

Schur Q -functions and the Capelli eigenvalue problem for the Lie superalgebra $\mathfrak{q}(n)$

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ABSTRACT. Let $\mathfrak{l} := \mathfrak{q}(n) \times \mathfrak{q}(n)$, where $\mathfrak{q}(n)$ denotes the queer Lie superalgebra. The associative superalgebra V of type $Q(n)$ has a left and right action of $\mathfrak{q}(n)$, and hence is equipped with a canonical \mathfrak{l} -module structure. We consider a distinguished basis $\{D_\lambda\}$ of the algebra of \mathfrak{l} -invariant super-polynomial differential operators on V , which is indexed by strict partitions of length at most n . We show that the spectrum of the operator D_λ , when it acts on the algebra $\mathcal{P}(V)$ of super-polynomials on V , is given by the factorial Schur Q -functions of Okounkov and Ivanov. As an application, we show that the radial projections of the spherical super-polynomials (corresponding to the diagonal symmetric pair $(\mathfrak{l}, \mathfrak{m})$, where $\mathfrak{m} := \mathfrak{q}(n)$) of irreducible \mathfrak{l} -submodules of $\mathcal{P}(V)$ are the classical Schur Q -functions. As a further application, we compute the Harish-Chandra images of the Nazarov basis $\{C_\lambda\}$ of the centre of $\mathfrak{U}(\mathfrak{q}(n))$.

1. Introduction

Let G/K be a Hermitian symmetric space of tube type. The *Shilov boundary* of G/K is of the form $G/P = K/M$, where $P = LN$ is the Siegel parabolic subgroup and $M = L \cap K$ is a symmetric subgroup of both K and L . Let \mathfrak{l} , \mathfrak{m} , and \mathfrak{n} be the complexified Lie algebras of L , M , and N , respectively. We set $V := \mathfrak{n}$ and regard V as an L -module. In this setting, V has the structure of a simple Jordan algebra.

The polynomial algebra $\mathcal{P}(V)$ decomposes as the multiplicity-free direct sum of simple L -modules V_λ , indexed naturally by partitions λ . In this situation one has canonical invariant ‘‘Capelli’’ differential operators of the form $\varphi^k \partial(\varphi)^k$, where φ is the Jordan norm polynomial. The spectrum of these operators was computed by Kostant and Sahi [11, 12], and a close connection with reducibility and composition factors of degenerate principal series was established by Sahi [17, 18, 20].

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Sahi showed [19] that the decomposition of $\mathcal{P}(V)$ in fact yields a distinguished basis $\{D_\lambda\}$, called the *Capelli basis*, of the subalgebra of L -invariant elements of the algebra $\mathcal{PD}(V)$ of differential operators on V with polynomial coefficients. Moreover, there is a polynomial c_λ , uniquely characterized by its degree, symmetry, and vanishing properties, such that D_λ acts on each simple summand V_μ by the scalar $c_\lambda(\mu)$. The problem of characterizing the spectrum of the operators D_λ is referred to as the *Capelli eigenvalue problem*.

In fact, Sahi [19] introduced a universal multi-parameter family of inhomogeneous polynomials that serve as a common generalization of the spectral polynomials c_λ across all Hermitian symmetric spaces of rank n . Later, Knop and Sahi [10] studied a one-parameter subfamily of these polynomials, which already contains all the spectral polynomials. They showed that these polynomials are eigenfunctions of a class of difference operators extending the Debiard–Sekiguchi differential operators. It follows that the top degree terms of the Knop–Sahi polynomials are Jack polynomials, which for special choices of the parameter become spherical functions.

These polynomials were later studied from a different point of view by Okounkov and Olshanski, who referred to them as shifted Jack polynomials.

Subsequently, supersymmetric analogs of the Knop–Sahi shifted Jack polynomials were constructed by Sergeev and Veselov in [28]. More recently, two of us (Sahi and Salmasian [22]) have extended this circle of ideas to the setting of the triples $(\mathfrak{l}, \mathfrak{m}, V)$ of the form

$$(1.1) \quad \begin{aligned} &(\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n), \mathfrak{gl}(m|n), \text{Mat}_{m|n}(\mathbb{C})), \\ &(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n), S^2(\mathbb{C}^{m|2n})). \end{aligned}$$

In each of these situations one has, once again, a canonical Capelli basis of differential operators, and [22] establishes a precise connection to the abstract Capelli problem of Howe and Umeda [6]. It is further shown in Ref. [22] that the spectrum of the Capelli basis is given by specialisations of super analogues of Knop–Sahi polynomials, defined earlier by Sergeev and Veselov [28]. In the case of the triple $(\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n), \mathfrak{gl}(m|n), \text{Mat}_{m|n}(\mathbb{C}))$, these results follow from earlier work of Molev [15], however the case $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n), S^2(\mathbb{C}^{m|2n}))$ is harder and requires new ideas.

The Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ are examples of basic classical Lie superalgebras. Such an algebra admits an even non-degenerate invariant bilinear form and an even Cartan subalgebra, and many results for ordinary Lie algebras extend to this setting, see for instance Ref. [2], where spherical representations for the corresponding symmetric pairs are studied. In this paper, we show that the ideas of Ref. [22] can actually be extended to *non-basic* Lie superalgebras. More precisely, we consider the case of the *queer Lie superalgebra* $\mathfrak{q}(n)$, usually defined as the subalgebra of $\mathfrak{gl}(n|n)$ of matrices commuting with an odd involution [9]. For the present purposes, it is convenient to work with a slightly different realization of $\mathfrak{q}(n)$, which we describe below.

Let \mathcal{E} be the \mathbb{C} -algebra generated by an odd element ε , with $\varepsilon^2 = 1$; thus as a superspace, $\mathcal{E} \cong \mathbb{C}^{1|1} \cong \mathbb{C} \oplus \mathbb{C}\varepsilon$. Let \mathcal{A} be the associative superalgebra of $n \times n$ matrices with entries in \mathcal{E} . Then \mathcal{A} is the associative superalgebra of type $Q(n)$, and $\mathfrak{q}(n)$ is isomorphic to \mathcal{A} regarded as a Lie superalgebra via the graded commutator

$$[x, y] := xy - (-1)^{|x||y|}yx.$$

In fact \mathscr{A} is also a Jordan superalgebra via the graded anticommutator, and an \mathscr{A} -bimodule via left and right multiplication. This bimodule structure induces a $\mathfrak{q}(n) \times \mathfrak{q}(n)$ -module structure on $V := \mathscr{A}$.

In this paper, we consider the Capelli eigenvalue problem for the “diagonal” triple

$$(1.2) \quad (\mathfrak{l}, \mathfrak{m}, V) := (\mathfrak{q}(n) \times \mathfrak{q}(n), \mathfrak{q}(n), \mathscr{A}).$$

We establish a close connection with the Schur Q -functions Q_λ and their inhomogeneous analogues, the *factorial Schur Q -functions* Q_λ^* , which were originally defined by Okounkov and studied by Ivanov [7]. Our main results are as follows. From Ref. [4], it is known that the space $\mathscr{P}(V)$ of super-polynomials on V decomposes as a multiplicity-free direct sum of certain \mathfrak{l} -modules V_λ , which are parametrised by strict partitions λ of length at most n . It follows that $\mathscr{P}(V^*)$ decomposes as a direct sum of the contragredient \mathfrak{l} -modules V_λ^* . In Section 4.3, we describe a certain even linear slice \mathfrak{t}^* to the M -orbits on V^* . If p is an \mathfrak{m} -invariant super-polynomial on V^* , then it is uniquely determined by its restriction to \mathfrak{t}^* . This restriction is an ordinary polynomial, and we call it the *\mathfrak{m} -radial part* of p .

THEOREM 1.1. *For every λ , the \mathfrak{l} -module V_λ^* contains an \mathfrak{m} -spherical super-polynomial p_λ^* , which is unique up to a scalar multiple. Moreover, up to a scalar, the \mathfrak{m} -radial part of p_λ^* is the Schur Q -function Q_λ .*

This is proved in Theorem 4.5 below. Now consider the algebra $\mathscr{PD}(V)$ of polynomial coefficient differential operators on V . It has an \mathfrak{l} -module decomposition

$$\mathscr{PD}(V) \cong \bigoplus_{\lambda, \mu} V_\mu \otimes V_\lambda^* \cong \bigoplus_{\lambda, \mu} \underline{\mathrm{Hom}}_{\mathfrak{l}}(V_\lambda, V_\mu),$$

and we write D_λ for the differential operator corresponding to the identity map $\mathrm{id}_{V_\lambda} \in \underline{\mathrm{Hom}}_{\mathfrak{l}}(V_\lambda, V_\mu)$. The D_λ are the Capelli operators, and they form a basis for the \mathfrak{l} -invariant differential operators acting on $\mathscr{P}(V)$. The operator D_λ acts on each irreducible component V_μ of $\mathscr{P}(V)$ by a scalar eigenvalue $c_\lambda(\mu)$.

THEOREM 1.2. *The eigenvalues of the Capelli operator D_λ are given by the factorial Schur Q -function Q_λ^* . More precisely, for all λ, μ , we have*

$$c_\lambda(\mu) = \frac{Q_\lambda^*(\mu)}{Q_\lambda^*(\lambda)}.$$

In fact, we prove Theorem 1.2 *first* (see Theorem 3.8 below) and then use it to prove Theorem 1.1.

Compared to the cases considered in Equation (1.1), the situation in Equation (1.2) is more complicated. First, since the Cartan subalgebra of $\mathfrak{q}(n)$ is not purely even, the highest weight space of an irreducible finite dimensional $\mathfrak{q}(n)$ -module is not necessarily one-dimensional. Second, unlike the basic classical cases, the tensor product of two irreducible $\mathfrak{q}(n)$ -modules is not necessarily an irreducible $\mathfrak{q}(n) \times \mathfrak{q}(n)$ -module, and sometimes decomposes as a direct sum of two modules which are isomorphic up to parity change. Third, the \mathfrak{m} -spherical vectors in $\mathscr{P}(V)$ are purely odd, whereas the \mathfrak{m} -spherical vectors in $\mathscr{P}(V^*)$ are purely even. These issues add to the difficulties that arise in the proofs in the case of the symmetric pair in Equation (1.2).

In [27, Theorem 3], Sergeev introduced the $\mathfrak{q}(n)$ -analogue of the Harish-Chandra isomorphism

$$(1.3) \quad \eta : \mathcal{Z}(\mathfrak{q}(n)) \longrightarrow \mathcal{P}(\mathfrak{h}_{\bar{0}}),$$

where $\mathfrak{h}_{\bar{0}}$ denotes the even part of the Cartan subalgebra of $\mathfrak{q}(n)$. Theorem 1.2 can be reformulated in terms of the map η , as follows. The image of η can be naturally identified with the space of n -variable Q -symmetric polynomials (see Section 3.2). We denote the actions of the first and second factors of $\mathfrak{l} = \mathfrak{q}(n) \times \mathfrak{q}(n)$ on $\mathcal{P}(V)$ by L and R , respectively. Since the typical “ Q -shift” for the Sergeev–Harish-Chandra isomorphism is equal to zero, we obtain the following reformulation of Theorem 1.2.

THEOREM 1.3. *For every Capelli operator D_λ , there exists a unique central element $z_\lambda \in \mathcal{Z}(\mathfrak{q}(n))$ such that $L(z_\lambda) = D_\lambda$. Furthermore,*

$$\eta(z_\lambda)(\mu) = \frac{Q_\lambda^*(\mu)}{Q_\lambda^*(\lambda)}.$$

The setting of the present paper was also considered by Nazarov, who constructed [16, Eq. (4.7)] a family of invariant differential operators $\{I_\lambda\}$ using characters of the Sergeev algebra [27]. Nazarov also defined [16, Eq. (4.6)] certain explicit “Capelli” elements $\{C_\lambda\}$ in $\mathcal{Z}(\mathfrak{q}(n))$, and proved [16, Cor. 4.6] that $I_\lambda = \gamma(C_\lambda)$, where γ is the left action of $\mathfrak{q}(n)$ on V .

Although our operators $\{D_\lambda\}$ and central elements $\{z_\lambda\}$ are *different* from the $\{I_\lambda\}$ and $\{C_\lambda\}$ defined by Nazarov, one can make an *a posteriori* connection using our Proposition 3.6 below. This allows us to compute the Harish-Chandra image of Nazarov’s central elements $\{C_\lambda\}$. The following result follows immediately from Theorem 4.9.

THEOREM 1.4. *The Harish-Chandra image of the operator C_λ is given by*

$$\eta(C_\lambda)(\mu) = k_\lambda Q_\lambda^*(-\mu),$$

where

$$k_\lambda := (-1)^{|\lambda|} \lambda_1! \cdots \lambda_{\ell(\lambda)}! \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}.$$

We would like to mention that the polynomials Q_λ occur in a further different scenario related to the Lie superalgebra $\mathfrak{q}(n)$. In [26, Theorem 1.7], Sergeev showed that the radial parts of the bispherical matrix coefficients on $\mathfrak{q}(n) \times \mathfrak{q}(n)$ with respect to the diagonal and twisted-diagonal embeddings of $\mathfrak{q}(n)$ in $\mathfrak{q}(n) \times \mathfrak{q}(n)$, are Schur Q -polynomials. It will be interesting to explore possible connections between our work and Sergeev’s result.

We remark that it is possible to extend the results of the present paper and of Ref. [22] to the common setting of multiplicity-free actions on Jordan superalgebras. This will be established in a forthcoming paper [24]. In addition, recently Sahi and Salmasian [23] constructed quadratic analogues of Capelli operators on Grassmannian manifolds by lifting the Capelli basis of [19] via a double fibration. In the near future they plan to consider the analogous problem in the super setting.

We conclude this introduction with a brief synopsis of our paper. In Section 2, we realise the Lie superalgebras relevant to us in terms of supermatrices. In Section 3, we identify the action of \mathfrak{l} on $\mathcal{P}(V)$, construct the Capelli basis, and determine the eigenvalue polynomials (Theorem 3.8). Finally, in Section 4, we study the open orbits in V and V^* , show the existence of \mathfrak{m} -invariant functionals for the simple

summands of $\mathcal{P}(V)$, and prove that the spherical polynomials thus defined are the classical Schur Q -functions (Theorem 4.5).

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2. Lie superalgebras

The triple $(\mathfrak{l}, \mathfrak{m}, V)$ given in Equation (1.2) can be embedded inside the Lie superalgebra $\mathfrak{q}(2n)$, which can be further embedded inside $\mathfrak{gl}(2n|2n)$. This provides a concrete realisation which allows us to express the Lie superalgebras of interest as $4n \times 4n$ matrices. In order to describe it, it will be convenient to consider three commuting involutions of the algebra $\mathfrak{gl}(2n|2n)$. To this end, first we equip the space $\text{Mat}_{4n \times 4n}(\mathbb{C})$ of $4n \times 4n$ complex matrices with a Lie superalgebra structure isomorphic to $\mathfrak{gl}(2n|2n)$. Instead of supermatrices in standard format, we prefer to consider those of the shape

$$(2.1) \quad x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \mathfrak{gl}(n|n).$$

Equipped with the signed matrix commutator, the space of such matrices forms a Lie superalgebra \mathfrak{g} isomorphic to $\mathfrak{gl}(2n|2n)$, the isomorphism being given by conjugation by the $4n \times 4n$ matrix

$$\tilde{I} := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

where $I := I_{n \times n}$ denotes the $n \times n$ identity matrix. Next let σ, φ , and θ be three involutions on \mathfrak{g} , given respectively by conjugation by the matrices

$$\Sigma := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}, \quad \Phi := \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix}, \quad \Theta := \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}.$$

It is straightforward to verify that $\Sigma\Phi = \Phi\Sigma$, $\Phi\Theta = \Theta\Phi$, and $\Sigma\Theta = -\Theta\Sigma$. Hence, the involutions σ, φ , and θ commute with each other.

2.1. The involution σ . The subspace \mathfrak{g}^σ of fixed points of σ equals $\mathfrak{gl}(n|n) \times \mathfrak{gl}(n|n)$ with elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \quad \text{where } A, D \in \mathfrak{gl}(n|n).$$

The subspace $\mathfrak{g}^{-\sigma}$ of fixed points of $-\sigma$ consists of matrices of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \quad \text{where } B, C \in \mathfrak{gl}(n|n).$$

Thus, as a super-vector space, $\mathfrak{g}^{-\sigma} = \mathfrak{g}_+^{-\sigma} \oplus \mathfrak{g}_-^{-\sigma}$ where $\mathfrak{g}_{\pm}^{-\sigma}$ are respectively the spaces of $4n \times 4n$ matrices of the form

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

with $B, C \in \mathfrak{gl}(n|n)$. In fact, $\mathfrak{g}_{\pm}^{-\sigma}$ are the ± 2 -eigenspaces of $\text{ad}(\Sigma)$, where we think of Σ as an element of \mathfrak{g} . Therefore, $\mathfrak{g}_{\pm}^{-\sigma}$ are abelian subalgebras of \mathfrak{g} , and together with \mathfrak{g}^{σ} , they form a \mathbb{Z} -grading of \mathfrak{g} . The action of \mathfrak{g}^{σ} on $\mathfrak{g}_+^{-\sigma}$ is given explicitly by

$$(2.2) \quad [x, v] = \begin{pmatrix} 0 & AB - (-1)^{|B||D|}BD \\ 0 & 0 \end{pmatrix},$$

for all homogeneous

$$x = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g}^{\sigma}, \quad v = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_+^{-\sigma}.$$

In what follows, we set $U := \mathfrak{g}_+^{-\sigma}$ and identify it as a super-vector space with $\mathfrak{gl}(n|n)$, *via* the map

$$(2.3) \quad \mathfrak{gl}(n|n) \longrightarrow U : B \longmapsto \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

2.2. The involution φ . The involution φ is induced by the parity reversing automorphism of $\mathbb{C}^{2n|2n}$, and therefore the subalgebra \mathfrak{g}^{φ} of fixed points of φ is isomorphic to $\mathfrak{q}(2n)$. It consists of all $x \in \mathfrak{g}$ as in Equation (2.1) such that the blocks $A, B, C, D \in \mathfrak{q}(n)$. Furthermore,

$$\mathfrak{l} := \mathfrak{g}^{\varphi, \sigma} \cong \mathfrak{q}(n) \times \mathfrak{q}(n).$$

We define $V := \mathfrak{g}^{\varphi} \cap U$. Then it is clear that

$$[\mathfrak{l}, V] \subseteq \mathfrak{g}^{\varphi} \cap [\mathfrak{g}^{\sigma}, \mathfrak{g}_+^{-\sigma}] \subseteq \mathfrak{g}^{\varphi} \cap \mathfrak{g}_+^{-\sigma} = V.$$

The restriction of the map defined in Equation (2.3) yields an identification of V with a subspace of $\mathfrak{gl}(n|n)$ which carries the structure of $\mathfrak{q}(n)$.

2.3. The involution θ . The algebra $\mathfrak{g}^{\sigma, \theta}$ of fixed points of both σ and θ is isomorphic to $\mathfrak{gl}(n|n)$, and realised by supermatrices

$$(2.4) \quad \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad A \in \mathfrak{gl}(n|n).$$

From Equation (2.2), it follows that the action of $\mathfrak{g}^{\sigma, \theta}$ on U is precisely the adjoint action of $\mathfrak{gl}(n|n)$. Furthermore,

$$\mathfrak{m} := \mathfrak{l}^{\theta} = \mathfrak{g}^{\varphi, \sigma, \theta} \cong \mathfrak{q}(n).$$

It is realised by supermatrices of the form as in Equation (2.4) where in addition $A \in \mathfrak{q}(n)$. Moreover, the action of \mathfrak{m} on V is precisely the adjoint action of $\mathfrak{q}(n)$. For the following lemma, let $e \in V \subseteq U$ be the element corresponding to the matrix

$$\begin{pmatrix} 0 & I_{2n \times 2n} \\ 0 & 0 \end{pmatrix},$$

and set

$$\mathfrak{s} := \mathfrak{l}^{-\theta} = \mathfrak{g}^{\varphi, \sigma, -\theta}.$$

LEMMA 2.1. *Let $\mathfrak{p} : \mathfrak{g}^{\sigma} \longrightarrow U$ be the linear map defined by $x \longmapsto x \cdot e$.*

(i) The map \mathfrak{p} is a surjection onto U with kernel $\mathfrak{g}^{\sigma,\theta}$. Its restriction to $\mathfrak{g}^{\sigma,-\theta}$ is a linear isomorphism.

(ii) The restriction of \mathfrak{p} to \mathfrak{l} is a surjection onto V with kernel \mathfrak{m} . Its restriction to \mathfrak{s} is a linear isomorphism onto V .

(iii) Equipped with the binary operation $(x \cdot e) \circ (y \cdot e) := [x, [y, e]]$ for $x, y \in \mathfrak{g}^{\sigma,-\theta}$ (respectively, $x, y \in \mathfrak{s}$) the super-vector space U (respectively, V) becomes a Jordan superalgebra.

PROOF. Parts (i) and (ii) follow from straightforward calculations. For (iii), it suffices to prove the statement for U . Note that according to Equation (2.2), we have

$$x \cdot e = \begin{pmatrix} 0 & 2A \\ 0 & 0 \end{pmatrix}, \quad \forall x = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \in \mathfrak{g}^{\sigma,-\theta}.$$

Thus, a direct calculation shows that up to normalization, $A \circ B$ coincides for $A, B \in U$ with the super-anticommutator $AB + (-1)^{|A||B|}BA$. \square

3. The eigenvalue polynomials

3.1. The action of \mathfrak{l} on polynomials. Let $\mathcal{P}(V)$ denote the superalgebra of super-polynomials on V . Recall that $\mathcal{P}(V)$ is by definition equal to $S(V^*)$. As \mathfrak{g}^σ acts on V , we obtain an induced locally finite \mathfrak{g}^σ -action on $\mathcal{P}(V^*) = S(V)$. Similar statements apply to $\mathcal{P}(V)$. We shall identify this action in terms of differential operators. To that end, we consider the complex supermanifold $\mathbb{A}(V)$, defined as the locally ringed space with underlying topological space V_0 and sheaf of superfunctions $\mathcal{O}_{\mathbb{A}(V)} := \mathcal{H}_{V_0} \otimes \wedge(V_1^*)$, where \mathcal{H} denotes the sheaf of holomorphic functions. There is a natural inclusion

$$V^* \longrightarrow \Gamma(\mathcal{O}_{\mathbb{A}(V)})$$

(Γ denoting global sections), allowing us to identify linear forms on V with certain superfunctions on $\mathbb{A}(V)$. In particular, $\mathcal{P}(V)$ is a subsuperalgebra of $\Gamma(\mathcal{O}_{\mathbb{A}(V)})$.

Recall that on a supermanifold X , the vector fields on X , defined on an open set $O \subseteq X_0$ of the underlying topological space, are defined to be the superderivations of $\mathcal{O}_X|_O$ [1, Definition 4.1]. For any homogeneous basis (x_a) of V , the dual basis (x^a) is a coordinate system on $\mathbb{A}(V)$, and [1, Proposition 4.5] there are unique (and globally defined) vector fields $\frac{\partial}{\partial x^a}$ on $\mathbb{A}(V)$ of parity $|x^a|$, determined by

$$\frac{\partial}{\partial x^a}(x^b) = \delta_{ab}, \quad \forall a, b.$$

The linear action of \mathfrak{l} on V determines, for $v \in \mathfrak{l}$, vector fields \mathbf{a}_v on $\mathbb{A}(V)$ by

$$(3.1) \quad \mathbf{a}_v(x^b) = -v \cdot x^b, \quad \forall b.$$

The sign stems from the fact that these are the fundamental vector fields for a Lie supergroup action on $\mathbb{A}(V)$, as we shall see later. By construction, for any $v \in \mathfrak{l}$, the action of $-a_v$ on $\mathcal{P}(V)$ coincides with the action of v defined in the first paragraph of this subsection.

We now make this action explicit. Let $(e_k, \varepsilon_k)_{k=1, \dots, n}$ be the standard basis of $\mathbb{C}^{n|n}$. A homogeneous basis of the super-vector space $V \cong \mathfrak{q}(n)$ is determined by

$$u_{k\ell} := \begin{pmatrix} E_{k\ell} & 0 \\ 0 & E_{k\ell} \end{pmatrix}, \quad \xi_{k\ell} := \begin{pmatrix} 0 & E_{k\ell} \\ E_{k\ell} & 0 \end{pmatrix},$$

where $k, \ell = 1, \dots, n$ and $E_{k\ell}$ are the usual elementary matrices. Then

$$\begin{aligned} u_{k\ell}(e_j) &= \delta_{\ell j} e_k, & u_{k\ell}(\varepsilon_j) &= \delta_{\ell j} \varepsilon_k, \\ \xi_{k\ell}(e_j) &= \delta_{\ell j} \varepsilon_k, & \xi_{k\ell}(\varepsilon_j) &= \delta_{\ell j} e_k. \end{aligned}$$

Let $(u^{k\ell}, \xi^{k\ell})$ be the dual basis of $\mathfrak{q}(n)^* = V^*$. This determines vector fields

$$\frac{\partial}{\partial u^{k\ell}}, \quad \frac{\partial}{\partial \xi^{k\ell}}$$

on $\mathbb{A}(V)$, by the recipe given above.

Moreover, let $(a_{k\ell}, \alpha_{k\ell})$ and $(b_{k\ell}, \beta_{k\ell})$, respectively, be the copies of $(u_{k\ell}, \xi_{k\ell})$ in the first and second factor of $\mathfrak{l} = \mathfrak{q}(n) \times \mathfrak{q}(n)$. By Equation (2.2), as a module over the second factor, V is isomorphic to $\mathbb{C}^n \otimes (\mathbb{C}^{n|n})^*$ where $\mathfrak{q}(n)$ acts on $\mathbb{C}^{n|n}$ in the standard way. Hence, the second factor acts on V^* as on $\mathbb{C}^n \otimes \mathbb{C}^{n|n}$:

$$\begin{aligned} b_{k\ell}(u^{pq}) &= \delta_{\ell q} u^{pk}, & b_{k\ell}(\xi^{pq}) &= \delta_{\ell q} \xi^{pk}, \\ \beta_{k\ell}(u^{pq}) &= \delta_{\ell q} \xi^{pk}, & \beta_{k\ell}(\xi^{pq}) &= \delta_{\ell q} u^{pk}. \end{aligned}$$

It follows that

$$(3.2) \quad -\mathbf{a}_{b^{k\ell}} = \sum_{p=1}^n u^{pk} \frac{\partial}{\partial u^{p\ell}} + \xi^{pk} \frac{\partial}{\partial \xi^{p\ell}}, \quad -\mathbf{a}_{\beta^{k\ell}} = \sum_{p=1}^n \xi^{pk} \frac{\partial}{\partial u^{p\ell}} + u^{pk} \frac{\partial}{\partial \xi^{p\ell}}.$$

Similarly, the first factor of \mathfrak{l} acts on V as on $\mathbb{C}^{n|n} \otimes \mathbb{C}^n$, and hence on V^* as on $(\mathbb{C}^{n|n})^* \otimes \mathbb{C}^n$. Reasoning as above, this implies

$$(3.3) \quad -\mathbf{a}_{a^{k\ell}} = \sum_{p=1}^n u^{p\ell} \frac{\partial}{\partial u^{pk}} + \xi^{p\ell} \frac{\partial}{\partial \xi^{pk}}, \quad -\mathbf{a}_{\alpha^{k\ell}} = \sum_{p=1}^n -\xi^{p\ell} \frac{\partial}{\partial u^{pk}} + u^{p\ell} \frac{\partial}{\partial \xi^{pk}}.$$

We will presently decompose the \mathfrak{l} -module $\mathcal{P}(V)$. To that end, we introduce a labelling set. Let Λ be the set of *partitions*, that is, of all finite sequences $\lambda = (\lambda_1, \dots, \lambda_m)$ of non-negative integers λ_j such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. Here, we identify λ with any partition $(\lambda_1, \dots, \lambda_m, 0, \dots, 0)$ obtained from λ by appending a finite number of zeros at its tail. If λ can be written in the form $(\lambda_1, \dots, \lambda_\ell)$ where $\lambda_\ell > 0$, then we say λ has *length* $\ell(\lambda) = \ell$. Let $\Lambda^\ell \subseteq \Lambda$ be the set of partitions of length $\ell(\lambda) \leq \ell$. We also set $|\lambda| := \lambda_1 + \dots + \lambda_\ell$ if $\ell(\lambda) \leq \ell$. A partition λ of length $\ell(\lambda) = \ell$ is called *strict* if $\lambda_1 > \dots > \lambda_\ell > 0$. The set of all strict partitions will be denoted by $\Lambda_{>0}$, and we write $\Lambda_{>0}^\ell$ for the set of strict partitions of length at most ℓ ; that is, $\Lambda_{>0}^\ell := \Lambda^\ell \cap \Lambda_{>0}$.

For every strict partition λ such that $\ell(\lambda) \leq n$, let F_λ be the $\mathfrak{q}(n)$ -highest weight module with highest weight $\lambda_1 \varepsilon_1 + \dots + \lambda_n \varepsilon_n$, where the ε_i are the standard characters of the even part of the Cartan subalgebra of $\mathfrak{q}(n)$. For every strict partition λ , set $\delta(\lambda) := 0$ when $\ell(\lambda)$ is even, and $\delta(\lambda) := 1$ otherwise. We now define an \mathfrak{l} -module V_λ as follows. It is shown in [4, Section 2] that, as an \mathfrak{l} -module, the exterior tensor product $(F_\lambda)^* \boxtimes F_\lambda$ is irreducible when $\delta(\lambda) = 0$, and decomposes into a direct sum of two irreducible isomorphic \mathfrak{l} -modules (via an odd map) if $\delta(\lambda) = 1$. Following the notation of Ref. [4], we set

$$V_\lambda := \frac{1}{2^{\delta(\lambda)}} (F_\lambda)^* \boxtimes F_\lambda,$$

that is, we take V_λ to be the irreducible component of $(F_\lambda)^* \boxtimes F_\lambda$ that appears in the decomposition of the super-polynomial algebra over the natural $(\mathfrak{q}(n), \mathfrak{q}(n))$ -module (see Proposition 3.1). The \mathfrak{l} -module V_λ is always of type M, that is, it is

irreducible as an ungraded representation. It follows that in the \mathbb{Z}_2 -graded sense,

$$(3.4) \quad \underline{\text{Hom}}_{\mathfrak{l}}(V_\lambda, V_\mu) = \delta_{\lambda\mu} \cdot \mathbb{C},$$

where *a priori*, $\underline{\text{Hom}}_{\mathfrak{l}}$ denotes the set of all \mathfrak{l} -equivariant linear maps (of any parity). In particular, all non-zero \mathfrak{l} -equivariant endomorphisms of V_λ are even.

PROPOSITION 3.1. *Under the action of \mathfrak{l} , $\mathcal{P}^k(V)$ decomposes as the multiplicity-free direct sum of simple modules V_λ , where λ ranges over elements of $\Lambda_{>0}^n$ which satisfy $|\lambda| = k$.*

PROOF. Recall that $\mathcal{P}(V) \cong S(V^*)$ as \mathfrak{l} -modules. The proposition follows from the description of the actions of the left and right copies of $\mathfrak{q}(n)$ on V given above, and the results of [4, Section 3]. \square

3.2. Invariant polynomial differential operators. On a complex supermanifold X , the *differential operators* on X defined on an open set $O \subseteq X_0$ of the underlying topological space are generated as a subsuperalgebra of the \mathbb{C} -linear endomorphisms of $\mathcal{O}_X|_O$ by vector fields and functions. This gives a \mathbb{C} -algebra sheaf \mathcal{D}_X that is an \mathcal{O}_X -bimodule, filtered by order.

Here, a differential operator is of *order* $\leq n$ if it can be expressed as a product of some functions and at most n vector fields. A differential operator of order $\leq n$ is uniquely determined by its action on monomials of order $\leq n$ in some given system of coordinate functions. This follows in the usual way from the Hadamard Lemma [13, Lemma 2.1.8] and implies that \mathcal{D}_X is locally free as a left \mathcal{O}_X -supermodule.

For the supermanifold $\mathbb{A}(V)$, we have a \mathbb{C} -superalgebra map

$$\partial : S(V) \longrightarrow \Gamma(\mathcal{D}_{\mathbb{A}(V)}).$$

It is determined by the linear map which sends any homogeneous $v \in V$ to the unique vector field $\partial(v)$ such that

$$(3.5) \quad \partial(v)(\mu) = (-1)^{|\mu||v|} \mu(v)$$

for all homogeneous $\mu \in V^*$. If (x_a) is a homogeneous basis of V , then

$$\partial(x_a) = (-1)^{|x_a|} \frac{\partial}{\partial x^a}.$$

The image of ∂ , denoted by $D(V)$, is the superalgebra of *constant-coefficient* differential operators on V . The map

$$(3.6) \quad \mathcal{P}(V) \otimes D(V) \longrightarrow \Gamma(\mathcal{D}_{\mathbb{A}(V)}) : p \otimes D \longmapsto pD$$

is an isomorphism onto a $\mathcal{P}(V)$ -submodule of $\Gamma(\mathcal{D}_{\mathbb{A}(V)})$ denoted by $\mathcal{P}\mathcal{D}(V)$. Indeed, $\mathcal{P}\mathcal{D}(V)$ is a subsuperalgebra, the algebra of *polynomial differential operators*. As \mathfrak{l} acts by linear vector fields, we have the bracket relation

$$[-\mathbf{a}_x, p] = -\mathbf{a}_x(p) = x \cdot p, \quad \forall x \in \mathfrak{l}, p \in \mathcal{P}(V).$$

The bilinear form

$$(3.7) \quad \mathcal{P}^k(V) \otimes D^k(V) \longrightarrow \mathbb{C} : p \otimes D \longmapsto Dp$$

is a non-degenerate pairing which is \mathfrak{l} -equivariant, and therefore results in a canonical \mathfrak{l} -module isomorphism $D^k(V) \cong \mathcal{P}^k(V)^*$. Hence the following corollary to Proposition 3.1 holds.

COROLLARY 3.2. *The space $\mathcal{P}\mathcal{D}(V)^\mathfrak{l}$ of \mathfrak{l} -invariant polynomial differential operators decomposes as follows:*

$$\mathcal{P}\mathcal{D}(V)^\mathfrak{l} = \bigoplus_{\lambda \in \Lambda_{>0}^n} (V_\lambda \otimes (V_\lambda)^*)^\mathfrak{l}.$$

There is a natural \mathfrak{l} -equivariant isomorphism $V_\lambda \otimes (V_\lambda)^* \cong \underline{\text{End}}_{\mathbb{C}}(V_\lambda)$, so the identity element id_{V_λ} of V_λ determines an even \mathfrak{l} -invariant polynomial differential operator

$$D_\lambda \in \mathcal{P}\mathcal{D}(V)^\mathfrak{l}.$$

That is, if (p_j) is a homogeneous basis of $V_\lambda \subseteq \mathcal{P}(V)$ and (D_j) is its dual basis for $(V_\lambda)^* \subseteq D(V)$, then

$$(3.8) \quad D_\lambda := \sum_j p_j D_j.$$

Schur's Lemma implies that for every $\mu \in \Lambda_{>0}^n$, there is a complex scalar $c_\lambda(\mu) \in \mathbb{C}$ such that D_λ acts by the scalar $c_\lambda(\mu)$ on $V_\lambda \subseteq \mathcal{P}(V)$.

COROLLARY 3.3. *The operators D_λ , where λ ranges over $\Lambda_{>0}^n$, form a basis of the space $\mathcal{P}\mathcal{D}(V)^\mathfrak{l}$ of \mathfrak{l} -invariant differential operators. Moreover, $c_\lambda(\lambda) = 1$, while $c_\lambda(\mu) = 0$ whenever $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.*

PROOF. The first statement follows from Corollary 3.2 and Equation (3.4). Since the order of the operator D_λ is not $\leq |\lambda| - 1$, it vanishes on $\mathcal{P}^k(V)$ for $k < |\lambda|$. Next assume $k = |\lambda| = |\mu|$. If $\lambda \neq \mu$ and D_λ does not vanish on V_μ , then the restriction of the bilinear form (3.7) to $V_\mu \times (V_\lambda)^*$ will be a nonzero \mathfrak{l} -equivariant form, hence $V_\lambda \cong V_\mu$, which is a contradiction.

It remains to compute the action of D_λ on V_λ . Let (p_j) and (D_j) denote the dual bases of V_λ and $(V_\lambda)^*$. Then $D_\lambda p_i = \sum_j p_j D_j p_i = \sum_j p_j \delta_{ij} = p_i$ for every i . \square

To determine $c_\lambda(\mu)$ (which will be done in the next subsection), we first need to see that it extends to a Q -symmetric polynomial. To that end, let Γ_n denote the ring of Q -symmetric polynomials, that is, n -variable symmetric polynomials $p(x_1, \dots, x_n)$ such that $p(t, -t, x_3, \dots, x_n)$ does not depend on t . (For $n = 1$, the latter condition is vacuous.)

PROPOSITION 3.4. *For all $\lambda \in \Lambda_{>0}^n$, there exists a polynomial $q_\lambda^* \in \Gamma_n$ of degree at most $|\lambda|$ such that $c_\lambda(\mu) = q_\lambda^*(\mu)$ for all $\mu \in \Lambda_{>0}^n$.*

PROOF. Recall that L and R , respectively, denote the actions of the first and second factor of $\mathfrak{l} = \mathfrak{q}(n) \times \mathfrak{q}(n)$ on $\mathcal{P}(V)$. As $\mathcal{P}(V)$ is the multiplicity-free direct sum of simple modules of the universal enveloping algebra $\mathfrak{U}(\mathfrak{l}) = \mathfrak{U}(\mathfrak{q}(n)) \otimes \mathfrak{U}(\mathfrak{q}(n))$, it follows that $L(\mathfrak{U}(\mathfrak{q}(n)))$ and $R(\mathfrak{U}(\mathfrak{q}(n)))$ are mutual commutants in $\mathcal{P}\mathcal{D}(V)$ (this double commutant property is also mentioned in [4]). In particular, we have

$$\mathcal{P}\mathcal{D}(V)^\mathfrak{l} \subseteq R(\mathfrak{U}(\mathfrak{q}(n))).$$

As R is faithful, it follows that in fact

$$\mathcal{P}\mathcal{D}(V)^\mathfrak{l} = R(\mathcal{Z}(\mathfrak{q}(n))),$$

where \mathcal{Z} denotes the centre of \mathfrak{U} . The latter statement also follows from the explicit construction of the Capelli operators in the work of Nazarov [16].

Furthermore, the simple module V_λ occurring in the decomposition of $\mathcal{P}(V)$ is contained in the external tensor product $(F_\lambda)^* \boxtimes F_\lambda$. By Sergeev's Harish-Chandra

isomorphism for $\mathfrak{q}(n)$ [27, Theorem 3] (see also [5, Theorem 2.46]), for any $u \in \mathcal{Z}(\mathfrak{q}(n))$, there exists a Q -symmetric polynomial $q_u^* \in \Gamma_n$ such that for all $\lambda \in \Lambda_{>0}^n$, u acts on F_λ by $q_u^*(\lambda)$.

Fix $\lambda \in \Lambda_{>0}^n$. Then the order of D_λ is $\leq |\lambda|$. As L is faithful, there is a unique $z_\lambda \in \mathcal{Z}(\mathfrak{q}(n))$ such that $L(z_\lambda) = D_\lambda$. Since $\mathfrak{q}(n) \subseteq \mathfrak{l}$ acts by linear vector fields, z_λ lies in the $|\lambda|$ -th part of the standard increasing filtration of $\mathfrak{L}(\mathfrak{q}(n))$. Then $q_\lambda^* := q_{z_\lambda}^*$ has degree at most $|\lambda|$, see [5, 27]. The assertion follows. \square

DEFINITION 3.5. We call q_λ^* the *eigenvalue polynomial* of D_λ for $\lambda \in \Lambda_{>0}^n$.

3.3. Schur Q -functions. Our next goal is to identify the eigenvalue polynomials $q_\lambda^* \in \Gamma_n$. We first recall the definitions of certain elements Q_λ of Γ_n , called the Q -functions of Schur, and their shifted analogues, the *factorial Q -functions* Q_λ^* originally defined by Okounkov, see Ref. [7].

Given a sequence $(a_n)_{n \geq 1}$ of complex numbers, we define for any non-negative integer k the k th *generalized power* of x by

$$(x | a)^k := \prod_{i=1}^k (x - a_i),$$

where we set $(x|a)^0 = 1$. For every $\lambda \in \Lambda_{>0}^n$, we set

$$F_\lambda(x_1, \dots, x_n | a) := \prod_{i=1}^{\ell(\lambda)} (x | a)^{\lambda_i} \prod_{\substack{i \leq \ell(\lambda) \\ i < j \leq n}} \frac{x_i + x_j}{x_i - x_j}.$$

We now define

$$Q_\lambda(x_1, \dots, x_n | a) := \frac{2^{\ell(\lambda)}}{(n - \ell(\lambda))!} \sum_{\sigma \in S_n} F_\lambda(x_{\sigma(1)}, \dots, x_{\sigma(n)} | a).$$

We remark that $Q_\lambda(x_1, \dots, x_n | a)$ can be expressed as a ratio of an antisymmetric polynomial by the Vandermonde polynomial, and therefore it is also a polynomial. Two special cases of interest are

$$Q_\lambda(x_1, \dots, x_n) := Q_\lambda(x_1, \dots, x_n | \underline{0}),$$

where $\underline{0} := (0, 0, 0, \dots)$, and

$$Q_\lambda^*(x_1, \dots, x_n) := Q_\lambda(x_1, \dots, x_n | \delta),$$

where $\delta := (0, 1, 2, 3, \dots)$, called, respectively, the *Schur Q -function*, and the *factorial Schur Q -function*.

PROPOSITION 3.6. Let $\lambda \in \Lambda_{>0}^n$.

(i) We have $Q_\lambda, Q_\lambda^* \in \Gamma_n$. Furthermore, Q_λ is homogeneous of degree $|\lambda|$ and

$$\deg(Q_\lambda^* - Q_\lambda) < |\lambda|.$$

(ii) Both $(Q_\mu)_{\mu \in \Lambda_{>0}^n}$ and $(Q_\mu^*)_{\mu \in \Lambda_{>0}^n}$ are bases for the vector space Γ_n .

(iii) We have $Q_\lambda^*(\mu) = 0$ for every $\mu \in \Lambda_{>0}^n$ such that $|\mu| \leq |\lambda|$, $\lambda \neq \mu$; moreover,

$$Q_\lambda^*(\lambda) = Q_\lambda(\lambda) = H(\lambda) := \lambda! \prod_{i < j} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j},$$

where $\lambda! := \lambda_1! \cdots \lambda_n!$, and it is understood that the product extends only up to $\ell(\lambda)$.

(iv) *The unique element of Γ_n of degree at most $|\lambda|$ which satisfies Equation (iii) is precisely Q_λ^* .*

PROOF. Parts (i)–(iii) follow from [7, § 1] (see also [29, § 6]). Thus, we only sketch the argument for item (iv). Set $k := |\lambda|$ and let $\Gamma_n(k) \subseteq \Gamma_n$ consist of the polynomials of degree at most k . From item (ii), it follows that $\dim(\Gamma_n(k)) = \#\Lambda_{>0}^n(k)$, where

$$\Lambda_{>0}^n(k) := \{\lambda \in \Lambda_{>0}^n \mid |\lambda| \leq k\}.$$

For every $\mu \in \Lambda_{>0}^n(k)$, we consider the linear functional $\varepsilon_\mu \in \Gamma_n^*$ defined by

$$\varepsilon_\mu(f) := f(\mu), \quad \forall f \in \Gamma_n.$$

Let the order \subseteq on Λ be defined by

$$\mu \subseteq \nu \quad \text{if and only if} \quad \forall j : \mu_j \leq \nu_j.$$

We choose a total order \preceq on $\Lambda_{>0}^n(k)$ such that $\mu \prec \nu$ if either $|\mu| < |\nu|$ or $|\mu| = |\nu|$ and $\nu \not\subseteq \mu$. Then by [7, Proposition 1.16], in terms of \preceq , the $k \times k$ matrix

$$(\varepsilon_\mu(Q_\nu^*))_{\mu, \nu \in \Lambda_{>0}^n(k)}$$

is triangular with no zeros on the diagonal, and therefore, invertible. It follows that the linear system

$$\varepsilon_\mu(f) = \begin{cases} H(\lambda) & \text{if } \mu = \lambda, \\ 0 & \text{if } \mu \neq \lambda. \end{cases}$$

has a unique solution in $\Gamma_n(k)$. In view of item (iii), this solution is Q_λ^* . \square

REMARK 3.7. The polynomials Q_λ also appear in Ref. [14]. They were initially introduced by Schur in connection with the projective representations of the symmetric group. According to [7, Proposition 1.16], the vanishing property of Proposition 3.6 (iii) holds in a stronger form: $Q_\lambda^*(\mu) = 0$ whenever $\lambda \not\subseteq \mu$.

THEOREM 3.8. *Let q_λ^* for $\lambda \in \Lambda_{>0}^n$ be the eigenvalue polynomial of Proposition 3.4. Then*

$$q_\lambda^* = \frac{1}{\lambda!} \prod_{i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} Q_\lambda^*.$$

PROOF. By Proposition 3.4, $q_\lambda^* \in \Gamma_n$ satisfies the same degree, symmetry, and vanishing properties as Q_λ^* , see Proposition 3.6 (iii). The statement now follows from Proposition 3.6 (iv). \square

REMARK 3.9. Nazarov [16, Proposition 4.8] constructs certain Capelli elements $C_\lambda \in \mathcal{Z}(\mathfrak{q}(n))$, and proves that the top degree part of their Harish-Chandra image is the classical Schur Q -function. Even though he refers to Ivanov's work [7] for the eigenvalue polynomials (see [16, p. 871]), his results do not include Theorem 3.8 explicitly.

4. The spherical polynomials

In this final section, we define \mathfrak{m} -spherical polynomials associated with the representations V_λ . We will show that up to a scalar multiple, these spherical polynomials are indeed the Schur Q -functions. As the Schur Q -function Q_λ is the top-degree homogeneous part of Q_λ^* , our strategy is to prove that the spherical polynomials are equal to the top-degree homogeneous parts of the eigenvalue polynomials q_λ^* .

4.1. Open orbits. We begin by globalising the action of \mathfrak{l} on V . Let G be the complex Lie supergroup corresponding *via* [3, Theorem 7.4.5] to the supergroup pair (\mathfrak{g}, G_0) , where G_0 consists of the matrices of the form

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}, \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{C}),$$

and acts on \mathfrak{g} by conjugation. Thus, $G_0 \cong \mathrm{GL}(2n, \mathbb{C}) \times \mathrm{GL}(2n, \mathbb{C})$. The automorphisms φ , σ , and θ of \mathfrak{g} integrate to G , as their restrictions to \mathfrak{g}_0 integrate to G_0 and

$$\mathrm{Ad}(\phi(g))(\phi(x)) = \phi(\mathrm{Ad}(g)x), \quad \forall x \in \mathfrak{g}, g \in G_0,$$

for every choice of $\phi \in \{\varphi, \sigma, \theta\}$ [3, Theorem 7.4.5]. In particular, there are fixed subsupergroups G^σ , $L := G^{\sigma, \varphi}$, and $M := G^{\sigma, \varphi, \theta}$ of G . One sees directly that

$$\begin{aligned} G_0^\sigma &= \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}), \\ L_0 &= \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}), \quad M_0 = \mathrm{GL}(n, \mathbb{C}) \end{aligned}$$

are connected.

Write the elements of G_0^σ in the form

$$(A, D), \quad A = \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}, \quad a, a', d, d' \in \mathrm{GL}(n, \mathbb{C}).$$

Then the action

$$G_0^\sigma \times U \longrightarrow U : (A, D, x) \longmapsto Ax D^{-1}$$

integrates the action of \mathfrak{g}_0^σ on $U = \mathfrak{gl}(n|n)$ in such a way that

$$g \cdot (x \cdot v) = \mathrm{Ad}(g)(x) \cdot (g \cdot v), \quad \forall g \in G_0^\sigma, x \in \mathfrak{g}^\sigma, u \in U,$$

see Equation (2.2). This gives rise to a Lie supergroup action

$$G^\sigma \times \mathbb{A}(U) \longrightarrow \mathbb{A}(U),$$

where $\mathbb{A}(U)$ is the complex supermanifold corresponding to U , as in Section 3.1. Passing to φ -fixed points, this implies that there is an action

$$\mathbf{a} : L \times \mathbb{A}(V) \longrightarrow \mathbb{A}(V)$$

such that the fundamental vector fields

$$\mathbf{a}_x := (x \otimes 1) \circ \mathbf{a}^\sharp, \quad x \in \mathfrak{l},$$

are those determined by Equation (3.1), see [3, Proposition 8.3.3]. Here, \mathbf{a}^\sharp is the morphism of sheaves of superalgebras that corresponds to \mathbf{a} . We equally obtain an L -action on $\mathbb{A}(V^*)$ integrating the given \mathfrak{l} -action.

Recall from Section 3.1 that $\mathbb{A}(V) = (V_{\bar{0}}, \mathcal{O}_{\mathbb{A}(V)})$ as a locally ringed space.

PROPOSITION 4.1. *The L -orbit $L \cdot e$ through the point $e \in V_{\bar{0}}$ is the homogeneous supermanifold L/M . The canonical morphism $j_V : L \cdot e \longrightarrow \mathbb{A}(V)$ is an L -equivariant open embedding.*

PROOF. As M is connected, and by Lemma 2.1, the isotropy of L at e is M , see [3, Proposition 8.4.7]. Thus, $L \cdot e = L/M$ by definition, and the orbit morphism $\mathbf{a}_e : L \longrightarrow \mathbb{A}(V)$ factors into $\pi : L \longrightarrow L/M$ and $j_V : L \cdot e = L/M \longrightarrow \mathbb{A}(V)$ where

π is a surjective submersion and j_V is an injective immersion, both L -equivariant, by [1, Theorem 4.24]. By Lemma 2.1, the kernel of

$$T_1(\mathbf{a}_e) : T_1(L) = \mathfrak{l} \longrightarrow T_e(\mathbb{A}(V)) = V$$

is \mathfrak{m} and its restriction to \mathfrak{s} an isomorphism $\mathfrak{s} \longrightarrow V$. Thus, the inverse function theorem [13, Theorem 2.3.1] applies, and j_V is an open embedding. \square

REMARK 4.2. It is not hard to identify the open orbit from Proposition 4.1. Indeed, identifying V with $\mathfrak{q}(n)$ via the isomorphism from Equation (2.3), $L \cdot e$ equals $Q(n, \mathbb{C}) \subseteq \mathbb{A}(\mathfrak{q}(n))$, defined to be the open subspace corresponding to the open subset $GL(n, \mathbb{C}) \subseteq \mathfrak{gl}(n) = \mathfrak{q}(n)_{\bar{0}}$. Observe that $Q(n, \mathbb{C})$ has the structure of a complex Lie supergroup with Lie superalgebra $\mathfrak{q}(n)$.

The orbit morphism $L \longrightarrow L \cdot e$ is given by

$$(A, D) \longmapsto AD^{-1}$$

on T -valued points, where T is any complex supermanifold.

4.2. Spherical vectors and polynomials. Let $\lambda \in \Lambda_{>0}^n$. Define $p_\lambda^* \in (V_\lambda)^*$ by

$$(4.1) \quad \langle p_\lambda^*, \ell \rangle := (-1)^k k! j_V^\#(\ell)(e), \quad \forall \ell \in V_\lambda \subseteq \mathcal{P}(V),$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing of $(V_\lambda)^*$ with V_λ , and $j_V^\#$ is pullback along the morphism $j_V : L \cdot e \longrightarrow \mathbb{A}(V)$.

PROPOSITION 4.3. *For every $\lambda \in \Lambda_{>0}^n$, p_λ^* is the up to scalars unique non-zero \mathfrak{m} -invariant vector of $(V_\lambda)^*$.*

PROOF. Uniqueness of the \mathfrak{m} -invariant in V_λ follows from the fact that by the hom-tensor adjunction formula

$$((V_\lambda)^*)^{\mathfrak{m}} = \underline{\text{Hom}}_{\mathfrak{m}}(V_\lambda, \mathbb{C}) \cong \frac{1}{2^{\delta(\lambda)}} \underline{\text{Hom}}_{\mathfrak{m}}(F_\lambda, F_\lambda),$$

combined with the fact that $\dim \underline{\text{End}}_{\mathfrak{m}}(F_\lambda) = 2^{\delta(\lambda)}$. (Recall that F_λ is of type Q if and only if $\delta(\lambda) = 1$.)

Next we prove that p_λ^* is the desired \mathfrak{m} -invariant. Since j_V is L -equivariant and M fixes e , it follows that p_λ^* is \mathfrak{m} -invariant. We need to see that $p_\lambda^* \neq 0$. Above, we have noted the commutative diagram

$$\begin{array}{ccc} L & & \\ \pi \downarrow & \searrow \mathbf{a}_e & \\ L \cdot e & \xrightarrow{j_V} & \mathbb{A}(V) \end{array}$$

where j_V is an open embedding, π is a surjective submersion, and

$$\mathbf{a}_e = \mathbf{a} \circ (\text{id} \times j_e),$$

where j_e is the embedding $* \longrightarrow \mathbb{A}(V)$ of the singleton space defined by e . The action of L is linear, i.e.

$$\mathbf{a}^\#(V^*) \subseteq \Gamma(\mathcal{O}_L) \otimes V^*,$$

and hence, because $\mathbf{a}^\#$ is a superalgebra morphism,

$$\mathbf{a}^\#(\mathcal{P}(V)) \subseteq \Gamma(\mathcal{O}_L) \otimes \mathcal{P}(V).$$

If $W \subseteq \mathcal{P}(V)$ is a graded subspace invariant under L_0 and \mathfrak{l} , then

$$\mathbf{a}^\sharp(W) \subseteq \Gamma(\mathcal{O}_L) \otimes W.$$

Indeed, we may identify

$$\Gamma(\mathcal{O}_L) = \underline{\text{Hom}}_{\mathfrak{l}_0}(\mathfrak{U}(\mathfrak{l}), \Gamma(\mathcal{O}_{L_0})),$$

so we may consider superfunctions f on L as functions $f(u; g)$ of $u \in \mathfrak{U}(\mathfrak{l})$ and $g \in L_0$. Consider the canonical extension \mathbf{a}_u of the anti-homomorphism $x \mapsto \mathbf{a}_x$ to $\mathfrak{U}(\mathfrak{l})$ and $\mathbf{a}_g := \mathbf{a} \circ (j_g \times \text{id})$ for $g \in L_0$. Then by [3, Proposition 8.3.3], we have for $w \in W$, $u \in \mathfrak{U}(\mathfrak{l})$, and $g \in L_0$:

$$\mathbf{a}^\sharp(w)(u; g) = \mathbf{a}_u \mathbf{a}_g^\sharp(w) = S_{\mathfrak{l}}(u) \cdot g^{-1} \cdot w \in W,$$

with $S_{\mathfrak{l}}$ denoting the antipodal anti-automorphism of $\mathfrak{U}(\mathfrak{l})$.

Now, seeking a contradiction, assume that $p_\lambda^* = 0$, so that

$$j_e^\sharp(\ell) = j_V^\sharp(\ell)(e) = 0, \quad \forall \ell \in V_\lambda.$$

Let $\ell \in L_V^\sharp$ be arbitrary. Then

$$\pi^\sharp j_V^\sharp(\ell) = \mathbf{a}_e^\sharp(\ell) = (\text{id} \otimes j_e)^\sharp \mathbf{a}^\sharp(\ell) \subseteq (\text{id} \otimes j_e)^\sharp(\Gamma(\mathcal{O}_L) \otimes V_\lambda) = 0.$$

Since π is a surjective submersion, π^\sharp is injective, so that $j_V^\sharp(\ell) = 0$. But j_V is an open embedding and ℓ is a superpolynomial, so $\ell = 0$, contradiction! This proves the assertion. \square

4.3. Determination of the spherical polynomials. The Lie superalgebra $\mathfrak{g} \cong \mathfrak{q}(2n)$ carries a non-degenerate invariant odd supersymmetric bilinear form $b : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$, the *odd trace form*. In terms of the matrix realisation we have chosen in Equation (2.1), it is given by

$$b(x, x') := \frac{1}{2} \text{tr}(x \Phi x'), \quad x, x' \in \mathfrak{g}$$

or explicitly,

$$b(x, x') = \text{tr}(a\alpha' + \alpha a' + c\beta' + \beta c' + b\gamma' + \gamma b' + d\delta' + \delta d'),$$

$$x = \begin{pmatrix} a & \alpha & b & \beta \\ \alpha & a & \beta & b \\ \gamma & c & \delta & d \\ c & \gamma & d & \delta \end{pmatrix}, \quad x' = \begin{pmatrix} a' & \alpha' & b' & \beta' \\ \alpha' & a' & \beta' & b' \\ \gamma' & c' & \delta' & d \\ c' & \gamma' & d' & \delta' \end{pmatrix}.$$

The odd form restricts to an \mathfrak{l} -invariant odd non-degenerate pairing of V and V^- , allowing us to identify the \mathfrak{l} -modules V^* and $\Pi(V^-)$.

We let $\mathfrak{t}^* := \mathbb{C}^n$. Then $\Pi(\mathfrak{t}^*)$ may be identified with a subspace of V^- via the odd map

$$\mathfrak{t}^* \longrightarrow V^- : x \longmapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \text{diag}(x) & 0 & 0 \\ \text{diag}(x) & 0 & 0 & 0 \end{pmatrix}$$

This determines an even injective linear map $\mathfrak{t}^* \rightarrow V^*$ and hence, a morphism $\mathbb{A}(\mathfrak{t}^*) \rightarrow \mathbb{A}(V^*)$. Thereby, we confer a meaning to $p|_{\mathfrak{t}^*}$ for polynomials p in $\mathcal{P}(V^*) = S(V)$, as the pullback along this morphism.

Identifying $\Pi(V^-)$ with V^* , $p_\lambda^*|_{\mathfrak{t}^*}$ is the ordinary n -variable polynomial given by

$$(4.2) \quad p_\lambda^*|_{\mathfrak{t}^*}(x_1, \dots, x_n) = p_\lambda^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \text{diag}(x) & 0 & 0 \\ \text{diag}(x) & 0 & 0 & 0 \end{pmatrix}, \quad \forall x_j \in \mathbb{C}.$$

By Proposition 3.1, $p_\lambda^*|_{\mathfrak{t}^*}$ is homogeneous of degree $|\lambda|$ in $\mathcal{P}(V^*)$, so $p_\lambda^*|_{\mathfrak{t}^*}$ is likewise homogeneous of degree $|\lambda|$.

DEFINITION 4.4. We call $p_\lambda^*|_{\mathfrak{t}^*}$ the *spherical polynomial* for $\lambda \in \Lambda_{>0}^n$.

The remainder of this subsection is devoted to the proof of the following result.

THEOREM 4.5. *Let $\lambda \in \Lambda_{>0}^n$ and $k := |\lambda|$. Then the spherical polynomial $p_\lambda^*|_{\mathfrak{t}^*}$ is the homogeneous part of degree k of the eigenvalue polynomials q_λ^* , so*

$$p_\lambda^*|_{\mathfrak{t}^*} = \frac{1}{\lambda!} \prod_{i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} Q_\lambda.$$

Let \mathfrak{n}^+ be the subalgebra of \mathfrak{l} formed by the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad A = \begin{pmatrix} a & \alpha \\ \alpha & a \end{pmatrix}, \quad D = \begin{pmatrix} d & \delta \\ \delta & d \end{pmatrix}$$

where a and α are strictly upper triangular and d and δ are strictly lower triangular. Similarly, let \mathfrak{a} be the subspace of \mathfrak{l} formed by the matrices

$$h_{a,\alpha} := \frac{1}{2} \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad A = \begin{pmatrix} a & \alpha \\ \alpha & a \end{pmatrix},$$

where a and α are diagonal. Then we have the vector space decomposition

$$\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

Although \mathfrak{a}_0 is a subalgebra, \mathfrak{a} is not. We still consider the basis of \mathfrak{a} given by

$$h_i := h_{e_i,0} = \frac{1}{2} \begin{bmatrix} E_{ii} & 0 & 0 & 0 \\ 0 & E_{ii} & 0 & 0 \\ 0 & 0 & -E_{ii} & 0 \\ 0 & 0 & 0 & -E_{ii} \end{bmatrix}, \quad \eta_i := h_{0,e_i} = \frac{1}{2} \begin{bmatrix} 0 & E_{ii} & 0 & 0 \\ E_{ii} & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_{ii} \\ 0 & 0 & -E_{ii} & 0 \end{bmatrix},$$

where $i = 1, \dots, n$.

Let $\omega : S(\mathfrak{l}) \rightarrow \mathfrak{U}(\mathfrak{l})$ denote the canonical supersymmetrisation map, given by

$$\omega(x_1 \cdots x_m) := \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^{\#\{(i,j) | i < j, \sigma(i) > \sigma(j), |x_i| = |x_j| = \bar{1}\}} x_{\sigma(1)} \cdots x_{\sigma(m)}$$

for all homogeneous $x_1, \dots, x_m \in \mathfrak{l}$. Then, by the Poincaré–Birkhoff–Witt theorem [25, Chapter I, § 2.3, Corollary 1 to Theorem 1], every $u \in \mathfrak{U}(\mathfrak{l})$ can be written in a unique way as

$$(4.3) \quad u = u' + u_{\mathfrak{a}},$$

where

$$u' \in \mathfrak{m}\mathfrak{U}(\mathfrak{l}) + \mathfrak{U}(\mathfrak{l})\mathfrak{n}^+ = \mathfrak{m}\mathfrak{U}(\mathfrak{l}) \oplus \omega(S(\mathfrak{a} \oplus \mathfrak{n}^+))\mathfrak{n}^+,$$

and $u_{\mathfrak{a}}$ can be written as a linear combination of monomials in the basis of \mathfrak{a} , that is,

$$u_{\mathfrak{a}} = \sum_{J \in \mathcal{J}} c_J h^J,$$

where

$$\mathcal{J} := \{(j_1, \dots, j_n, j'_1, \dots, j'_n) \mid \forall 1 \leq i \leq n : j_i \in \mathbb{N}, j'_i = 0, 1\},$$

the constants $c_J \in \mathbb{C}$, and

$$h^{\mathcal{J}} := \prod_{i=1}^n h_i^{j_i} \prod_{i=1}^n \eta_i^{j'_i}.$$

Recall that z_λ is the unique element of $\mathcal{Z}(\mathfrak{q}(n))$ such that $L(z_\lambda) = D_\lambda$.

LEMMA 4.6. *In the decomposition $z_\lambda = z'_\lambda + z_{\lambda, \mathfrak{a}}$ of Equation (4.3), we have $z_\lambda \in \mathfrak{U}(\mathfrak{a}_0)$. In particular, there exists a unique polynomial $\gamma(z_\lambda) \in \mathbb{C}[x_1, \dots, x_n]$ such that*

$$(4.4) \quad z_{\lambda, \mathfrak{a}} = \gamma(z_\lambda)(h_1, \dots, h_n).$$

PROOF. Let $\mathfrak{q}(n) = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}^+$ be the standard triangular decomposition of $\mathfrak{q}(n)$. Furthermore, let $\mathfrak{U}(\mathfrak{q}(n))^0$ denote the centralizer of \mathfrak{h} in $\mathfrak{U}(\mathfrak{q}(n))$. As usual, the Harish-Chandra projection gives rise to an $\text{ad}_{\mathfrak{h}}$ -invariant direct sum decomposition

$$\mathfrak{U}(\mathfrak{q}(n))^0 = (\mathfrak{u}^- \mathfrak{U}(\mathfrak{q}(n)) \cap \mathfrak{U}(\mathfrak{q}(n))\mathfrak{u}^+) \oplus \mathfrak{U}(\mathfrak{h}).$$

Now write $z_\lambda := z_\lambda^0 + z_{\lambda, \mathfrak{h}}$, according to the latter decomposition. From $\text{ad}_{\mathfrak{h}}$ -invariance of the decomposition, it follows that $[\mathfrak{h}_1, z_{\lambda, \mathfrak{h}}] = 0$, and therefore that indeed $z_{\lambda, \mathfrak{h}} \in \mathfrak{U}(\mathfrak{h}_0)$. We can write z_λ^0 as a sum of monomials of the form $x_1 \cdots x_r$, where $x_1, \dots, x_r \in \mathfrak{q}(n)$, such that $x_1 \in \mathfrak{u}^-$ and $x_r \in \mathfrak{u}^+$. Next, by the natural embedding of $\mathfrak{U}(\mathfrak{q}(n))$ into $\mathfrak{U}(\mathfrak{l})$ (as the left factor) we can consider the decomposition $z_\lambda := z_\lambda^0 + z_{\lambda, \mathfrak{h}}$ as one in $\mathfrak{U}(\mathfrak{l})$, and clearly under this embedding the monomial $x_1 \cdots x_r$ is mapped to

$$(x_1, 0) \cdots (x_r, 0) \in \mathfrak{U}(\mathfrak{l})\mathfrak{n}^+.$$

Similarly, since $z_{\lambda, \mathfrak{h}} \in \mathfrak{U}(\mathfrak{h}_0)$, every monomial $x_1 \cdots x_r$ of $z_{\lambda, \mathfrak{h}}$ can be written as

$$x_1 \cdots x_r = x'_1 \cdots x'_r,$$

where $x'_i := \frac{1}{2}((x_i, x_i) + (x_i, -x_i))$, with $(x_i, x_i) \in \mathfrak{m}$ and $(x_i, -x_i) \in \mathfrak{a}_0$. Since \mathfrak{h}_0 is commutative, we obtain $z_{\lambda, \mathfrak{h}} \equiv \frac{1}{2^r} \prod_{i=1}^r (x_i, -x_i)$ modulo $\mathfrak{m}\mathfrak{U}(\mathfrak{l}) + \mathfrak{U}(\mathfrak{l})\mathfrak{n}^+$. The claim of the lemma now follows from uniqueness of the decomposition $z_\lambda = z'_\lambda + z_{\lambda, \mathfrak{a}}$. \square

LEMMA 4.7. *Consider $e \in V$ as a linear polynomial in $\mathcal{P}(V^*)$. Then, for every non-negative integer $k \geq 0$, we have $\mathfrak{m} \cdot e^k = 0$ and*

$$(\mathfrak{n}^+ \omega(S(\mathfrak{a} \oplus \mathfrak{n}^+)) \cdot e^k)|_{\mathfrak{t}^*} = 0.$$

PROOF. We may assume that $k \geq 1$. For $x \in \mathfrak{m}$, we have

$$x \cdot e^k = k(x \cdot e)e^{k-1} = 0,$$

as $x \cdot e = 0$. Next, Equations (3.2) and (3.3) imply that \mathfrak{n}^+ acts by vector fields that lie in the span of

$$u_{pk} \frac{\partial}{\partial u_{p\ell}}, \quad \xi_{pk} \frac{\partial}{\partial \xi_{p\ell}}, \quad u_{pk} \frac{\partial}{\partial \xi_{p\ell}}, \quad \xi_{pk} \frac{\partial}{\partial u_{p\ell}},$$

where $k > \ell$, whereas the action of \mathfrak{a} is given by vector fields of the same shape where now $k = \ell$. Let \mathcal{A}^+ be the subalgebra of $\mathcal{P}(V^*)$ generated by

$$u_{\ell k}, \quad \xi_{\ell k}, \quad \forall k \geq \ell,$$

and \mathcal{I}^+ the ideal of \mathcal{A}^+ generated by

$$u_{\ell k}, \quad \xi_{\ell k}, \quad \forall k > \ell.$$

Then $\omega(S(\mathfrak{a} \oplus \mathfrak{n}^+))$ leaves \mathcal{A}^+ invariant and \mathfrak{n}^+ maps \mathcal{A}^+ into \mathcal{I}^+ .

On the other hand, e is the sum of all the u_{kk} and $\xi_{\ell\ell}$, and hence e^k is contained in \mathcal{A}^+ . Thus, $\mathfrak{n}^+ \omega(S(\mathfrak{a} \oplus \mathfrak{n}^+)) \cdot e^k$ is contained in \mathcal{I}^+ . As the restriction of \mathcal{I}^+ to \mathfrak{t}^* vanishes, this proves the claim. \square

For $\lambda \in \Lambda_{>0}^n$, let $D_\lambda^* \in \mathcal{P}\mathcal{D}(V^*)$ be the operator dual to D_λ , i.e.

$$\langle D_\lambda^*(p), q \rangle := \langle p, D_\lambda(q) \rangle, \quad \forall p \in \mathcal{P}(V^*), q \in \mathcal{P}(V).$$

PROPOSITION 4.8. *For all $\lambda \in \Lambda_{>0}^n$ and $k := |\lambda|$, we have*

$$(4.5) \quad p_\lambda^* = (-1)^k D_\lambda^*(e^k).$$

PROOF. The polynomial $e^k \in \mathcal{P}(V^*)$ is homogeneous of degree k . Thus, by Proposition 3.1 and Corollary 3.3, we see that $D_\lambda^*(e^k) \in (V_\lambda)^*$. Furthermore, for every $\ell \in V_\lambda \subseteq \mathcal{P}(V)$, we have

$$\langle D_\lambda^*(e^k), \ell \rangle = c_\lambda(\lambda) \langle e^k, \ell \rangle = k! j_V^\sharp(\ell)(e) = (-1)^k \langle p_\lambda^*, \ell \rangle,$$

in view of Equation (4.1), and because ℓ is homogeneous of degree k . \square

PROOF OF THEOREM 4.5. From the proof of Proposition 3.4, recall that there is an element $z_\lambda \in \mathcal{Z}(\mathfrak{q}(n)) \subseteq \mathcal{Z}(\mathfrak{l})$ (necessarily even) of order $\leq k$ such that $L(z_\lambda) = D_\lambda$. Because of Lemma 4.6, we have

$$z_{\lambda, \mathfrak{a}} = \gamma(z_\lambda)(h_1, \dots, h_n) \in \mathfrak{U}(\mathfrak{a}_0) = S(\mathfrak{a}_0)$$

for a unique polynomial $\gamma(z_\lambda) \in \mathbb{C}[x_1, \dots, x_n]$. By Equation (2.2), we have

$$h_{a,0} \cdot e = a, \quad \forall a = \text{diag}(a_1, \dots, a_n),$$

so that

$$(4.6) \quad (z_{\lambda, \mathfrak{a}} \cdot e^k)|_{\mathfrak{t}^*}(x_1, \dots, x_n) = \gamma(z_\lambda)(x_1, \dots, x_n).$$

Let $S_{\mathfrak{l}} : \mathfrak{U}(\mathfrak{l}) \rightarrow \mathfrak{U}(\mathfrak{l})$ denote the antipodal anti-automorphism of $\mathfrak{U}(\mathfrak{l})$, defined by $S_{\mathfrak{l}}(x) := -x$ for $x \in \mathfrak{l}$. Then

$$S_{\mathfrak{l}}(z_\lambda) \equiv (-1)^k z_{\lambda, \mathfrak{a}} \pmod{\mathfrak{U}(\mathfrak{l})\mathfrak{m} \oplus \mathfrak{n}^+ \omega(S(\mathfrak{a} \oplus \mathfrak{n}^+))},$$

as \mathfrak{a}_0 is Abelian. Recall that we consider $\mathfrak{t}^* = \mathbb{C}^n$ as a subspace of V^* . Thus, for any $\mu \in \mathfrak{t}^*$, we may form $\mu^k \in \mathcal{P}^k(V) = S^k(V^*)$. In these terms, we compute by the use of Proposition 4.8 and Lemma 4.7:

$$\begin{aligned} p_\lambda^*|_{\mathfrak{t}^*}(\mu) &= \frac{1}{k!} \langle p_\lambda^*, \mu^k \rangle = \frac{(-1)^k}{k!} \langle D_\lambda^*(e^k), \mu^k \rangle \\ &= \frac{(-1)^k}{k!} \langle S_{\mathfrak{l}}(z_\lambda)(e^k), \mu^k \rangle = \frac{1}{k!} \langle z_{\lambda, \mathfrak{a}} \cdot e^k, \mu^k \rangle. \end{aligned}$$

By Equation (4.6), this is the value of the k -homogeneous part of $(z_{\lambda, \mathfrak{a}} \cdot e^k)|_{\mathfrak{t}^*}$ at the point $\mu \in \mathfrak{t}^*$.

Let $\mu \in \Lambda_{>0}^n$. By the highest weight theory for $\mathfrak{q}(n)$ [5, § 2.3], there is a non-zero vector $v_\mu \in V_\mu$ such that $\mathfrak{n}^+ v_\mu = 0$ and

$$h_{a,0} \cdot v_\mu = \sum_{j=1}^n \mu_j a_j \cdot v_\mu, \quad \forall a = \text{diag}(a_1, \dots, a_n).$$

In view of Equation (4.4), and because $\eta_i^2 = \frac{1}{2}[\eta_i, \eta_i] \in \mathfrak{m}$, we find that V_μ is generated as an \mathfrak{m} -module by the vectors $v_\mu^i := \eta_i v_\mu$, $i = 1, \dots, n$. Thus, there must be some i such that $\langle p_\mu^*, v_\mu^i \rangle \neq 0$.

As $\mathfrak{n}^+ \cdot v_\mu^i = 0$ and $\mathfrak{m} \cdot p_\mu^* = 0$, we see that

$$c_\lambda(\mu) \langle p_\mu^*, v_\mu^i \rangle = \langle p_\mu^*, z_\lambda \cdot v_\mu^i \rangle = \langle p_\mu^*, z_{\lambda, \mathfrak{a}} \cdot v_\mu^i \rangle = \gamma(z_\lambda)(\mu_1, \dots, \mu_n) \langle p_\mu^*, v_\mu^i \rangle.$$

We conclude

$$q_\lambda^*(\mu) = c_\lambda(\mu) = \gamma(z_{\lambda, \mathfrak{a}})(\mu_1, \dots, \mu_n) = (z_{\lambda, \mathfrak{a}} \cdot e^k)|_{\mathfrak{t}^*}(\mu_1, \dots, \mu_n),$$

by Equation (4.6), and with the convention from Equation (4.2). As $\Lambda_{>0}^n$ is Zariski dense in \mathbb{C}^n , $p_\lambda^*|_{\mathfrak{t}^*}$ is the k -homogeneous part of q_λ^* . The assertion now follows directly from Theorem 3.8. \square

In the next theorem, we use the notation introduced in Section 1. In particular, recall that $\eta : \mathcal{Z}(\mathfrak{q}(n)) \rightarrow \mathcal{P}(\mathfrak{h}_{\bar{0}})$ denotes the Harish-Chandra homomorphism of Equation (1.3), and we identify the image of η with the algebra of Q -symmetric polynomials in n variables. Let $S : \mathfrak{U}(\mathfrak{q}(n)) \rightarrow \mathfrak{U}(\mathfrak{q}(n))$ denote the canonical anti-automorphism of the enveloping algebra $\mathfrak{U}(\mathfrak{q}(n))$ obtained by extending the map $\mathfrak{q}(n) \rightarrow \mathfrak{q}(n) : x \mapsto -x$.

THEOREM 4.9. *For every $\lambda \in \Lambda_{>0}$, there exists a unique element $z_\lambda \in \mathcal{Z}(\mathfrak{q}(n))$ such that $L(z_\lambda) = D_\lambda$. Furthermore,*

$$(4.7) \quad z_\lambda = r_\lambda S(C_\lambda), \text{ where } r_\lambda := \frac{(-1)^{|\lambda|}}{2^{\ell(\lambda)} (\lambda_1! \cdots \lambda_{\ell(\lambda)}!)^2} \prod_{1 \leq i < j \leq \ell(\lambda)} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right)^2.$$

PROOF. Existence and uniqueness of z_λ follow as in the proof of Proposition 3.4. Note that Nazarov considers a slightly different action of \mathfrak{l} on a polynomial space which decomposes as a direct sum of \mathfrak{l} -modules of the form $2^{-\delta(\lambda)} F_\lambda \otimes F_\lambda$. Therefore, it follows from [16, Proposition 4.3] that $(-1)^{|\lambda|} L(S(C_\lambda))$ acts on V_λ with the same scalar that C_λ acts on Nazarov's W_λ (see [16, Section 4]).

For the proof of Equation (4.7), note first that the “ ϱ -shift” for Sergeev’s Harish-Chandra homomorphism is zero, and that $L(z_\lambda)$ acts on the modules $V_\mu \subseteq \mathcal{P}(V)$ by a scalar given by a Q -symmetric polynomial in μ_1, \dots, μ_n of degree at most $|\mu|$. From Proposition 3.6 and Corollary 3.3, it follows that $\eta(z_\lambda) = q_\lambda^*$. In particular, the action of $L(z_\lambda)$ on V_μ is given by a polynomial in μ whose leading coefficient is $(Q_\lambda^*(\lambda))^{-1}$. In addition, it follows from [16, Proposition 4.8] that the action of $(-1)^{|\lambda|} L(S(C_\lambda))$ is by a polynomial in μ with leading coefficient

$$\lambda_1! \cdots \lambda_{\ell(\lambda)}! \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}.$$

Equation (4.7) now follows from a comparison of leading coefficients and the formula for $Q_\lambda^*(\lambda)$ given in Ref. [8]. \square

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