rho^{\mathcal{B}}(x)}v, \overline{\mathrm{d}\pi}(y)w \rangle$ is continuous, i.e., $\overline{\rho^{\mathcal{B}}(x)}v \in \mathcal{D}\big((\overline{\mathrm{d}\pi}(y)\big|_{\mathcal{B}})^*\big)$. Since $\overline{\mathrm{d}\pi}(y)\big|_{\mathcal{B}} = \rho^{\mathcal{B}}(y)$ is essentially skew-adjoint, from Lemma 6.6(i) it follows that $(\overline{\mathrm{d}\pi}(y)\big|_{\mathcal{B}})^* = -\overline{\mathrm{d}\pi}(y)$, i.e., $\overline{\rho^{\mathcal{B}}(x)}v \in \mathcal{D}(\overline{\mathrm{d}\pi}(y))$. This proves (15) for $n = 1$.

For $n > 1$ the proof of (15) can be completed by induction. Let $x_1, \ldots, x_n \in \mathfrak{g}_{\overline{0}}$ and $v \in \mathcal{D}^{n+1}$. Using the induction hypothesis, for every $w \in \mathcal{B}$ we can write

$$\langle \overline{\mathrm{d}\pi}(x_{n-1})\cdots\overline{\mathrm{d}\pi}(x_1)\overline{\rho^{\mathcal{B}}(x)}v, \overline{\mathrm{d}\pi}(x_n)w \rangle = e^{(n-\frac{1}{2})\pi i}\langle v, \rho^{\mathcal{B}}(x)\rho^{\mathcal{B}}(x_1)\cdots\rho^{\mathcal{B}}(x_n)w \rangle$$

$$= e^{(n-\frac{1}{2})\pi i}\langle v, \rho^{\mathcal{B}}([x,x_1])\rho^{\mathcal{B}}(x_2)\cdots\rho^{\mathcal{B}}(x_n)w + \rho^{\mathcal{B}}(x_1)\rho^{\mathcal{B}}(x)\rho^{\mathcal{B}}(x_2)\cdots\rho^{\mathcal{B}}(x_n)w \rangle$$

$$= \langle \overline{\mathrm{d}\pi}(x_n)\cdots\overline{\mathrm{d}\pi}(x_2)\overline{\rho^{\mathcal{B}}([x,x_1])}v, w \rangle + \langle \overline{\mathrm{d}\pi}(x_n)\cdots\overline{\mathrm{d}\pi}(x_2)\overline{\rho^{\mathcal{B}}(x)\mathrm{d}\pi}(x_1)v, w \rangle.$$

An argument similar to the case $n = 1$ proves that

$$\overline{\mathrm{d}\pi}(x_{n-1})\cdots\overline{\mathrm{d}\pi}(x_1)\overline{\rho^{\mathcal{B}}(x)}v \in \mathcal{D}(\overline{\mathrm{d}\pi}(x_n)).$$

Consequently, $\overline{\rho^{\mathcal{B}}(x)}v \in \mathcal{D}^n$.

(ii) (cf. [MNS11, Prop. 4.6]) First we show that $\widetilde{\rho}^{\mathcal{B}}$ is linear. By Lemma 6.6(i) and the definition of $\widetilde{\rho}^{\mathcal{B}}$, it is enough to prove that, for every $v \in \mathcal{D}^{\infty}$, the map

$$\mathfrak{g}_{\overline{1}} \to \mathcal{H}, \quad x \mapsto \overline{\rho^{\mathscr{B}}(x)}v$$

is $\mathbb{R}$-linear. Let $x \in \mathfrak{g}_{\overline{1}}$ and $a \in \mathbb{R}$. Then the equality

$$\overline{\rho^{\mathcal{B}}(ax)}v = a\overline{\rho^{\mathcal{B}}(x)}v \tag{16}$$

holds for every $v \in \mathcal{B}$, and therefore, by Lemma 6.8, it also holds for every $v \in \mathcal{D}^{\infty}$. A similar reasoning proves that, if $x, y \in \mathfrak{g}_{\overline{1}}$, then, for every $v \in \mathcal{D}^{\infty}$, we have

$$\overline{\rho^{\mathcal{B}}(x+y)}v = \overline{\rho^{\mathcal{B}}(x)}v + \overline{\rho^{\mathcal{B}}(y)}v. \tag{17}$$

It suffices to prove the commutation relation for $x, y \in \mathfrak{g}$. Depending on the parities of x an y, there are four cases to consider, but the argument for all of them is essentially the same. For example, if $x \in \mathfrak{g}_{\overline{0}}$ and $y \in \mathfrak{g}_{\overline{1}}$, then we define two operators P_1 and P_2 with domains $\mathcal{D}(P_1) = \mathcal{D}(P_2) = \mathcal{D}^{\infty}$ as follows. For $v \in \mathcal{D}^{\infty}$ we set

$$P_1 v = e^{-\frac{\pi i}{4}}\overline{\rho^{\mathcal{B}}([x,y])}v \quad \text{and} \quad P_2 v = e^{-\frac{\pi i}{4}}\left(\overline{\rho^{\mathcal{B}}(x)}\,\overline{\rho^{\mathcal{B}}(y)}v - \overline{\rho^{\mathcal{B}}(y)}\,\overline{\rho^{\mathcal{B}}(x)}v\right).$$

Then P_1 and P_2 are both symmetric, $P_1\big|_{\mathcal{B}} = P_2\big|_{\mathcal{B}}$, and by Lemma 6.6(ii), the operator $P_1\big|_{\mathcal{B}}$ is essentially self-adjoint. Lemma 6.8 implies that $P_1 = P_2$.

(iii) The linearity of ω_v follows from (ii). The continuity of ω_v on $\mathfrak{g}_{\overline{0}}$ follows from the definition of a smooth vector. Therefore it remains to show that $\omega_v|_{\mathfrak{g}_{\overline{1}}}$ is continuous in 0. This can be reduced to continuity of ω_v on $\mathfrak{g}_{\overline{0}}$ using the estimate

$$\|\overline{\rho^{\mathcal{B}}(y)}v\| = \langle \overline{\rho^{\mathcal{B}}(y)}v, \overline{\rho^{\mathcal{B}}(y)}v\rangle^{\frac{1}{2}} \tag{18}$$
$$= |\langle v, \overline{\rho^{\mathcal{B}}(y)}^2 v\rangle|^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}}\|v\|^{1/2}\|\overline{\mathrm{d}\pi}([y,y])v\|^{1/2}, \quad y \in \mathfrak{g}_{\overline{1}}, v \in \mathcal{D}^2$$

which is a consequence of the Cauchy–Schwarz inequality. $\qquad\square$

Lemma 6.10. *Let $(G, \mathfrak{g})$ be a Fréchet–Lie supergroup such that G has the Trotter property. Let $(\pi, \mathcal{H}, \mathcal{B}, \rho^{\mathcal{B}})$ be a pre-representation of G and $v \in \mathcal{H}^{\infty}$. For every $x \in \mathfrak{g}$ the map*

$$G \times \mathfrak{g}_{\overline{0}} \to \mathcal{H}, \quad (g, y) \mapsto \overline{\rho^{\mathcal{B}}(x)}\overline{\mathrm{d}\pi}(y)\pi(g)v \tag{19}$$

is continuous.

Proof. Since G has the Trotter property, by Theorem 5.3 and Proposition 6.9(ii) we have $\widetilde{\rho}^{\mathcal{B}}(z)\mathcal{H}^{\infty} \subseteq \mathcal{H}^{\infty}$ for every $z \in \mathfrak{g}^{\mathbb{C}}$.

We can assume x is homogeneous. First assume $x \in \mathfrak{g}_{\overline{0}}$. By Remark 4.2(c), the maps

$$G \times \mathfrak{g}_{\overline{0}} \to \mathcal{H} \ , \ (g, x_1) \mapsto \pi(g)\overline{\mathrm{d}\pi}(x_1)v$$

and

$$G \times \mathfrak{g}_{\overline{0}} \times \mathfrak{g}_{\overline{0}} \to \mathcal{H} \ , \ (g, x_1, x_2) \mapsto \pi(g)\overline{\mathrm{d}\pi}(x_1)\overline{\mathrm{d}\pi}(x_2)v$$

are continuous. Therefore continuity of (19) follows from

$$\overline{\mathrm{d}\pi}(x)\overline{\mathrm{d}\pi}(y)\pi(g)v = \pi(g)\overline{\mathrm{d}\pi}(\mathrm{Ad}(g^{-1})x)\overline{\mathrm{d}\pi}(\mathrm{Ad}(g^{-1})y)v$$

and smoothness of the map $G \to \mathfrak{g}_{\overline{0}}$, $g \mapsto \mathrm{Ad}(g)z$ for every $z \in \mathfrak{g}_{\overline{0}}$.

Next assume $x \in \mathfrak{g}_{\overline{1}}$. From the preceding argument it follows that the maps

$$G \times \mathfrak{g}_{\overline{0}} \to \mathcal{H} , \quad (g, x_1) \mapsto \overline{\mathrm{d}\pi}([x, x])\overline{\mathrm{d}\pi}(x_1)\pi(g)v \tag{20}$$

and

$$G \times \mathfrak{g}_{\overline{0}} \to \mathcal{H} , \quad (g, x_1) \mapsto \overline{\mathrm{d}\pi}(x_1)\pi(g)v \tag{21}$$

are continuous. From (18) it follows that

$$\|\overline{\rho^{\mathcal{B}}(x)}\big(\overline{\mathrm{d}\pi}(y_1)\pi(g_1)v - \overline{\mathrm{d}\pi}(y_2)\pi(g_2)v\big)\|$$
$$\leq \frac{1}{\sqrt{2}}\|\overline{\mathrm{d}\pi}(y_1)\pi(g_1)v - \overline{\mathrm{d}\pi}(y_2)\pi(g_2)v\|^{\frac{1}{2}} \cdot \|\overline{\mathrm{d}\pi}([x, x])(\overline{\mathrm{d}\pi}(y_1)\pi(g_1)v - \overline{\mathrm{d}\pi}(y_2)\pi(g_2)v)\|^{\frac{1}{2}}$$

for $g_1, g_2 \in G$ and $y_1, y_2 \in \mathfrak{g}_{\overline{0}}$. Consequently, continuity of (19) follows from continuity of (20) and (21). $\qquad\square$

Remark 6.11. Let $\varepsilon > 0$ and $\alpha : (-\varepsilon, \varepsilon) \to G$ be a smooth curve. The right logarithmic derivative of α is the smooth curve $\breve{\delta}(\alpha) : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ defined by

$$\breve{\delta}(\alpha)_s := \mathrm{d}\varrho_{\alpha(s)^{-1}}(\alpha(s))(\alpha'(s))$$

where $\varrho_g : G \to G$ is the right multiplication by $g \in G$. The product rule for $\breve{\delta}$ is given by

$$\breve{\delta}(\alpha\beta)_s = \breve{\delta}(\alpha)_s + \mathrm{Ad}(\alpha(s))\breve{\delta}(\beta)_s.$$

If $(\pi, \mathcal{H})$ is a unitary representation of G and $v \in \mathcal{H}^\infty$ then $\pi(\alpha(s))v \in \mathcal{H}^\infty$ and therefore

$$\lim_{h \to 0} \frac{1}{h}\big(\pi(\alpha(s + h))v - \pi(\alpha(s))v\big)$$
$$= \lim_{h \to 0} \frac{1}{h}\big(\pi(\alpha(s + h)\alpha(s)^{-1})\pi(\alpha(s))v - \pi(\alpha(s))v\big) = \overline{\mathrm{d}\pi}(\breve{\delta}(\alpha)_s)\pi(\alpha(s))v.$$

Similarly, we obtain with the left logarithmic derivative $\delta(\alpha)$:

$$\lim_{h \to 0} \frac{1}{h}\big(\pi(\alpha(s + h))v - \pi(\alpha(s))v\big)$$
$$= \pi(\alpha(s)) \lim_{h \to 0} \frac{1}{h}\big(\pi(\alpha(s)^{-1}\alpha(s + h))v - v\big) = \pi(s)\overline{\mathrm{d}\pi}(\delta(\alpha)_s)v.$$

Our next goal is to prove Proposition 6.13 below. The proof of this proposition is based on a variation of a subtle lemma from [JM84, Chap. 3] (see [Me11] and [MNS11] as well).

Lemma 6.12. *Let $(\pi, \mathcal{H})$ be a smooth unitary representation of a Lie group G. Let B be a closable operator on $\mathcal{H}$ such that $\mathcal{H}^\infty \subseteq \mathcal{D}(B)$ and $B\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$. Let $\alpha : (-\varepsilon, \varepsilon) \to G$ be a smooth curve. Let $v \in \mathcal{H}^\infty$ and set*

$$\eta : (-\varepsilon, \varepsilon) \to \mathcal{H} \ , \ \eta(s) := B\pi(\alpha(s))v.$$

Assume that the map

$$\beta : (-\varepsilon, \varepsilon) \to \mathcal{H} \ , \ \beta(s) := B\overline{\mathrm{d}\pi}(\breve{\delta}(\alpha)_s)\pi(\alpha(s))v$$

is continuous. Then η is differentiable and $\dfrac{d}{ds}\eta(s) = \beta(s)$ for every $s \in (-\varepsilon, \varepsilon)$.

Proof. Set $\gamma(s) := \pi(\alpha(s))v$. Since v is a smooth vector and α a smooth curve, γ is also smooth. Remark 6.11 implies that $\gamma'(s) = \overline{\mathrm{d}\pi}(\breve{\delta}(\alpha)_s)\pi(\alpha(s))v$, and therefore

$$\pi(\alpha(s))v - \pi(\alpha(0))v = \gamma(s) - \gamma(0) = \int_0^s \left(\overline{\mathrm{d}\pi}(\breve{\delta}(\alpha)_s)\pi(\alpha(s))v\right)dt. \tag{22}$$

Let $\mathcal{H}_B$ denote the completion of $\mathcal{H}^\infty$ with respect to the norm $\|w\|_B := \|w\| + \|Bw\|$. Then $\mathcal{H}_B$ is a Banach space and $B : \mathcal{H}_B \to \mathcal{H}$ is a continuous linear map. Continuity of β implies that the map $\gamma' : (-\varepsilon, \varepsilon) \to \mathcal{H}_B$ is continuous, and therefore (22) holds in $\mathcal{H}_B$. Since $B : \mathcal{H} \to \mathcal{H}_B$ is linear and continuous, we have

$$\eta(s) - \eta(0) = B\pi(\alpha(s))v - B\pi(\alpha(0))v = B(\gamma(s) - \gamma(0)) = \int_0^s \left(B\overline{\mathrm{d}\pi}(\breve{\delta}(\alpha)_s)\pi(\alpha(s))v\right)dt.$$

The last equality immediately implies that $\dfrac{d}{ds}\eta(s) = \beta(s)$ for every $s \in (-\varepsilon, \varepsilon)$. $\qquad\square$

Proposition 6.13. *Let $(G, \mathfrak{g})$ be a Fréchet–Lie supergroup. Assume that G is connected and has the Trotter property. Let $(\pi, \mathcal{H}, \mathcal{B}, \rho^\mathcal{B})$ be a pre-representation of $(G, \mathfrak{g})$. Then for every $g \in G$, every $x \in \mathfrak{g}_{\bar{1}}^\mathbb{C}$, and every $v \in \mathcal{H}^\infty$ we have*

$$\pi(g)\widetilde{\rho}^\mathcal{B}(x)\pi(g)^{-1}v = \widetilde{\rho}^\mathcal{B}(\mathrm{Ad}(g)x)v. \tag{23}$$

Proof. By Theorem 5.3 and Proposition 6.9(ii) we have $\widetilde{\rho}^\mathcal{B}(z)\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$ for every $z \in \mathfrak{g}^\mathbb{C}$. Let $\alpha : \mathbb{R} \to G$ be a smooth curve such that $\alpha(0) = \mathbf{1}$ and $\alpha(1) = g$. Set $A(s) := \pi(g\alpha(s)^{-1})$ and $\gamma(s) := \mathrm{Ad}(\alpha(s))x$ for $s \in \mathbb{R}$. For $v \in \mathcal{H}^\infty$ and $s \in \mathbb{R}$ define

$$K(s) : \mathfrak{g}_{\bar{1}}^\mathbb{C} \to \mathcal{H} \, , \ K(s)x = A(s)\widetilde{\rho}^\mathcal{B}(x)A(s)^{-1}v.$$

The left hand side of (23) is equal to $K(0)\gamma(0)$ and the right hand side of (23) is equal to $K(1)\gamma(1)$. Therefore it suffices to prove that the map $s \mapsto K(s)\gamma(s)$ is constant. To this end, we will prove that it is differentiable and that its derivative is identically zero.

By Proposition 6.9(iii), for every $t \in \mathbb{R}$, the linear map $K(t)$ is continuous. From Lemma 6.10 we derive that for every $y \in \mathfrak{g}_{\bar{1}}^\mathbb{C}$, the curve

$$\mathbb{R} \to \mathcal{H} \, , \ s \mapsto \widetilde{\rho}^\mathcal{B}(y)\overline{\mathrm{d}\pi}(\breve{\delta}(\alpha)_s)A(s)^{-1}v$$

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is continuous. It follows from Lemma 6.6(ii) and Lemma 6.12 that the curve

$$\eta_y : \mathbb{R} \mapsto \mathcal{H}, \, \eta(s) = \widetilde{\rho}^{\mathcal{B}}(y)A(s)^{-1}v = \widetilde{\rho}^{\mathcal{B}}(y)\pi(\alpha(s))\pi(g)^{-1}v$$

is differentiable, and $\eta'_y(s) = \widetilde{\rho}^{\mathcal{B}}(y)\overline{\mathrm{d}\pi}(\check{\delta}(\alpha)_s)A(s)^{-1}v = \widetilde{\rho}^{\mathcal{B}}(y)\widetilde{\rho}^{\mathcal{B}}(\check{\delta}(\alpha)_s)A(s)^{-1}v.$

Next we show that, for every $y \in \mathfrak{g}_{\overline{\mathsf{T}}}^{\mathbb{C}}$, the map $K^y : \mathbb{R} \to \mathcal{H}, s \mapsto K(s)y$ is differentiable and we compute its derivative. Observe that

$$(K^y)'(s) = \frac{d}{ds}(A(s)\eta_y(s)) = \lim_{h\to 0}\frac{1}{h}\big(A(s+h)\eta_y(s+h) - A(s)\eta_y(s)\big)$$

and

$$\frac{1}{h}\big(A(s+h)\eta_y(s+h) - A(s)\eta_y(s)\big)$$
$$= A(s+h)\Big(\frac{1}{h}\big(\eta_y(s+h) - \eta_y(s)\big)\Big) + \frac{1}{h}\big(A(s+h)\eta_y(s) - A(s)\eta_y(s)\big). \qquad (24)$$

Since the map $\mathbb{R} \times \mathcal{H} \to \mathcal{H}, (s, v) \mapsto A(s)v$ is continuous, when $h \to 0$ we obtain

$$A(s+h)\Big(\frac{1}{h}\big(\eta_y(s+h) - \eta_y(s)\big)\Big) \to A(s)\eta'_y(s).$$

Since $\eta_y(s) \in \mathcal{H}^\infty$, from Remark 6.11 and the relation $\delta(\alpha^{-1})_s = -\check{\delta}(\alpha)_s$ it follows that, as $h \to 0$, the second term in (24) converges to

$$-A(s)\widetilde{\rho}^{\mathcal{B}}(\check{\delta}(\alpha)_s)\eta_y(s) = -A(s)\widetilde{\rho}^{\mathcal{B}}(\check{\delta}(\alpha)_s)\widetilde{\rho}^{\mathcal{B}}(y)A(s)^{-1}v.$$

By Proposition 6.9(ii) we conclude that

$$(K^y)'(s) = A(s)\eta'_y(s) - A(s)\widetilde{\rho}^{\mathcal{B}}(\check{\delta}(\alpha)_s)\widetilde{\rho}^{\mathcal{B}}(y)A(s)^{-1}v = A(s)[\widetilde{\rho}^{\mathcal{B}}(y), \widetilde{\rho}^{\mathcal{B}}(\check{\delta}(\alpha)_s)]A(s)^{-1}v. \quad (25)$$

Finally we prove that $\frac{d}{ds}(K(s)\gamma(s)) = 0$ for every $s \in \mathbb{R}$. We have

$$\frac{1}{h}\big(K(s+h)\gamma(s+h) - K(s)\gamma(s)\big)$$
$$= K(s+h)\Big(\frac{1}{h}\big(\gamma(s+h) - \gamma(s)\big)\Big) + \frac{1}{h}\big(K(s+h)\gamma(s) - K(s)\gamma(s)\big). \qquad (26)$$

The first term in (26) can be written as

$$K(s+h)\Big(\frac{1}{h}\big(\gamma(s+h) - \gamma(s)\big) - \gamma'(s)\Big) + K(s+h)\gamma'(s).$$

Differentiability of the map K^y implies that it is continuous. Thus, if I is a compact interval containing s, then $\sup_{t\in I}\|K(t)y\| < \infty$. Since $\mathfrak{g}$ is Fréchet, the Banach–Steinhaus Theorem implies that $(K(t))_{t\in I}$ is equicontinuous. Since $\gamma'(s) = [\check{\delta}(\alpha)_s, \mathrm{Ad}(\alpha(s))x]$, it follows that, for $h \to 0$, the first term in (26) converges to

$$K(s)\gamma'(s) = A(s)\widetilde{\rho}^{\mathcal{B}}([\check{\delta}(\alpha)_s, \mathrm{Ad}(\alpha(s))x])A(s)^{-1}v = A(s)\widetilde{\rho}^{\mathcal{B}}([\check{\delta}(\alpha)_s, \gamma(s)])A(s)^{-1}v.$$

By (25), the second term in (26) converges to

$$A(s)[\widetilde{\rho}^{\mathcal{B}}(\gamma(s)), \widetilde{\rho}^{\mathcal{B}}(\check{\delta}(\alpha)_s)]A(s)^{-1}v = A(s)\widetilde{\rho}^{\mathcal{B}}([\gamma(s), \check{\delta}(\alpha)_s])A(s)^{-1}v.$$

We conclude that $\frac{d}{ds}(K(s)\gamma(s)) = 0$. $\qquad\qquad\qquad\square$

We can now prove the following theorem, which asserts that every pre-representation of a Lie supergroup corresponds to a unique unitary representation.

Theorem 6.14. (Stability Theorem) *Let $(\pi, \mathcal{H}, \mathcal{B}, \rho^{\mathcal{B}})$ be a pre-representation of the Fréchet–Lie supergroup $(G, \mathfrak{g})$ and assume that G has the Trotter property.*
(a) There exists a unique linear map

$$\rho^{\pi} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}^{\infty})$$

such that $\rho^{\pi}(x)\big|_{\mathcal{B}} = \rho^{\mathcal{B}}(x)$ and $(\pi, \rho^{\pi}, \mathcal{H})$ is a smooth unitary representation of $(G, \mathfrak{g})$.
(b) If the representation $(\pi, \mathcal{H})$ of G is analytic, then there exists a unique map

$$\rho^{\pi} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}^{\omega})$$

such that $\overline{\rho^{\pi}(x)}\big|_{\mathcal{B}} = \rho^{\mathcal{B}}(x)$ and $(\pi, \rho^{\pi}, \mathcal{H})$ is an analytic unitary representation of $(G, \mathfrak{g})$.

Proof. First we recall from Theorem 5.3 that the Trotter property of G and the smoothness of the representation imply that $\mathcal{D}^{\infty} = \mathcal{H}^{\infty}$.

(a) To prove the existence of ρ^{π}, we set $\rho^{\pi}(x) = \widetilde{\rho}^{\mathcal{B}}(x)$ for every $x \in \mathfrak{g}$. Proposition 6.9 implies that $\widetilde{\rho}^{\mathcal{B}}(x) \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}^{\infty})$ and $\widetilde{\rho}^{\mathcal{B}}$ is a representation of the Lie superalgebra $\mathfrak{g}$. To prove the conjugacy invariance relation of (SR5), for every element of G/G° we take a coset representative $g \in G$ which satisfies the condition (PR6) and apply Lemma 6.8 with $P_1 = e^{-\frac{\pi i}{4}}\pi(g)\widetilde{\rho}^{\mathcal{B}}(x)\pi(g)^{-1}$, $P_2 = e^{-\frac{\pi i}{4}}\widetilde{\rho}^{\mathcal{B}}(\mathrm{Ad}(g)x)$, and $\mathscr{L} = \mathcal{B}$.

To prove uniqueness, it suffices to show that, if $(\pi, \rho^{\pi}, \mathcal{H})$ is a smooth unitary representation such that for every $x \in \mathfrak{g}$ we have $\rho^{\pi}(x)\big|_{\mathcal{B}} = \rho^{\mathcal{B}}(x)$ then for every $x \in \mathfrak{g}$ we have

$$\rho^{\pi}(x)\big|_{\mathcal{H}^{\infty}} = \widetilde{\rho}^{\mathcal{B}}(x). \tag{27}$$

It suffices to prove (27) when x is homogeneous. If $x \in \mathfrak{g}_{\overline{1}}$ then by Lemma 6.6(ii) the operator $e^{-\frac{\pi i}{4}}\widetilde{\rho}^{\mathcal{B}}(x)\big|_{\mathcal{B}}$ is essentially self-adjoint. Therefore (27) follows from setting $P_1 = e^{-\frac{\pi i}{4}}\rho^{\pi}(x)\big|_{\mathcal{H}^{\infty}}$, $P_2 = e^{-\frac{\pi i}{4}}\widetilde{\rho}^{\mathcal{B}}(x)$ and $\mathscr{L} = \mathcal{B}$ in Lemma 6.8. The argument for $x \in \mathfrak{g}_{\overline{0}}$ is similar.

(b) The proof of uniqueness is similar to the one given for the smooth case. For the existence, it remains to verify that if $(\pi, \rho^{\pi}, \mathcal{H})$ is the smooth representation obtained in (a), then $\rho^{\pi}(y)\mathcal{H}^{\omega} \subseteq \mathcal{H}^{\omega}$ for every $y \in \mathfrak{g}_{\overline{1}}$. Let $y \in \mathfrak{g}_{\overline{1}}$, $v \in \mathcal{H}^{\omega}$ and $w := \overline{\rho^{\mathcal{B}}(y)}v$. We have to show that $w \in \mathcal{H}^{\omega}$, and, in view of [Ne11, Thm. 5.2], it suffices to verify that the function

$$G \to \mathbb{C} \ , \ g \mapsto \langle \pi(g)w, w \rangle$$

is analytic. Note that

$$\langle \pi(g)w, w \rangle = \langle \pi(g)\overline{\rho^{\mathcal{B}}(y)}v, w \rangle = \langle \overline{\rho^{\mathcal{B}}(\mathrm{Ad}(g)y)}\pi(g)v, w \rangle = -i\langle \pi(g)v, \overline{\rho^{\mathcal{B}}(\mathrm{Ad}(g)y)}w \rangle.$$

The orbit map of v is analytic, and by Proposition 6.9(iii) the linear map $\mathfrak{g} \to \mathcal{H}, z \mapsto \overline{\rho^{\mathcal{B}}(z)}w$ is continuous. The assertion follows from analyticity of the map $G \to \mathfrak{g}, g \mapsto \mathrm{Ad}(g)y$. $\qquad\square$

A   Some results on Lie groups of maps

The results of this appendix will be used in Appendix B. Let E be a locally convex space, M be a smooth finite dimensional manifold (possibly with boundary) and K be a Lie group (possibly infinite dimensional) with Lie algebra $\mathfrak{k}$. In the following we write $\Omega^1_{\mathcal{C}^r}(M, E)$ for the space of E-valued 1-forms on M defining $\mathcal{C}^r$-functions $TM \to E$. The space of E-valued smooth forms will be denoted by $\Omega^1(M, E)$.

We endow $\Omega^1_{\mathcal{C}^r}(M, E)$ with the topology induced by the embedding

$$\Omega^1_{\mathcal{C}^r}(M, E) \hookrightarrow \mathcal{C}^r(TM, E),$$

where TM is the tangent bundle and $\mathcal{C}^r(TM, E)$ is endowed with the compact open $\mathcal{C}^r$-topology, so that $\Omega^1_{\mathcal{C}^r}(M, E)$ is a closed subspace of $\mathcal{C}^r(TM, E)$. The space $\Omega^1(M, E)$ is endowed with the topology induced by the diagonal embedding

$$\Omega^1(M, E) \hookrightarrow \prod_{r=1}^{\infty} \Omega^1_{\mathcal{C}^r}(M, E).$$

Lemma A.1. *Let M be a compact smooth manifold (possibly with boundary) and K a Lie group with Lie algebra $\mathfrak{k}$. Then, for each $r \in \mathbb{N}_0 \cup \{\infty\}$, the action of the Lie group $\mathcal{C}^r(M, K)$ on $\Omega^1_{\mathcal{C}^r}(M, \mathfrak{k})$ by $(g, \alpha) \mapsto \mathrm{Ad}(g)\alpha$ is smooth.*

Proof. Assume that $d = \dim M$. Every covering $(U_i)_{i \in I}$ of M by compact submanifolds with boundary, which are diffeomorphic to d-dimensional balls and whose interiors define an atlas, yields an embedding

$$\Omega^1_{\mathcal{C}^r}(M, \mathfrak{k}) \hookrightarrow \prod_{i \in I} \Omega^1_{\mathcal{C}^r}(U_i, \mathfrak{k}) \cong \prod_{i \in I} \mathcal{C}^r(U_i, \mathfrak{k})^d.$$

Therefore it suffices to show that the action of $\mathcal{C}^r(M, K)$ on each space $\mathcal{C}^r(U_i, \mathfrak{k})$, given by

$$(g, f) \mapsto \mathrm{Ad}(g)f = \sigma_{\mathrm{Ad}} \circ (g, f)$$

is smooth. This action factors through the Lie group morphisms

$$\mathcal{C}^r(M, K) \to \mathcal{C}^r(U_i, K),$$

and for the Lie groups $G := \mathcal{C}^r(U_i, K)$, it coincides with the adjoint action on $\mathbf{L}(G) \cong \mathcal{C}^r(U_i, \mathfrak{k})$, which is smooth. This proves the lemma. $\qquad\square$

Lemma A.2. *([GN12]) Let E_1 and E_2 be locally convex spaces, $U_1 \subseteq E_1$ open, M a compact smooth manifold (possibly with boundary), and $\varphi \colon U_1 \to E_2$ be a smooth map. Then, for each $r \in \mathbb{N} \cup \{\infty\}$, the map*

$$\varphi_* \colon \mathcal{C}^r(M, U_1) \to \mathcal{C}^r(M, E_2), \quad f \mapsto \varphi \circ f$$

is smooth.

Lemma A.3. *Let U be an open subset of a locally convex space E, F a locally convex space, M a compact manifold (possibly with boundary), $r \in \mathbb{N} \cup \{\infty\}$, and $\alpha \in \Omega^1(U, F)$. Then the map*

$$\mathcal{C}^r(M, U) \to \Omega^1_{\mathcal{C}^{r-1}}(M, F), \quad f \mapsto f^* \alpha$$

is smooth if $\mathcal{C}^r(M, U)$ is considered as an open subset of $\mathcal{C}^r(M, E)$.

Proof. Let $\pi \colon TM \to M$ denote the bundle projection. Then both components of the map

$$
\begin{aligned}
\mathcal{C}^r(M, U) &\to \mathcal{C}^{r-1}(TM, TU) \cong \mathcal{C}^{r-1}(TM, U) \times \mathcal{C}^{r-1}(TM, E), \\
f &\mapsto Tf = (f \circ \pi, \mathrm{d}f)
\end{aligned}
$$

are restrictions of continuous linear maps, hence smooth. Since $f^* \alpha = \alpha \circ Tf$, smoothness of α and Lemma A.2 imply that

$$\alpha_* \colon \mathcal{C}^{r-1}(TM, TU) \to \mathcal{C}^{r-1}(TM, F), \quad h \mapsto \alpha \circ h$$

is smooth, from which the assertion follows (recall that we topologize $\Omega^1_{\mathcal{C}^{r-1}}(M, F)$ as a closed subspace of $\mathcal{C}^{r-1}(TM, F)$). $\qquad\square$

Proposition A.4. *For any Lie group K with Lie algebra $\mathfrak{k}$, any compact manifold M (possibly with boundary) and any $r \in \mathbb{N} \cup \{\infty\}$, the left logarithmic derivative*

$$\delta \colon \mathcal{C}^r(M, K) \to \Omega^1_{\mathcal{C}^{r-1}}(M, \mathfrak{k})$$

is a smooth map with respect to the Lie group structure on $\mathcal{C}^r(M, K)$, and

$$T_{\mathbf{1}}(\delta) = \mathrm{d} \colon \mathcal{C}^r(M, \mathfrak{k}) \to \Omega^1_{\mathcal{C}^{r-1}}(M, \mathfrak{k}), \quad \xi \mapsto \mathrm{d}\xi. \tag{28}$$

Proof. From Lemma A.1, we already know that the action of the Lie group $G := \mathcal{C}^r(M, K)$ on $\Omega^1_{\mathcal{C}^r}(M, \mathfrak{k})$ by $f \cdot \alpha := \mathrm{Ad}(f)\alpha$ is smooth. Since the inclusion map $\mathcal{C}^r(M, K) \to \mathcal{C}^{r-1}(M, K)$ is a smooth morphism of Lie groups, the action of G on on $\Omega^1_{\mathcal{C}^{r-1}}(M, \mathfrak{k})$ is also smooth.

The product rule $\delta(\eta\gamma) = \delta(\gamma) + \mathrm{Ad}(\gamma)^{-1}\delta(\eta)$ means that δ is a right crossed homomorphism for the smooth action of $\mathcal{C}^r(M, K)$ on $\Omega^1_{\mathcal{C}^{r-1}}(M, \mathfrak{k})$. It therefore suffices to verify its smoothness in a neighborhood of the identity.

Let (φ_K, U_K) be a chart of an identity neighborhood of K with $T_{\mathbf{1}}(\varphi_K) = \mathrm{id}_{\mathfrak{k}}$, so that (φ_G, U_G) with

$$U_G := \lfloor M, U_K \rfloor := \{\gamma \in G \colon \gamma(M) \subseteq U_K\}, \quad \varphi_G(\gamma) := \varphi_K \circ \gamma$$

is a chart of an identity neighborhood of the Lie group G. If $\kappa_K \in \Omega^1(K, \mathfrak{k})$ denotes the left Maurer–Cartan form of K, then we have a map

$$
\begin{aligned}
\mathcal{C}^r(M, \varphi_K(U_K)) &\to \Omega^1_{\mathcal{C}^{r-1}}(M, \mathfrak{k}), \\
\xi &\mapsto \delta(\varphi_K^{-1} \circ \xi) = (\varphi_K^{-1} \circ \xi)^* \kappa_K = \xi^* (\varphi_K^{-1})^* \kappa_K
\end{aligned}
$$

whose smoothness follows from Lemma A.3.

Set $\beta := (\varphi_K^{-1})^* \kappa_K$. Then $\beta_0 = \mathrm{id}_{\mathfrak{k}}$ and (28) follows from the fact that for every $m \in M$ and $v \in T_m(M)$, we have

$$\frac{d}{dt}\bigg|_{t=0} (t\xi)^* \beta v = \frac{d}{dt}\bigg|_{t=0} \beta_{t\xi(m)}(t\mathrm{d}\xi(m)v) = \lim_{t \to 0} \beta_{t\xi(m)} \mathrm{d}\xi(m)v = \beta_0 \mathrm{d}\xi(m)v = \mathrm{d}\xi(m)v. \quad \square$$

B $\mathcal{C}^k$-regularity is an extension property

In this appendix we generalize the result that regularity of Lie groups is an extension property to the stronger notion of $\mathcal{C}^k$-regularity for $k \in \mathbb{N}_0$.

Throughout this section $I := [0,1]$. If $\gamma : (-\varepsilon, \varepsilon) \to G$ is a $\mathcal{C}^1$ curve, then the left logarithmic derivative $\delta(\gamma) : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ is defined by

$$\delta(\gamma)_t := \mathsf{d}\ell_{\gamma(t)^{-1}}(\gamma(t))\gamma'(t)$$

where $\ell_g : G \to G$ denotes the left translation $\ell_g(x) := gx$.

Definition B.1. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A Lie group G with Lie algebra $\mathfrak{g}$ is called $\mathcal{C}^k$-*regular*, if for each $\xi \in \mathcal{C}^k(I, \mathfrak{g})$, the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \delta(\gamma) = \xi \tag{29}$$

has a solution γ_ξ, which is then contained in $\mathcal{C}^{k+1}(I, G)$, and the corresponding evolution map

$$\mathrm{evol}_G : \mathcal{C}^k(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth. The solutions of (29) are unique whenever they exist (cf. [Ne06]). If G is $\mathcal{C}^k$-regular, we write

$$\mathrm{Evol}_G : \mathcal{C}^k(I, \mathfrak{g}) \to \mathcal{C}^{k+1}(I, G), \quad \xi \mapsto \gamma_\xi$$

for the corresponding map on the level of Lie group-valued curves. This map is also smooth (cf. [Gl12a, Thm. A]). The group G is called *regular* if it is C^∞-regular.

Remark B.2. (a) Any regular Lie group G has a smooth exponential function

$$\exp_G : \mathfrak{g} \to G \quad \text{by} \quad \exp_G(x) := \gamma_x(1),$$

where $x \in \mathfrak{g}$ is considered as a constant function $I \to \mathfrak{g}$. As a restriction of the smooth function evol_G to the topological subspace $\mathfrak{g} \subseteq C^\infty(I, \mathfrak{g})$ of constant functions, the exponential function is smooth.

(b) For $k \leq r$, the $\mathcal{C}^k$-regularity of a Lie group G implies its $\mathcal{C}^r$-regularity because the inclusion map $\mathcal{C}^r(I, \mathfrak{g}) \to \mathcal{C}^k(I, \mathfrak{g})$ is continuous linear, hence smooth.

Lemma B.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the prescription*

$$\xi * \gamma := \delta(\gamma) + \mathrm{Ad}(\gamma)^{-1}\xi$$

defines a smooth affine right action of the group $\mathcal{C}^{k+1}(I, G)$ on $\mathcal{C}^k(I, \mathfrak{g})$.

Proof. That we have an action follows from

$$(\xi * \gamma_1) * \gamma_2 = \delta(\gamma_2) + \mathrm{Ad}(\gamma_2)^{-1}(\delta(\gamma_1) + \mathrm{Ad}(\gamma_1)^{-1}\xi) = \delta(\gamma_1\gamma_2) + \mathrm{Ad}(\gamma_1\gamma_2)^{-1}\xi = \xi * (\gamma_1\gamma_2).$$

Since I is a compact manifold with boundary, the smoothness of the action follows from the smoothness of δ (see Proposition A.4) and Lemma A.1 (note that we can identify $\Omega_{\mathcal{C}^k}(I, \mathfrak{g})$ with $\mathcal{C}^k(I, \mathfrak{g})$). $\qquad\square$

Remark B.4. If $\xi = \delta(\eta)$ for some smooth function $h\colon M \to G$, then the Product Rule implies that $\delta(\eta\gamma) = \xi * \gamma$, so that the action from above corresponds to the right multiplication action on the level of group-valued functions.

Lemma B.5. (Local regularity criterion) *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. Suppose that (29) has a solution for each ξ in an open 0-neighborhood $U \subseteq \mathcal{C}^k(I, \mathfrak{g})$. Then it has a solution for each $\xi \in \mathcal{C}^k(I, \mathfrak{g})$. If the evolution map*

$$\mathrm{evol}_G\colon \mathcal{C}^k(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth in U, then it is smooth on all of $\mathcal{C}^k(I, \mathfrak{g})$.

Proof. Let $\xi \in \mathcal{C}^k(I, \mathfrak{g})$. For $n \in \mathbb{N}$ and $i \in \{0, \ldots, n-1\}$ we define

$$\xi_i^n \in \mathcal{C}^k(I, \mathfrak{g}) \quad \text{by} \quad \xi_i^n(t) := \frac{1}{n}\xi\left(\frac{i+t}{n}\right).$$

Then for each $m \le k$ we have $\frac{d^m}{dt^m}\xi_i^n(t) = \frac{1}{n^{m+1}}\xi^{(m)}\left(\frac{i+t}{n}\right)$. We conclude that, for $n \to \infty$, the sequence $\frac{d^m}{dt^m}\xi_i^n$ tends to 0 in $\mathcal{C}^0(I, \mathfrak{g})$, uniformly in i, and hence that ξ_i^n tends to 0 in $\mathcal{C}^k(I, \mathfrak{g})$, uniformly in i. In particular, there exists some $N \in \mathbb{N}$ for which $\xi_i^N \in U$ for $i = 0, 1, \ldots, N-1$.

We define a path $\gamma_\xi\colon I \to G$ by

$$\gamma_\xi(t) := \gamma_{\xi_0^N}(1) \cdots \gamma_{\xi_{i-1}^N}(1)\gamma_{\xi_i^N}(nt - i) \quad \text{for} \quad \frac{i}{n} \le t \le \frac{i+1}{n}$$

and observe that $\delta(\gamma_\xi)$ exists on all of I and equals ξ. We now put

$$\mathrm{Evol}_G(\xi) := \gamma_\xi \quad \text{and} \quad \mathrm{evol}_G(\xi) := \gamma_\xi(1).$$

Since, for each i, the assignment $\xi \mapsto \xi_i^N$ is linear and continuous, there exists an open neighborhood V of ξ such that $\eta_i^N \in U$ holds for each $\eta \in V$ and $i = 0, 1, \ldots, N-1$. It suffices to show that evol_G is smooth on V, but this follows from the fact that

$$\mathrm{evol}_G(\eta) = \mathrm{evol}_G(\eta_0^N) \cdots \mathrm{evol}_G(\eta_{N-1}^N)$$

is a product of N smooth functions. $\qquad\square$

For the definition of an initial Lie subgroup see [Ne06, Def. II.6.1].

Proposition B.6. *Let G be a $\mathcal{C}^k$-regular Lie group with Lie algebra $\mathfrak{g}$ and $H \le G$ an initial Lie subgroup with Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$, for which there exists an open identity neighborhood $U \subseteq G$ and a smooth function $f\colon U \to E$ into some locally convex space E, such that f is constant on $U \cap gH$ for each $g \in U$, and $H \cap U = f^{-1}(0) \cap U$. Then H is $\mathcal{C}^k$-regular.*

Proof. The $\mathcal{C}^k$-regularity of G implies the existence of a smooth evolution map

$$\mathrm{evol}_H\colon \mathcal{C}^k(I, \mathfrak{h}) \to G, \quad \xi \mapsto \gamma_\xi(1),$$

and since H is initial, it suffices to see that the range of this map lies in H.

If $\xi \in \mathcal{C}^k(I, \mathfrak{h})$ such that $\mathrm{im}(\gamma_\xi) \subseteq U$, then for every $t \in I$,

$$(f \circ \gamma_\xi)'(t) = \mathrm{d}f(\gamma_\xi(t))\gamma_\xi'(t) = 0$$

because $\gamma_\xi'(t) = \mathrm{d}\ell_{\gamma_\xi(t)}(\mathbf{1})\xi(t)$ is the derivative of a curve in the set $U \cap \gamma_\xi(t)H$, on which f is constant. Therefore $f \circ \gamma_\xi$ is constant, which leads to $\mathrm{im}(\gamma_\xi) \subseteq f^{-1}(f(\mathbf{1})) = f^{-1}(0) = U \cap H$.

If $\xi \in \mathcal{C}^k(I, \mathfrak{h})$ is arbitrary, we apply the preceding argument to the curves $t \mapsto \gamma_\xi(t_0)^{-1}\gamma_\xi(t)$ on sufficiently small intervals $[t_0, t_0 + \varepsilon]$ and see that $\mathrm{im}(\gamma_\xi)$ is contained in H. $\qquad\square$

Theorem B.7. ($\mathcal{C}^k$-regularity is an extension property) *Let $q\colon \widehat{G} \to G$ be an extension of the Lie group G by the Lie group N and $k \in \mathbb{N}_0$. Then $\widehat{G}$ is $\mathcal{C}^k$-regular if and only if N and G are $\mathcal{C}^k$-regular.*

Proof. **Step 1.** We assume that G and N are $\mathcal{C}^k$-regular and show that this implies the $\mathcal{C}^k$-regularity of $\widehat{G}$. Since G is $\mathcal{C}^k$-regular, the evolution map

$$\mathrm{Evol}_G\colon \mathcal{C}^k(I, \mathfrak{g}) \to \mathcal{C}_*^{k+1}(I, G)$$

is smooth [Gl12a, Thm A].

Let $U_G \subseteq G$ be an open $\mathbf{1}$-neighborhood for which we have a smooth section $\sigma\colon U_G \to \widehat{G}$ with $\sigma(\mathbf{1}_G) = \mathbf{1}_{\widehat{G}}$ and

$$p\colon \mathcal{C}^k(I, \widehat{\mathfrak{g}}) \to \mathcal{C}^k(I, \mathfrak{g}), \quad \xi \mapsto q_{\mathfrak{g}} \circ \xi$$

be the projection map. Then $V := p^{-1}(\mathrm{Evol}_G^{-1}(\mathcal{C}^{k+1}(I, U_G)))$ is an open 0-neighborhood in $\mathcal{C}^k(I, \widehat{\mathfrak{g}})$. Further, by Lemma A.2 the map

$$\Phi\colon V \to \mathcal{C}_*^{k+1}(I, \widehat{G}), \quad \xi \mapsto \sigma \circ \mathrm{Evol}_G(q_{\mathfrak{g}} \circ \xi)$$

is smooth. For $\xi \in \mathcal{C}^k(I, \widehat{\mathfrak{g}})$, we find

$$q_{\mathfrak{g}}(\xi * \Phi(\xi)^{-1}) = q_{\mathfrak{g}}\Big(\mathrm{Ad}(\Phi(\xi))(\xi - \delta(\Phi(\xi))) \Big)$$
$$= \mathrm{Ad}(q_G(\Phi(\xi)))\Big(q_{\mathfrak{g}} \circ \xi - \delta(q_G(\Phi(\xi))) \Big) = \mathrm{Ad}(q_G(\Phi(\xi)))\big(q_{\mathfrak{g}} \circ \xi - q_{\mathfrak{g}} \circ \xi \big) = 0.$$

This means that $\xi * \Phi(\xi)^{-1} \in \mathcal{C}^k(I, \mathfrak{n})$. Now Lemma B.3, applied to the action of $\mathcal{C}^{k+1}(I, \widehat{G})$ on $\Omega_{\mathcal{C}^k}^1(I, \widehat{\mathfrak{g}}) \cong \mathcal{C}^k(I, \widehat{\mathfrak{g}})$, shows that it depends smoothly on ξ. We thus obtain a smooth map

$$\widehat{E}\colon V \to \widehat{G}, \quad \widehat{E}(\xi) := \mathrm{evol}_N(\xi * \Phi(\xi)^{-1}) \cdot \Phi(\xi)(1).$$

The curve $\gamma := \mathrm{Evol}_N(\xi * \Phi(\xi)^{-1})\Phi(\xi)$ satisfies

$$\delta(\gamma) = \delta(\Phi(\xi)) + \mathrm{Ad}(\Phi(\xi))^{-1}\big(\delta(\Phi(\xi)^{-1}) + \mathrm{Ad}(\Phi(\xi))\xi\big) = \xi.$$

Therefore $\widehat{E}$ is a smooth local evolution map for $\widehat{G}$, and Lemma B.5 implies that $\widehat{G}$ is $\mathcal{C}^k$-regular.

Step 2. Conversely, we show that the $\mathcal{C}^k$-regularity of $\widehat{G}$ implies the $\mathcal{C}^k$-regularity of N and G. To see that N is $\mathcal{C}^k$-regular, we choose a chart (φ_G, U_G) of G and consider the map

$$f := \varphi_G \circ q\colon q^{-1}(U_G) \to \varphi_G(U_G),$$

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which is constant on the left cosets of N, lying in this set. Therefore Proposition B.6 implies that N is $\mathcal{C}^k$-regular because it is a submanifold of $\widehat{G}$, hence in particular an initial submanifold.

To see that G is $\mathcal{C}^k$-regular, we first choose a continuous linear section $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$, which induces a continuous linear section

$$\sigma_*\colon \mathcal{C}^k(I, \mathfrak{g}) \to \mathcal{C}^k(I, \widehat{\mathfrak{g}}), \quad \xi \mapsto \sigma \circ \xi.$$

Then, for each $\xi \in \mathcal{C}^k([0,1], \mathfrak{g})$, the curve $\gamma_\xi := q \circ \mathrm{Evol}_{\widehat{G}}(\sigma \circ \xi)$ satisfies

$$\delta(\gamma_\xi) = \mathbf{L}(q) \circ \sigma \circ \xi = \xi,$$

so that $\mathrm{evol}_G = q \circ \mathrm{evol}_{\widehat{G}} \circ \sigma$ is a composition of smooth maps, hence smooth. $\qquad\square$

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