

THE GELFAND–NAIMARK–SEGAL CONSTRUCTION FOR UNITARY REPRESENTATIONS OF \mathbb{Z}_2^n -GRADED LIE SUPERGROUPS

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Abstract. We establish a Gelfand–Naimark–Segal construction which yields a correspondence between cyclic unitary representations and positive definite superfunctions of a general class of \mathbb{Z}_2^n -graded Lie supergroups.

1. Introduction. It is by now well known that unitary representations of Lie supergroups appear in various areas of mathematical physics related to supersymmetry (see [8], [10], [9], [17] as some important examples among numerous references). A mathematical approach to analysis on unitary representations of Lie supergroups was pioneered in [3], where unitary representations are defined based on the notion of the Harish-Chandra pair associated to a Lie supergroup.

More recently, there is growing interest in studying generalized supergeometry, that is, geometry of graded manifolds where the grading group is not \mathbb{Z}_2 , but $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. The foundational aspects of the theory of \mathbb{Z}_2^n -supermanifolds were recently established in the works of Covolo–Grabowski–Poncin (see [5], [6]) and Covolo–Kwok–Poncin (see [7]).

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In this short note, we make a first attempt to extend the theory of unitary representations to the \mathbb{Z}_2^n -graded setting. To this end, we use the concept of a \mathbb{Z}_2^n -graded Harish-Chandra pair, which consists of a pair (G_0, \mathfrak{g}) where G_0 is a Lie group and \mathfrak{g} is a \mathbb{Z}_2^n -graded generalization of a Lie superalgebra, sometimes known as a *Lie color algebra*. Extending one of the main results of [15], in Theorem 5.4 we prove that under a “perfectness” condition on \mathfrak{g} , there exists a Gelfand–Naimark–Segal (GNS) construction which yields a correspondence between positive definite smooth super-functions and cyclic unitary representations of (G_0, \mathfrak{g}) . Theorem 5.4 is applicable to interesting examples, such as \mathbb{Z}_2^n -Lie supergroups of classical type, e.g., the Harish-Chandra pair corresponding to $\mathfrak{gl}(V)$ defined in Example 2.5, where V is a \mathbb{Z}_2^n -graded vector space.

The key technical tool in the proof of Theorem 5.4 is a Stability Theorem (see Theorem 4.3) which guarantees the existence of a unique unitary representation associated to a weaker structure, called a *pre-representation*. Such a Stability Theorem holds unconditionally when $n = 1$. But for $n > 1$, it is not true in general. We are able to retrieve a variation of the Stability Theorem under the aforementioned extra condition on \mathfrak{g} . However, this still leaves the question of a general GNS construction open for further investigation. We defer the latter question, presentation of explicit examples, as well as some proof details, to a future work.

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2. \mathbb{Z}_2^n -supergeometry. We begin by reviewing the basic concepts of \mathbb{Z}_2^n -graded supergeometry, in the sense of [6]. Let $\Gamma := \mathbb{Z}_2^n := \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ where $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$, and let $\mathfrak{b} : \Gamma \times \Gamma \rightarrow \mathbb{Z}_2$ be a non-degenerate symmetric \mathbb{Z}_2 -bilinear map. By a result of Albert (see [1, Thm 6] or [11, Sec. 1–10]), if $\mathfrak{b}(\cdot, \cdot)$ is of alternate type (i.e., $\mathfrak{b}(a, a) = \bar{0}$ for every $a \in \Gamma$), then $\mathfrak{b}(\cdot, \cdot)$ is equivalent to the standard “symplectic” form

$$\mathfrak{b}_-(a, b) := \sum_{j=1}^n a_{2j-1}b_{2j} + a_{2j}b_{2j-1} \quad \text{for every } a = (a_1, \dots, a_n) \text{ and } b = (b_1, \dots, b_n) \in \Gamma,$$

whereas if $\mathfrak{b}(a, a) \neq \bar{0}$ for some $a \in \Gamma$, then $\mathfrak{b}(\cdot, \cdot)$ is equivalent to the standard symmetric form

$$\mathfrak{b}_+(a, b) := \sum_{j=1}^m a_j b_j \quad \text{for every } a = (a_1, \dots, a_n) \text{ and } b = (b_1, \dots, b_n) \in \Gamma.$$

Henceforth we assume that $\mathfrak{b}(\cdot, \cdot)$ is not of alternate type. Since an equivalence of $\mathfrak{b}(\cdot, \cdot)$ and $\mathfrak{b}_+(\cdot, \cdot)$ is indeed an automorphism of the finite abelian group Γ , without loss of generality from now on we can assume that $\mathfrak{b} = \mathfrak{b}_+$. In particular, from now on we represent an element $a \in \Gamma$ by $a := \sum_{j=1}^n a_j \mathbf{e}_j$, where $\{\mathbf{e}_j\}_{j=1}^n$ is an orthonormal basis of

Γ with respect to $\mathbf{b}(\cdot, \cdot)$.

We now define $\beta : \Gamma \times \Gamma \rightarrow \{\pm 1\}$ by $\beta(a, b) := (-1)^{\mathbf{b}(a, b)}$ for $a, b \in \Gamma$. We equip the category \mathbf{Vec}_Γ of Γ -graded complex vector spaces with the symmetry operator

$$\mathbb{S}_{V, W} : V \otimes W \rightarrow W \otimes V, \quad \mathbb{S}_{V, W}(v \otimes w) := \beta(|v|, |w|)w \otimes v, \quad (1)$$

where $|v| \in \Gamma$ denotes the degree of a homogeneous vector $v \in V$.

REMARK 2.1. As it is customary in supergeometry, equality (1) should be construed as a relation for homogeneous vectors that is subsequently extended by linearity to non-homogeneous vectors. In the rest of the manuscript we will stick to this convention.

Equipped with \mathbb{S} and the usual Γ -graded tensor product of vector spaces, \mathbf{Vec}_Γ is a symmetric monoidal category. This fact was also observed in [2, Prop. 1.5].

REMARK 2.2. Note that $\beta : \Gamma \times \Gamma \rightarrow \{\pm 1\}$ is a 2-cocycle. In fact it represents the obstruction to the lifting of the map $\Gamma \rightarrow \{\pm 1\}$, $(a_1, \dots, a_n) \mapsto (-1)^{\sum_{i=1}^n a_i}$, with respect to the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \{\pm 1, \pm i\} \xrightarrow{t \mapsto t^2} \{\pm 1\} \longrightarrow 1,$$

where $i := \sqrt{-1}$. More precisely, the lifting obstruction cocycle is naturally represented by

$$\delta : \Gamma \times \Gamma \rightarrow \{\pm 1\}, \quad (a_1, \dots, a_n) \mapsto (-1)^{\sum_{1 \leq i, j \leq n} a_i b_j},$$

but $\beta - \delta = d\eta$ where $\eta(a_1, \dots, a_n) := (-1)^{\sum_{1 \leq i \neq j \leq n} a_i a_j}$. We thank Professor Karl-Hermann Neeb for letting us know about this property of β .

Let \prec denote the lexicographic order on elements of Γ . That is, for $a := \sum_{j=1}^n a_j \mathbf{e}_j$ and $b := \sum_{j=1}^n b_j \mathbf{e}_j$, we set $a \prec b$ if and only if there exists some $1 \leq j \leq n$ such that $a_j = \bar{0}$, $b_j = \bar{1}$, and $a_k = b_k$ for all $k < j$. Thus we can express Γ as $\Gamma = \{\gamma_0 \prec \gamma_1 \prec \dots \prec \gamma_{2^n-1}\}$, where $\gamma_0 = 0$.

Let $p \in \mathbb{N} \cup \{0\}$ and let $\mathbf{q} := (q_1, \dots, q_{2^n-1})$ be a $(2^n - 1)$ -tuple such that $q_j \in \mathbb{N} \cup \{0\}$ for all j . We set $|\mathbf{q}| := \sum_{j=1}^{2^n-1} q_j$. By a Γ -superdomain of dimension $p|\mathbf{q}|$ we mean a locally ringed space (U, \mathcal{O}_U) such that $U \subset \mathbb{R}^p$ is an open set, and the structure sheaf \mathcal{O}_U is given by

$$\mathcal{O}_U(U') := C^\infty(U'; \mathbb{C})[[\xi_1, \dots, \xi_{|\mathbf{q}|}]] \quad \text{for every open set } U' \subseteq U,$$

where the right hand side denotes the Γ -graded algebra of formal power series in variables $\xi_1, \dots, \xi_{|\mathbf{q}|}$ with coefficients in $C^\infty(U'; \mathbb{C})$, such that $|\xi_r| = \gamma_k$ for $\sum_{j=1}^{k-1} q_j < r \leq \sum_{j=1}^k q_j$, subject to relations

$$\xi_r \xi_s = \beta(|\xi_r|, |\xi_s|) \xi_s \xi_r \quad \text{for every } 1 \leq r, s \leq |\mathbf{q}|.$$

By a smooth Γ -supermanifold M of dimension $p|\mathbf{q}|$ we mean a locally ringed space (M, \mathcal{O}_M) that is locally isomorphic to a $p|\mathbf{q}|$ -dimensional Γ -superdomain. From this viewpoint, a Γ -Lie supergroup is a group object in the category of smooth Γ -supermanifolds.

As one expects, to a Γ -Lie supergroup one can canonically associate a Γ -Lie superalgebra, which is an object of \mathbf{Vec}_Γ of the form $\mathfrak{g} = \bigoplus_{a \in \Gamma} \mathfrak{g}_a$, equipped with a Γ -superbracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that satisfies the following properties:

- (i) $[\cdot, \cdot]$ is bilinear, and $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{ab}$ for $a, b \in \Gamma$.
- (ii) $[x, y] = -\beta(a, b)[y, x]$ for $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$, where $a, b \in \Gamma$.
- (iii) $[x, [y, z]] = [[x, y], z] + \beta(a, b)\beta(a, c)[y, [z, x]]$ for $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b, z \in \mathfrak{g}_c$, where $a, b, c \in \Gamma$.

REMARK 2.3. We remark that a Γ -Lie superalgebra is more commonly known as a *Lie color algebra*. Nevertheless, in order to keep our nomenclature compatible with [6], we use the term Γ -Lie superalgebra instead.

From classical supergeometry (that is, when $n = 1$) one knows that the category of Lie supergroups can be replaced by another category with a more concrete structure, known as the category of *Harish-Chandra pairs* (see [12], [13]). A similar statement holds in the case of Γ -Lie supergroups.

DEFINITION 2.4. A Γ -*Harish-Chandra pair* is a pair (G_0, \mathfrak{g}) where G_0 is a Lie group, and $\mathfrak{g} = \bigoplus_{a \in \Gamma} \mathfrak{g}_a$ is a Γ -Lie superalgebra equipped with an action $\text{Ad} : G_0 \times \mathfrak{g} \rightarrow \mathfrak{g}$ of G_0 by linear operators that preserve the Γ -grading, which extends the adjoint action of G_0 on $\mathfrak{g}_0 \cong \text{Lie}(G_0)$.

EXAMPLE 2.5. Let $V := \bigoplus_{a \in \Gamma} V_a$ be a Γ -graded vector space. The Γ -Lie superalgebra $\mathfrak{gl}(V)$ is the vector space of linear transformations on V , with superbracket $[S, T] := ST - \beta(|S|, |T|)TS$. One can also consider the Γ -Harish-Chandra pair (G_0, \mathfrak{g}) , where $G_0 \cong \prod_{a \in \Gamma} \text{GL}(V_a)$.

Indeed the Γ -Harish-Chandra pairs form a category in a natural way. The morphisms of this category are pairs of maps $(\phi, \varphi) : (G, \mathfrak{g}) \rightarrow (H, \mathfrak{h})$, where $\phi : G \rightarrow H$ is a Lie group homomorphism and $\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a morphism in the category of Γ -Lie superalgebras such that $\varphi|_{\mathfrak{g}_0} = d\phi$. The following statement plays a key role in the study of Γ -Lie supergroups.

PROPOSITION 2.6. *The category of Γ -Lie supergroups is isomorphic to the category of Γ -Harish-Chandra pairs.*

Proof. The proof is a straightforward but lengthy modification of the argument for the analogous result in the case of ordinary supergeometry (see [12], [13], or [4]). Therefore we only sketch an outline of the proof. The functor from Γ -Lie supergroups to Γ -Harish-Chandra pairs is easy to describe: a Γ -Lie supergroup (G_0, \mathcal{O}_{G_0}) is associated to the Harish-Chandra pair (G_0, \mathfrak{g}) , where \mathfrak{g} is the Γ -Lie superalgebra of (G_0, \mathcal{O}_{G_0}) . As in the classical super case, the functor associates a homomorphism of Γ -Lie supergroups $(G_0, \mathcal{O}_{G_0}) \rightarrow (H_0, \mathcal{O}_{H_0})$ to the pair of underlying maps $G_0 \rightarrow H_0$ and the tangent map at identity $\mathfrak{g} \rightarrow \mathfrak{h}$. Conversely, from a Γ -Harish-Chandra pair (G_0, \mathfrak{g}) we construct a Lie supergroup (G_0, \mathcal{O}_{G_0}) as follows. For every open set $U \subseteq G_0$, we set $\overline{\mathcal{O}}_{G_0}(U) := \text{Hom}_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}), C^\infty(U; \mathbb{C}))$, where $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ denotes the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. The Γ -superalgebra structure on $\overline{\mathcal{O}}_{G_0}(U)$ is defined using the algebra structure of $C^\infty(U; \mathbb{C})$ and the Γ -coalgebra structure of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, exactly as in the classical super case. Using the PBW Theorem for Γ -Lie superalgebras (see [18]) one can see that $(G_0, \overline{\mathcal{O}}_{G_0})$ is indeed a Γ -supermanifold (see [4, Prop. 7.4.9]). The definition of the Γ -Lie supergroup structure of $(G_0, \overline{\mathcal{O}}_{G_0})$ is similar to the classical case as well (see [4,

Prop. 7.4.10]). Finally, to show that the two functors are inverse to each other, we need a sheaf isomorphism $\mathcal{O}_{G_0} \cong \overline{\mathcal{O}}_{G_0}$. This sheaf isomorphism is given by the maps

$$\mathcal{O}_{G_0}(U) \rightarrow \overline{\mathcal{O}}_{G_0}(U), \quad s \mapsto \left[D \mapsto \beta(|D|, |s|) \widetilde{L}_D s \right] \text{ for } D \in \mathfrak{U}(\mathfrak{g}_C),$$

where $|D|, |s| \in \Gamma$ are the naturally defined degrees, L_D denotes the left invariant differential operator on (G_0, \mathcal{O}_{G_0}) corresponding to D , and $\widetilde{L}_D s$ means evaluation of the section at points of U . The proof of bijective correspondence of morphisms in the two categories is similar to [4, Prop. 7.4.12]. ■

3. Γ -Hilbert superspaces and unitary representations. In order to define a unitary representation of a Γ -Harish-Chandra pair, one needs to obtain a well-behaved definition of Hilbert spaces in the category Vec_Γ . This is our first goal in this section.

For any $a = \sum_{j=1}^n a_j \mathbf{e}_j \in \Gamma$, set $\mathbf{u}(a) := |\{1 \leq j \leq n : a_j = \overline{1}\}|$ (for example, $\mathbf{u}(\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4) = 3$) and define

$$\alpha(a) := e^{\frac{\pi i}{2} \mathbf{u}(a)}. \quad (2)$$

DEFINITION 3.1. Let $\mathcal{H} \in \text{Obj}(\text{Vec}_\Gamma)$. We call \mathcal{H} a Γ -inner product space if and only if it is equipped with a non-degenerate sesquilinear form $\langle \cdot, \cdot \rangle$ that satisfies the following properties:

- i) $\langle \mathcal{H}_a, \mathcal{H}_b \rangle = 0$, for $a, b \in \Gamma$ such that $a \neq b$.
- ii) $\langle w, v \rangle = \beta(a, a) \overline{\langle v, w \rangle}$, for $v, w \in \mathcal{H}_a$ where $a \in \Gamma$.
- iii) $\alpha(a) \langle v, v \rangle \geq 0$, for every $v \in \mathcal{H}_a$ where $a \in \Gamma$.

REMARK 3.2. The choice of α in (2) is made as follows. The most natural property that one expects from Hilbert spaces is that the tensor product of (pre-)Hilbert spaces is a (pre-)Hilbert space. Thus, we are seeking $\alpha : \Gamma \rightarrow \mathbb{C}^\times$ such that the tensor product of two Γ -inner product spaces is also a Γ -inner product space. Clearly after scaling the values of α by positive real numbers we can assume that $\alpha(\Gamma) \subseteq \{\pm 1, \pm i\}$. For two Γ -inner product spaces \mathcal{H} and \mathcal{K} , one has

$$(\mathcal{H} \otimes \mathcal{K})_a := \bigoplus_{bc=a} \mathcal{H}_b \otimes \mathcal{K}_c,$$

equipped with the canonically induced sesquilinear form

$$\langle v \otimes w, v' \otimes w' \rangle_{\mathcal{H} \otimes \mathcal{K}} := \beta(|w|, |v'|) \langle v, v' \rangle_{\mathcal{H}} \langle w, w' \rangle_{\mathcal{K}}.$$

Closedness under tensor product implies that

$$\alpha(bc) = \beta(b, c) \alpha(b) \alpha(c), \quad \text{for all } b, c \in \Gamma. \quad (3)$$

In the language of group cohomology, this means that the 2-cocycle β satisfies $\beta = d\alpha$. Consequently, up to twisting by a group homomorphism $\Gamma \rightarrow \{\pm 1\}$, there exists a unique α which satisfies the latter relation.

REMARK 3.3. Associated to any Γ -inner product space \mathcal{H} , there is an inner product (in the ordinary sense) defined by

$$(v, w) := \begin{cases} 0 & \text{if } |v| \neq |w|, \\ \alpha(a) \langle v, w \rangle & \text{if } |v| = |w| = a \text{ where } a \in \Gamma. \end{cases}$$

The vector space \mathcal{H} , equipped with (\cdot, \cdot) , is indeed a pre-Hilbert space in the usual sense. Thus we can consider the completion of \mathcal{H} with respect to the norm $\|v\| := (v, v)^{\frac{1}{2}}$. By reversing the process of obtaining (\cdot, \cdot) from $\langle \cdot, \cdot \rangle$, we obtain a Γ -inner product on the completion of \mathcal{H} .

DEFINITION 3.4. Let \mathcal{H} be a Γ -inner product space, and let $T \in \text{End}_{\mathbb{C}}(\mathcal{H})$. We define the *adjoint* T^{\dagger} of T as follows. If $T \in \text{End}_{\mathbb{C}}(\mathcal{H})_a$ for some $a \in \Gamma$, we define T^{\dagger} by

$$\langle v, Tw \rangle = \beta(a, b) \langle T^{\dagger}v, w \rangle,$$

where $v \in \mathcal{H}_b$. We then extend the assignment $T \mapsto T^{\dagger}$ to a conjugate-linear map on $\text{End}_{\mathbb{C}}(\mathcal{H})$.

Now let \mathcal{H} be a Γ -inner product space, and let (\cdot, \cdot) be the ordinary inner product associated to \mathcal{H} as in Remark 3.3. Then for a linear map $T : \mathcal{H} \rightarrow \mathcal{H}$ we can define an adjoint with respect to (\cdot, \cdot) , by the relation

$$(Tv, w) = (v, T^*w) \quad \text{for every } v, w \in \mathcal{H}.$$

A straightforward calculation yields

$$T^* = \overline{\alpha(|T|)}T^{\dagger}.$$

It follows that $T^{\dagger\dagger} = T$. Furthermore, $(ST)^{\dagger} = \beta(|S|, |T|)T^{\dagger}S^{\dagger}$.

We are now ready to define unitary representations of Γ -Harish-Chandra pairs. The definition of a unitary representation of a Γ -Harish-Chandra pair is a natural extension of the one for the \mathbb{Z}_2 -graded case. First recall that for a unitary representation (π, \mathcal{H}) of a Lie group G on a Hilbert space \mathcal{H} , we denote the space of C^{∞} vectors by \mathcal{H}^{∞} . Thus the vector space \mathcal{H}^{∞} consists of all vectors $v \in \mathcal{H}$ for which the map $G \rightarrow \mathcal{H}$, $g \mapsto \pi(g)v$ is smooth. Furthermore, given $x \in \text{Lie}(G)$ and $v \in \mathcal{H}$, we set

$$d\pi(x) := \lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp(tx))v - v),$$

whenever the limit exists. We denote the domain of the unbounded operator $d\pi(x)$ by $\mathcal{D}(d\pi(x))$. The unbounded operator $-id\pi(x)$ is the self-adjoint generator of the one-parameter unitary representation $t \mapsto \pi(\exp(tx))$. For a comprehensive exposition of the theory of unitary representations and relevant facts from the theory of unbounded operators, see [19].

DEFINITION 3.5. A *smooth unitary representation* of a Γ -Harish-Chandra pair (G_0, \mathfrak{g}) is a triple $(\pi, \rho^{\pi}, \mathcal{H})$ that satisfies the following properties:

- (R1) (π, \mathcal{H}) is a smooth unitary representation of the Lie group G_0 on the Γ -graded Hilbert space \mathcal{H} by operators $\pi(g)$, $g \in G_0$, which preserve the Γ -grading.
- (R2) $\rho^{\pi} : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}^{\infty})$ is a representation of the Γ -Lie superalgebra \mathfrak{g} .
- (R3) $\rho^{\pi}(x) = d\pi(x)|_{\mathcal{H}^{\infty}}$ for $x \in \mathfrak{g}_0$.
- (R4) $\rho^{\pi}(x)^{\dagger} = -\rho^{\pi}(x)$ for $x \in \mathfrak{g}$.
- (R5) $\pi(g)\rho^{\pi}(x)\pi(g)^{-1} = \rho^{\pi}(\text{Ad}(g)x)$ for $g \in G_0$ and $x \in \mathfrak{g}$.

Unitary representations of a Γ -Harish-Chandra pair form a category $\text{Rep} = \text{Rep}(G_0, \mathfrak{g})$. A morphism in this category from $(\pi, \rho^{\pi}, \mathcal{H})$ to $(\sigma, \rho^{\sigma}, \mathcal{H})$ is a bounded linear map

$T : \mathcal{H} \rightarrow \mathcal{K}$ which respects the Γ -grading and satisfies $T\pi(g) = \sigma(g)T$ for $g \in G_0$ (from which it follows that $T\mathcal{H}^\infty \subseteq \mathcal{K}^\infty$) and $T\rho^\pi(x) = \rho^\sigma(x)T$ for $x \in \mathfrak{g}$.

REMARK 3.6. At first glance, it seems that the definition of a unitary representation of a Γ -Harish-Chandra pair depends on the choice of α . Nevertheless, it is not difficult to verify that for two coboundaries α, α' which satisfy (3), the corresponding categories Rep_α and $\text{Rep}_{\alpha'}$ are isomorphic. Indeed if $\chi : \Gamma \rightarrow \{\pm 1\}$ is a group homomorphism such that $\alpha' = \chi\alpha$, then we can define a functor $\mathcal{F} : \text{Rep}_\alpha \rightarrow \text{Rep}_{\alpha'}$ which maps $(\pi, \rho^\pi, \mathcal{H})$ to $(\pi', \rho^{\pi'}, \mathcal{H}')$ where $\mathcal{H}' := \mathcal{H}$ (but the Γ -inner product of \mathcal{H}' is defined by $\langle v, v \rangle_{\mathcal{H}'} = \chi(a)\langle v, v \rangle_{\mathcal{H}}$ for $v \in \mathcal{H}_a$ where $a \in \Gamma$), $\pi' := \pi$, and $\rho^{\pi'}(x) := \chi(a)\rho^\pi(x)$ for $x \in \mathfrak{g}_a$ and $a \in \Gamma$. The functor \mathcal{F} is defined to be identity on morphisms. That is, for a morphism $T : (\pi, \rho^\pi, \mathcal{H}) \rightarrow (\sigma, \rho^\sigma, \mathcal{K})$ we set $\mathcal{F}(T) := T$. It is straightforward to see that with $(\pi', \rho^{\pi'}, \mathcal{H}')$ and $(\sigma', \rho^{\sigma'}, \mathcal{K}')$ defined as above, the map $T : (\pi', \rho^{\pi'}, \mathcal{H}') \rightarrow (\sigma', \rho^{\sigma'}, \mathcal{K}')$ is still an intertwining map. The main point is that $\rho^{\pi'}$ and $\rho^{\sigma'}$ are obtained from ρ^π and ρ^σ via scaling by the same scalar. The inverse of \mathcal{F} is defined similarly.

4. The stability theorem. We now proceed towards the statement and proof of the Stability Theorem. In what follows, we will need the following technical definition (see [14]).

DEFINITION 4.1. Let (G_0, \mathfrak{g}) be a Γ -Harish-Chandra pair. By a *pre-representation* of (G_0, \mathfrak{g}) , we mean a 4-tuple $(\pi, \mathcal{H}, \mathcal{B}, \rho^\mathcal{B})$ that satisfies the following conditions:

- (PR1) (π, \mathcal{H}) is a smooth unitary representation of the Lie group G_0 on the Γ -graded Hilbert space \mathcal{H} by operators $\pi(g)$, $g \in G_0$, which preserve the Γ -grading.
- (PR2) \mathcal{B} is a dense, G_0 -invariant, and Γ -graded subspace of \mathcal{H} that is contained in $\bigcap_{x \in \mathfrak{g}_0} \mathcal{D}(\text{d}\pi(x))$.
- (PR3) $\rho^\mathcal{B} : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{B})$ is a representation of the Γ -Lie superalgebra \mathfrak{g} .
- (PR4) $\rho^\mathcal{B}(x) = \text{d}\pi(x)|_{\mathcal{B}}$ and $\rho^\mathcal{B}(x)$ is essentially skew adjoint for $x \in \mathfrak{g}_0$.
- (PR5) $\rho^\mathcal{B}(x)^\dagger = -\rho^\mathcal{B}(x)$ for $x \in \mathfrak{g}$.
- (PR6) $\pi(g)\rho^\mathcal{B}(x)\pi(g)^{-1} = \rho^\mathcal{B}(\text{Ad}(g)x)$ for $g \in G_0$ and $x \in \mathfrak{g}$.

REMARK 4.2. It is shown in [14, Rem. 6.5] that (PR2) and (PR3) imply that $\mathcal{B} \subseteq \mathcal{H}^\infty$.

Set

$$\Gamma_{\bar{0}} := \{a \in \Gamma : \mathfrak{b}(a, a) = \bar{0}\} \text{ and } \Gamma_{\bar{1}} := \{a \in \Gamma : \mathfrak{b}(a, a) = \bar{1}\}$$

THEOREM 4.3. (**Stability Theorem**) *Let $(\pi, \mathcal{H}, \mathcal{B}, \rho^\mathcal{B})$ be a pre-representation of a Γ -Harish-Chandra pair (G_0, \mathfrak{g}) . Assume that for every $a \in \Gamma_{\bar{0}} \setminus \{0\}$ we have*

$$\mathfrak{g}_a = \sum_{b, c \in \Gamma_{\bar{1}}, bc=a} [\mathfrak{g}_b, \mathfrak{g}_c].$$

Then there exists a unique extension of $\rho^\mathcal{B}$ to a linear map $\rho^\pi : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}^\infty)$ such that $(\pi, \rho^\pi, \mathcal{H})$ is a smooth unitary representation of (G_0, \mathfrak{g}) .

Proof. For every $a \in \Gamma_{\bar{1}}$ the direct sum $\mathfrak{g}_0 \oplus \mathfrak{g}_a$ is a Lie superalgebra (in the \mathbb{Z}_2 -graded

sense). We define a \mathbb{Z}_2 -grading of \mathcal{H} by

$$\mathcal{H}_{\bar{0}} := \bigoplus_{\mathfrak{b}(a,b)=\bar{0}} \mathcal{H}_b \quad \text{and} \quad \mathcal{H}_{\bar{1}} := \bigoplus_{\mathfrak{b}(a,b)=\bar{1}} \mathcal{H}_b,$$

and then we use the Stability Theorem in the \mathbb{Z}_2 -graded case (see [14, Thm 6.14]) to obtain that there exists a unique Γ -Lie superalgebra homomorphism $\rho^{\pi,a} : \mathfrak{g}_0 \oplus \mathfrak{g}_a \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}^\infty)$, such that $(\pi, \rho^{\pi,a}, \mathcal{H})$ is a unitary representation of the (\mathbb{Z}_2 -graded) Harish-Chandra pair $(G_0, \mathfrak{g}_0 \oplus \mathfrak{g}_a)$. Since the maps $\rho^{\pi,a}$ agree on \mathfrak{g}_0 , they give rise to a linear map

$$\mathfrak{g}_0 \oplus \left(\bigoplus_{a \in \Gamma_{\bar{1}}} \mathfrak{g}_a \right) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}^\infty).$$

It remains to obtain a suitable extension of the latter map to \mathfrak{g}_a for every $a \in \Gamma_{\bar{0}} \setminus \{0\}$.

Fix $x \in \mathfrak{g}_a$, $a \in \Gamma_{\bar{0}} \setminus \{0\}$. Then we can write $x = \sum_j [y_j, z_j]$ such that $y_j \in \mathfrak{g}_{b_j}$, $z_j \in \mathfrak{g}_{c_j}$, where $b_j, c_j \in \Gamma_{\bar{1}}$, and $b_j c_j = a$. For every $v \in \mathcal{H}^\infty$, we define

$$\rho^\pi(x)v := \sum_j \left(\rho^{\pi,b_j}(y_j) \rho^{\pi,c_j}(z_j)v - \beta(b_j, c_j) \rho^{\pi,c_j}(z_j) \rho^{\pi,b_j}(y_j)v \right). \quad (4)$$

Let us first verify that $\rho^\pi(x)v$ is well-defined. To this end, it suffices to show that if $\sum_j [y_j, z_j] = 0$, then the right hand side of (4) vanishes. To verify the latter assertion, observe that for every $w \in \mathcal{B}_s$, $s \in \Gamma$, we have

$$\begin{aligned} \langle w, \rho^\pi(x)v \rangle &= \sum_j \beta(b_j, s) \beta(c_j, sb_j) \langle \rho^{\pi,c_j}(z_j)^\dagger \rho^{\pi,b_j}(y_j)^\dagger w, v \rangle \\ &\quad - \sum_j \beta(c_j, s) \beta(b_j, sc_j) \beta(b_j, c_j) \langle \rho^{\pi,b_j}(y_j)^\dagger \rho^{\pi,c_j}(z_j)^\dagger w, v \rangle \\ &= \beta(a, s) \left\langle \sum_j \beta(b_j, c_j) \rho^{\pi,c_j}(z_j)^\dagger \rho^{\pi,b_j}(y_j)^\dagger w - \sum_j \rho^{\pi,b_j}(y_j)^\dagger \rho^{\pi,c_j}(z_j)^\dagger w, v \right\rangle \\ &= -\beta(a, s) \left\langle \sum_j \rho^{\mathcal{B}}([y_j, z_j])w, v \right\rangle = 0. \end{aligned}$$

The assertion now follows from density of \mathcal{B} in \mathcal{H} .

It remains to show that given any $x_a \in \mathfrak{g}_a$ and $y_b \in \mathfrak{g}_b$, where $a, b \in \Gamma$, we have

$$\rho^\pi([x_a, y_b])v = (\rho^\pi(x_a) \rho^\pi(y_b) - \beta(a, b) \rho^\pi(y_b) \rho^\pi(x_a))v \quad \text{for } v \in \mathcal{H}^\infty. \quad (5)$$

To verify the last equality note that for every $w \in \mathcal{B}_s$ where $s \in \Gamma$,

$$\begin{aligned} \langle w, \rho^\pi(x_a) \rho^\pi(y_b)v - \beta(a, b) \rho^\pi(y_b) \rho^\pi(x_a)v \rangle &= \beta(a, s) \beta(as, b) \langle \rho^\pi(y_b)^\dagger \rho^\pi(x_a)^\dagger w, v \rangle - \beta(b, s) \beta(s, a) \langle \rho^\pi(x_a)^\dagger \rho^\pi(y_b)^\dagger w, v \rangle \\ &= -\beta(a, s) \beta(b, s) \left\langle \rho^{\mathcal{B}}(x_a) \rho^{\mathcal{B}}(y_b)w - \beta(a, b) \rho^{\mathcal{B}}(y_b) \rho^{\mathcal{B}}(x_a)w, v \right\rangle \\ &= \beta(a, s) \beta(b, s) \left\langle \rho^{\mathcal{B}}([x_a, y_b])^\dagger w, v \right\rangle = \left\langle w, \rho^\pi([x_a, y_b])v \right\rangle. \end{aligned}$$

Again density of \mathcal{B} in \mathcal{H} implies that both sides of (5) are equal. Finally, uniqueness of $\rho^\pi(x)$ can be proved by the same technique. ■

EXAMPLE 4.4. For $n > 1$, one cannot expect the Stability Theorem to hold without any condition on \mathfrak{g} . For example, let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, let G_0 be the trivial group, and let $\mathfrak{g} = \mathfrak{g}_{00} \oplus \mathfrak{g}_{01} \oplus \mathfrak{g}_{10} \oplus \mathfrak{g}_{11}$ where

$$\mathfrak{g}_{11} := \mathbb{R} \text{ and } \mathfrak{g}_{00} := \mathfrak{g}_{01} := \mathfrak{g}_{10} := \{0\}.$$

Then a pre-representation of (G_0, \mathfrak{g}) is the same (up to scaling) as a symmetric operator defined on a dense subspace of a Hilbert space, whereas a unitary representation of (G_0, \mathfrak{g}) is the same (up to scaling) as a *bounded* self-adjoint operator. Therefore a pre-representation does not necessarily extend to a unitary representation.

5. The GNS representation. Our goal in this section is to extend the GNS construction of [15] to the setting of Γ -Harish-Chandra pairs. We begin by outlining some generalities about this construction. Let (G_0, \mathfrak{g}) be a Γ -Harish-Chandra pair, and let $\mathcal{G} := (G_0, \mathcal{O}_{G_0})$ denote the Γ -Lie supergroup corresponding to (G_0, \mathfrak{g}) . Our main goal, to be established in Theorem 5.4, is to construct a correspondence between unitary representations of (G_0, \mathfrak{g}) with a cyclic vector, and smooth positive definite functions on \mathcal{G} . The suitable definition of a unitary representation with a cyclic vector is given in Definition 5.3. The main subtlety is to define positive definite functions on \mathcal{G} . To this end, we use the method developed in [15] for the classical super case, which we describe below. Recall that the Γ -superalgebra $C^\infty(\mathcal{G})$ of smooth functions on \mathcal{G} is isomorphic to

$$\text{Hom}_{\mathfrak{g}_0}(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}), C^\infty(G_0; \mathbb{C})),$$

where \mathfrak{g}_0 acts on $C^\infty(G_0; \mathbb{C})$ by left invariant differential operators. We realize elements of the latter algebra as functions on a semigroup \mathcal{S} which is equipped with an involution. Then we use the abstract definition of a positive definite function on a semigroup with an involution (see Definition 5.1).

Given a unitary representation of (G_0, \mathfrak{g}) with a cyclic vector, the matrix coefficient of that cyclic vector will be a positive definite element of $C^\infty(\mathcal{G})$. Conversely, from a positive definite $f \in C^\infty(\mathcal{G})$, we construct a unitary representation of (G_0, \mathfrak{g}) by considering the reproducing kernel Hilbert space associated to f , now realized as a function on \mathcal{S} . The Lie group G_0 has a canonical action on the reproducing kernel Hilbert space. However, the Γ -Lie superalgebra \mathfrak{g} acts on a dense subspace of the latter Hilbert space, which is in general strictly smaller than the space of smooth vectors for the G_0 -action. The main part of the proof of Theorem 5.4 is to show that the action of \mathfrak{g} is well defined on the entire space of smooth vectors. For this we need Theorem 4.3, which is the reason for presence of the condition (8) in Theorem 5.4.

Set $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, that is, the quotient $T(\mathfrak{g}_{\mathbb{C}})/I$, where $T(\mathfrak{g}_{\mathbb{C}})$ denotes the tensor algebra of $\mathfrak{g}_{\mathbb{C}}$ in the category Vec_{Γ} , and I denotes the two-sided ideal of $T(\mathfrak{g}_{\mathbb{C}})$ generated by elements of the form

$$x \otimes y - \beta(|x|, |y|)y \otimes x - [x, y], \quad \text{for homogeneous } x, y \in \mathfrak{g}_{\mathbb{C}}.$$

See [16, Sec. 2.1] for further details.

Let $x \mapsto x^*$ be the (unique) conjugate-linear map on $\mathfrak{g}_{\mathbb{C}}$ that is defined by the relation

$$x^* := -\overline{\alpha(a)}x, \quad \text{for every } x \in \mathfrak{g}_a, a \in \Gamma. \quad (6)$$

We extend the map $x \mapsto x^*$ to a conjugate-linear anti-automorphism of the algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. Thus $(D_1 D_2)^* = D_2^* D_1^*$ for every $D_1, D_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. Such an extension is possible because of $*$ -invariance of I . We now define a monoid

$$\mathcal{S} := G_0 \ltimes \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}),$$

with a multiplication given by

$$(g_1, D_1)(g_2, D_2) := (g_1 g_2, (\text{Ad}(g_2^{-1})(D_1))D_2).$$

The neutral element of \mathcal{S} is $1_{\mathcal{S}} := (1_{G_0}, 1_{\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})})$. The map

$$\mathcal{S} \rightarrow \mathcal{S}, (g, D) \mapsto (g, D)^* := (g^{-1}, \text{Ad}(g)(D^*))$$

is an involution of \mathcal{S} .

The proof of the latter assertion is similar to the one in the \mathbb{Z}_2 -graded case (see [13], [12], [15, Thm 5.5.2]).

Next, for every $f \in C^\infty(\mathcal{G})$, we define a map

$$\check{f} : \mathcal{S} \rightarrow \mathbb{C}, (g, D) \mapsto f(D)(g).$$

Also, for any $a \in \Gamma$ we set $\mathcal{S}_a := \{(g, D) \in \mathcal{S} : |D| = a\}$.

DEFINITION 5.1. An $f \in C^\infty(\mathcal{G})$ is called *positive definite* if it satisfies the following two conditions:

- (i) $\check{f}(g, D) = 0$ unless $(g, D) \in \mathcal{S}_0$.
- (ii) $\sum_{1 \leq i, j \leq n} \bar{c}_i c_j \check{f}(s_i^* s_j) \geq 0$, for all $n \geq 1, c_1, \dots, c_n \in \mathbb{C}, s_1, \dots, s_n \in \mathcal{S}$.

Given a unitary representation $(\pi, \rho^\pi, \mathcal{H})$ of G , for any two vectors $v, w \in \mathcal{H}$ we define the matrix coefficient $\varphi_{v,w}$ to be the map

$$\varphi_{v,w} : \mathcal{S} \rightarrow \mathbb{C}, \varphi_{v,w}(g, D) := (\pi(g)\rho^\pi(D)v, w).$$

PROPOSITION 5.2. *Let $(\pi, \rho^\pi, \mathcal{H})$ be a smooth unitary representation of (G_0, \mathfrak{g}) , and let $v, w \in \mathcal{H}^\infty$ be homogeneous vectors such that $|v| = |w|$. Then there exists an $f \in C^\infty(\mathcal{G})$ such that $\check{f} = \varphi_{v,w}$. Furthermore, $\check{f}(s) = 0$ unless $s \in \mathcal{S}_0$. If $v = w$ then \check{f} is positive definite.*

Proof. Similar to [15, Prop. 6.5.2]. ■

We are now ready to describe the GNS construction for unitary representations of Γ -Harish-Chandra pairs. It is an extension of the one given in [15, Sec. 6] in the \mathbb{Z}_2 -graded case, and therefore we will skip the proof details.

Let $(\pi, \rho^\pi, \mathcal{H})$ be a smooth unitary representation of (G_0, \mathfrak{g}) . One can construct a $*$ -representation $\widetilde{\rho}^\pi$ of the monoid \mathcal{S} by setting

$$\widetilde{\rho}^\pi : \mathcal{S} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}^\infty), (g, D) \mapsto \pi(g)\rho^\pi(D).$$

Being a $*$ -representation means that $\widetilde{\rho}^\pi(s^*) = \widetilde{\rho}^\pi(s)^*$ for every $s \in \mathcal{S}$, and in particular

$$(\widetilde{\rho}^\pi(s)v, w) = (v, \widetilde{\rho}^\pi(s^*)w). \quad (7)$$

By Proposition 5.2, for every matrix coefficient $\varphi_{v,v}$, where $v \in \mathcal{H}$, there exists a positive definite $f \in C^\infty(\mathcal{G})$ such that $\varphi_{v,v} = \check{f}$.

Conversely, given a positive definite function $f \in C^\infty(\mathcal{G})$, one can associate a $*$ -representation of \mathcal{S} to f as follows. Set $\psi := \check{f}$, and for every $s \in \mathcal{S}$ let $\psi_s : \mathcal{S} \rightarrow \mathbb{C}$ be the map given by $\psi_s(t) := \psi(ts)$. We also set

$$\mathfrak{D}_\psi := \text{Span}_{\mathbb{C}} \{ \psi_s : s \in \mathcal{S} \}.$$

Note that \mathfrak{D}_ψ is a Γ -graded vector space of complex-valued functions on \mathcal{S} , where the homogeneous parts of the Γ -grading are defined by

$$\mathfrak{D}_{\psi,a} := \{ h \in \mathfrak{D}_\psi : h(s) = 0 \text{ unless } s \in \mathcal{S}_a \}.$$

The space \mathfrak{D}_ψ can be equipped with a sesquilinear form that is uniquely defined by the relation $(\psi_t, \psi_s) := \psi(s^*t)$. The completion \mathcal{H}_ψ of the resulting pre-Hilbert space is the *reproducing kernel Hilbert space* that corresponds to the kernel

$$K : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}, (t, s) \mapsto \psi(ts^*).$$

In other words, with respect to the inner product (\cdot, \cdot) on \mathcal{H}_ψ , we have

$$h(s) = (h, K_s) \quad \text{for } h \in \mathcal{H}_\psi, s \in \mathcal{S}.$$

There is a natural $*$ -representation $\widetilde{\rho}_\psi : \mathcal{S} \rightarrow \text{End}_{\mathbb{C}}(\mathfrak{D}_\psi)$ by right translation, given by

$$(\widetilde{\rho}_\psi(s)h)(t) := h(ts) \quad \text{for } s, t \in \mathcal{S}, h \in \mathfrak{D}_\psi.$$

If $s \in \mathcal{S}$ satisfies $ss^* = s^*s = 1_{\mathcal{S}}$, then $\widetilde{\rho}_\psi(s) : \mathfrak{D}_\psi \rightarrow \mathfrak{D}_\psi$ is an isometry and extends uniquely to a unitary operator on \mathcal{H}_ψ . Using the latter fact for elements of the form $(g, 1_{\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})})$ where $g \in G_0$, one obtains a unitary representation of G_0 on \mathcal{H}_ψ (in fact the vectors $K_s, s \in \mathcal{S}$, have smooth G_0 -orbits). Setting

$$\mathcal{B} := \mathfrak{D}_\psi, \mathcal{H} := \mathcal{H}_\psi, \rho^{\mathcal{B}}(x) := \widetilde{\rho}_\psi(1_{G_0}, x) \text{ for } x \in \mathfrak{g}, \text{ and } \pi(g) := \widetilde{\rho}_\psi(g, 0),$$

we obtain a pre-representation of (G_0, \mathfrak{g}) . Theorem 4.3 implies that this pre-representation corresponds to a unique unitary representation of (G_0, \mathfrak{g}) .

DEFINITION 5.3. By a *cyclic vector* in a unitary representation $(\pi, \rho^\pi, \mathcal{H})$ we mean a vector $v \in \mathcal{H}$ such that $\widetilde{\rho}^\pi(\mathcal{S})v$ is dense in \mathcal{H} .

The above construction results in Theorem 5.4 below.

THEOREM 5.4. *Let (G_0, \mathfrak{g}) be a Γ -Harish-Chandra pair such that*

$$\mathfrak{g}_a = \sum_{b,c \in \Gamma^{-1}, bc=a} [\mathfrak{g}_b, \mathfrak{g}_c]. \quad (8)$$

Also, let $f \in C^\infty(\mathcal{G})$ be positive definite.

- (i) *There exists a unitary representation $(\pi, \rho^\pi, \mathcal{H})$ of (G_0, \mathfrak{g}) with a cyclic vector $v_0 \in \mathcal{H}_0$ such that $\check{f} = \varphi_{v_0, v_0}$.*
- (ii) *Let $(\sigma, \rho^\sigma, \mathcal{H})$ be another unitary representation of (G_0, \mathfrak{g}) with a cyclic vector $w_0 \in \mathcal{H}_0$ such that $\check{f} = \varphi_{w_0, w_0}$. Then $(\pi, \rho^\pi, \mathcal{H})$ and $(\sigma, \rho^\sigma, \mathcal{H})$ are unitarily equivalent via an intertwining operator that maps v_0 to w_0 .*

Proof. The proof is an extension of the argument of [15, Thm 6.7.5]. ■

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