Conjugacy of maximal toral subalgebras of direct limits of loop algebras

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Dedicated to V. S. Varadarajan, on the occasion of his seventieth birthday.

ABSTRACT. We define and investigate maximal toral subalgebras of the loop algebras associated to classical simple direct limit Lie algebras which are defined and split over a field of characteristic zero. We show that any two such Lie subalgebras are conjugate under the automorphism group of the loop algebra.

1. Introduction

To say that the notion of diagonalizable subalgebras is among the most fundamental concepts of Lie theory is not an exaggeration. By a classical theorem, originally due to Chevalley [**Bour**, Chapter VII, §3], in a finite-dimensional split simple Lie algebra over a field of characteristic zero, any two split Cartan subalgebras are conjugate under the group of (elementary) automorphisms of the Lie algebra. Kac and Peterson proved a conjugacy theorem for Cartan (and Borel) subalgebras of Kac-Moody Lie algebras [**KaPe**]. In [**Pi1**], the author uses a marvellous connection between conjugacy of maximal toral subalgebras and triviality of certain equivariant torsors on affine schemes to obtain a conjugacy theorem for a general class of algebras which includes toroidal algebras over finite dimensional split simple Lie algebras.

For the standard direct limits of finite-dimensional split simple Lie algebras, i.e., the Lie algebras which are usually denoted by $\mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$, Neeb and Stumme defined splitting Cartan subalgebras and proved their conjugacy under the automorphism group of the Lie algebra [**NeSt1**], [**St**]. Their results are natural counterparts of the standard results in structure theory of semisimple Lie algebras, inlcuding Chevalley's theorem. Their proofs, though quite nontrivial, are devoid of advanced machinery. Nevertheless, somewhat mysteriously the discovery of this beautiful theory has been delayed for decades.

The main goal of this note is to prove a conjugacy theorem for maximal toral subalgebras of (universal central extensions of) loop algebras $\mathfrak{g} \otimes_k k[t, t^{-1}]$ where k is an algebraically closed field of characterestic zero, and $\mathfrak{g}$ is isomorphic to one

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of $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$ or $\mathfrak{so}_\infty$. Our main result implies that when $R = k[t, t^{-1}]$, the action of $\mathrm{Aut}_k(\mathfrak{g}(R))$ on maximal toral subalgebras of $\mathfrak{g}(R)$ has a small number of orbits (see section 5 for a precise statement). Indeed a subgroup of $\mathrm{Aut}_k(\mathfrak{g}(R))$ would suffice, which is the homomorphic image of a group $\mathfrak{G}$ that is described explicitly in sections 2.3-2.4. (When $\mathfrak{g} = \mathfrak{sl}_\infty$, the map $\mathfrak{G} \to \mathrm{Aut}_k(\mathfrak{g}(R))$ has a nontrivial kernel.) If we express $\mathfrak{g}$ as a direct limit $\varinjlim \mathfrak{g}_m$ of finite dimensional simple Lie algebras in a standard way, then the group $\mathfrak{G}$ can indeed be shown to be a certain topological completion of $\varinjlim \mathfrak{G}_m$, where $\mathfrak{G}_m$ is the R-points of a group scheme asociated to $\mathfrak{g}_m$. (We do not prove the latter statement in this article.) We will observe, somewhat surprisingly, that the case $\mathfrak{g} = \mathfrak{so}_\infty$ is more complicated and is related to classical K-theory. The passage to the universal central extension of $\mathfrak{g}(R)$ can be done using the method of [**Pi2**].

It is worth emphasizing that the geometric method of [**Pi1**] is based on the idea of conjugating a *regular* element of one toral subalgebra into another. However, when $\mathfrak{g}$ is infinite dimensional, maximal toral subalgebras of $\mathfrak{g} \otimes_k R$ are infinite dimensional as well, and it is not clear how to define a regular element in an infinite dimensional toral subalgebra. Therefore, the method of [**Pi1**] is not applicable when $\mathfrak{g}$ is infinite dimensional. It would be interesting to obtain an infinite dimensional version of the method of [**Pi1**], say based on the notion of ind-varieties. However, at this point this is merely speculative.

Our approach is to modify and combine the methods of [**NeSt1**], [**St**], [**NePe**] and [**Pi1**] suitably. Essentially, the hard part is obtaining a version of the technical lemmas of [**St**] over a ring, rather than a field.

2. Notation and preliminaries

Throughout k will be an algebraically closed field of characteristic zero. Let $R = k[t, t^{-1}]$, K be the quotient field of R, $R^\times$ be the set of nonzero elements of R, and $R^\bullet$ be the group of multiplicative units in R. For any Lie algebra $\mathfrak{g}$ over k, set $\mathfrak{g}(R) = \mathfrak{g} \otimes_k R$. All of our Lie algebras are over k. Thus, although $\mathfrak{g}(R)$ is a Lie algebra over R, we are interested in its Lie subalgebras over k, for example those of the form $\mathfrak{l} \otimes_k 1$, where $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g}$.

Let M be an arbitrary R-module. Then it is also a vector space over k. A basis of M as a vector space over k is called a *k-basis* of M. If M is a vector space over a field $k_1 \supseteq k$, then a basis of M as a vector space over k_1 is called a *k_1-basis* of M. If M is a free R-module, then a basis of M as an R-module is called an *R-basis* of M.

2.1. Classical simple direct limit Lie algebras. The Lie algebras $\mathfrak{sl}_\infty, \mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ are sometimes referred to as *classical simple direct limit Lie algebras*. Like their finite dimensional analogues, they can be described concretely using $\infty \times \infty$ matrices.

The description of $\mathfrak{so}_\infty$ sounds slighly tricky, as a priori it seems that there exist two possibilities for $\mathfrak{so}_\infty$: as a direct limit of the $\mathfrak{so}_{2n}$'s and also as a direct limit of the $\mathfrak{so}_{2n+1}$'s. Nevertheless, it is shown in [**NeSt1**, Lemma I.4] that the resulting Lie algebras are indeed isomorphic.

It can be shown that up to isomorphism, $\mathfrak{sl}_\infty, \mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ are the only locally finite simple Lie algebras of countable dimension over k which have a *splitting Cartan subalgebra* (i.e., a maximal abelian Lie subalgebra which yields a root decomposition of $\mathfrak{g}$). See [**NeSt1**] for further details.

For our purposes, it is most convenient to describe the classical direct limit Lie algebras and their standard representations simultaneously. In sections 2.3-2.5, we introduce several gadgets associated to any classical simple direct limit Lie algebra $\mathfrak{g}$: the loop algebras $\mathfrak{g}(R)$ and $\mathfrak{g}_m(R)$, standard representations V_R and $V_{m,R}$, an R-basis $\mathcal{B}$ of V_R, a nondegenerate pairing $\langle\cdot,\cdot\rangle$, and groups $\mathfrak{G}$ and $\mathfrak{G}_m$.

Let V be a vector space of countable dimension over k. Obviously, $\mathrm{End}_k(V)$ is a Lie algebra with the usual bracket $[X,Y]=XY-YX$. The vector space V is also a left $\mathrm{End}_k(V)$-module. Set $V_R=V\otimes_k R$. The free R-module V_R is a left $\mathrm{End}_R(V_R)$-module.

Let $\mathcal{B}$ be a k-basis of V. Obviously $\mathcal{B}$ serves as an R-basis of V_R as well. Using $\mathcal{B}$, we can represent elements of $\mathrm{End}_k(V)$ and $\mathrm{End}_R(V_R)$ by $\infty\times\infty$ matrices which have only finitely many nonzero entries in each column. An element of $\mathrm{End}_k(V)$ or $\mathrm{End}_R(V_R)$ is called *$\mathcal{B}$-finitary* if its matrix has only finitely many nonzero entries in each row. It is called *$\mathcal{B}$-finite* if its matrix has only finitely many nonzero entries.

2.2. The Lie algebras $\mathfrak{gl}(V)$ and $\mathfrak{gl}(V_R)$. Fix a k-basis $\mathcal{B}$ of V. We define the Lie algebra $\mathfrak{gl}(V)$ as the Lie subalgebra of $\mathcal{B}$-finite elements of $\mathrm{End}_k(V)$. In view of the canonical map

$$\mathrm{End}_k(V)\otimes_k R\to \mathrm{End}_R(V_R) \tag{2.1}$$

given by $T\otimes r\mapsto r\cdot T$, the Lie algebra $\mathfrak{gl}(V_R)$ is defined as the Lie subalgebra $\mathfrak{gl}(V)\otimes_k R$ of $\mathrm{End}_R(V_R)$.

2.3. The Lie algebras $\mathfrak{sl}_\infty$ and $\mathfrak{sl}_\infty(R)$. Choose an ordered k-basis

$$\mathcal{B}=\{e_1,e_2,e_3,\ldots\}$$

of V, and consider the corresponding Lie algebra $\mathfrak{gl}(V)$. In this case there is an obvious isomorphism between $\mathfrak{gl}(V)$ and the direct limit $\varinjlim \mathfrak{gl}_n$ of the direct system of Lie algebras with monomorphisms $i_n:\mathfrak{gl}_n\to\mathfrak{gl}_{n+1}$ given by

$$X\mapsto \begin{bmatrix} X & 0\\ 0 & 0\end{bmatrix}\qquad \text{for any } X\in\mathfrak{gl}_n. \tag{2.2}$$

Every element of $\mathfrak{gl}(V)$ has a well defined trace, and the Lie algebra $\mathfrak{sl}_\infty$ is defined as the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of elements of trace zero. The Lie algebra $\mathfrak{sl}_\infty(R)$ is the image of $\mathfrak{sl}_\infty\otimes_k R$ in $\mathfrak{gl}(V_R)$ under the map (2.1). One can also think of $\mathfrak{sl}_\infty(R)$ as the Lie algebra of $\infty\times\infty$ matrices $[a_{i,j}]_{1\le i,j<\infty}$ where $a_{i,j}\in R$, all but finitely many $a_{i,j}$'s are zero, and $\sum_{i=1}^{\infty}a_{i,i}=0$.

Using the basis $\mathcal{B}$, we can identify V (respectively V_R) with infinite column vectors $[a_1,a_2,a_3,\ldots]^T$ where $a_i\in k$ (respectively, $a_i\in R$) and all but finitely many of the a_i's are zero. The Lie algebra $\mathfrak{sl}_\infty(R)$ acts on V_R on the left by matrix multiplication. (Thus V_R is nothing but the standard representation of $\mathfrak{sl}_\infty(R)$.) At the same time, $\mathfrak{sl}_\infty(R)$ acts on V_R on the right by

$$v\cdot A=A^Tv \text{ for any } v\in V_R, A\in\mathfrak{sl}_\infty(R).$$

We represent the latter $\mathfrak{g}(R)$-module by V_R^*. It can be identified with the free R-module of infinite rows $[a_1,a_2,a_3,\ldots]$ where $a_i\in R$ and all but finitely many a_i's are zero, and $\mathfrak{sl}_\infty(R)$ acts on V_R^* on the right by matrix multiplication. The standard basis of V_R^* is denoted by

$$\mathcal{B}^*=\{e_1^*,e_2^*,e_3^*,\ldots\}.$$

There is an obvious R-linear pairing $\langle \cdot, \cdot \rangle : V_R \times V_R^* \to R$ given by $\langle x, y \rangle = yx$, where if $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i^*$ then $yx = \sum_i x_i y_i$. Thus $V_R \otimes_R V_R^*$ is an associative algebra with multiplication

$$x \otimes x' \cdot y \otimes y' = \langle y, x' \rangle x \otimes y'.$$

Thus $V_R \otimes_R V_R^*$ is a Lie algebra with the standard commutator $[a, b] = a \cdot b - b \cdot a$. This yields an R-linear isomorphism of Lie algebras

$$\begin{array}{rcl} \iota : V_R \otimes_R V_R^* & \to & \mathfrak{gl}(V_R) \\ v \otimes w & \mapsto & T_{v,w} \end{array}$$

where $T_{v,w}(x) = vwx$. Moreover, $\iota^{-1}(\mathfrak{sl}_\infty(R))$ is spanned by elements of the form $v \otimes w$ where $\langle w, v \rangle = 0$.

Let $\mathfrak{G}$ denote the set of elements $X \in \mathrm{End}_R(V_R)$ such that

(a) X is invertible.
(b) Both X and X^{-1} are $\mathcal{B}$-finitary.

It is not difficult to check that under composition $\mathfrak{G}$ forms a group. For any $X \in \mathfrak{G}$, the map $\pi_X : \mathrm{End}_R(V_R) \to \mathrm{End}_R(V_R)$ given by $\pi_X(T) = XTX^{-1}$ yields a k-linear (even R-linear) automorphism of the Lie algebra $\mathfrak{sl}_\infty(R)$.

For any positive integer m, let $\mathcal{B}_m = \{e_1, \ldots, e_m\}$, $V_m = \mathrm{Span}_k\{e_1, \ldots, e_m\}$ and $V_{m,R} = V_m \otimes_k R$. Set

$$\mathfrak{sl}_m = \{X \in \mathrm{End}_k(V_m) \mid \mathrm{tr}(X) = 0\}$$

and let $\mathfrak{sl}_m(R) = \mathfrak{sl}_m \otimes_k R$. We can see readily that $\mathfrak{sl}_\infty(R) = \varinjlim \mathfrak{sl}_m(R)$, where the direct system is defined in a fashion similar to (2.2). Set

$$\mathfrak{G}_m = \{X \in \mathrm{End}_R(V_{m,R}) \mid X \text{ is invertible and } \det(X) = 1\}.$$

Conjugation by an element of $\mathfrak{G}_m$ yields an R-linear automorphism of the Lie algebra $\mathfrak{sl}_m(R)$. There exists an obvious embedding of $\mathfrak{G}_m$ into $\mathfrak{G}$ given by $T \mapsto \tilde{T}$ where

$$\tilde{T} e_i = \begin{cases} Te_i & \text{if } i \le m, \\ e_i & \text{otherwise.} \end{cases}$$

2.4. The Lie algebras $\mathfrak{sp}_\infty$ and $\mathfrak{sp}_\infty(R)$. Let us choose the ordered k-basis $\mathcal{B} = \{e_1, e_2, \ldots, f_1, f_2, \ldots\}$ for V. We can represent an element $\sum_i a_i e_i + \sum_i b_i f_i$ of V or V_R by a column of the form $[a_1, a_2, a_3, \ldots; b_1, b_2, b_3, \ldots]^T$. The matrix of a k-linear map $T : V \to V$ or an R-linear map $T_R : V_R \to V_R$ in the basis $\mathcal{B}$ is of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

where the X_i's are $\infty \times \infty$ matrices of the form $[x_{r,s}]_{1 \le r,s < \infty}$.

Consider the skew symmetric k-bilinear form $\langle \cdot, \cdot \rangle$ on V which is identified by the relations

$$\langle e_i, e_j \rangle = 0 \ , \ \langle f_i, f_j \rangle = 0 \ , \ \text{ and } \ \langle e_i, f_j \rangle = \delta_{i,j}. \tag{2.3}$$

The form $\langle \cdot, \cdot \rangle$ extends to an R-bilinear form on V_R in a unique way. We denote the latter form by $\langle \cdot, \cdot \rangle$ as well.

The Lie algebra $\mathfrak{sp}_\infty$ (respectively, $\mathfrak{sp}_\infty(R)$) is the Lie subalgebra of $\mathfrak{gl}(V)$ (respectively, of $\mathfrak{gl}(V_R)$) consisting of elements X such that

$$\langle Xv, w \rangle + \langle v, Xw \rangle = 0 \text{ for any } \ v, w \in V \ \ \text{(respectively, for any } v, w \in V_R).$$

We can make $V_R \otimes_R V_R$ into an associative algebra with multiplication

$$x \otimes x' \cdot y \otimes y' = \langle x', y \rangle x \otimes y', \tag{2.4}$$

and then a Lie algebra with the standard commutator $[a, b] = a \cdot b - b \cdot a$. There exists an R-linear Lie algebra isomorphism

$$\begin{array}{rcl} \iota : V_R \otimes_R V_R & \to & \mathfrak{gl}(V_R) \\ v \otimes w & \mapsto & vw^T S \end{array}$$

with

$$S = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}$$

where $\mathrm{I} = [\delta_{i,j}]_{1 \le i,j < \infty}$. One can see that

$$\iota^{-1}(\mathfrak{sp}_\infty(R)) = \{v \otimes w + w \otimes v \mid v, w \in V_R\}.$$

Similar to what we did in section 2.3, we can consider the group $\mathfrak{G}$ consisting of elements $X \in \mathrm{End}_R(V_R)$ such that

(a) X is invertible.
(b) Both X and X^{-1} are $\mathcal{B}$-finitary.
(c) For any $v, w \in V_R$, $\langle Xv, Xw \rangle = \langle v, w \rangle$.

Again one can see that the map $\pi_X(T) = XTX^{-1}$ is an R-linear automorphism of $\mathfrak{sp}_\infty(R)$.

For any positive integer m, let $\mathcal{B}_m = \{e_1, \dots, e_m, f_1, \dots, f_m\}$, $V_m = \mathrm{Span}_k \mathcal{B}_m$ and $V_{m,R} = V_m \otimes_k R$. We can restrict $\langle \cdot, \cdot \rangle$ to V_m and $V_{m,R}$. Set

$$\mathfrak{sp}_m = \{X \in \mathrm{End}_k(V_m) \mid \text{ for every } v, w \in V_m, \langle Xv, w \rangle + \langle v, Xw \rangle = 0\} \tag{2.5}$$

and let $\mathfrak{sp}_m(R) = \mathfrak{sp}_m \otimes_k R$. Also, set

$$\mathfrak{G}_m = \{X \in \mathrm{End}_R(V_{m,R}) \mid \text{ for every } v, w \in V_{m,R}, \langle Xv, Xw \rangle = \langle v, w \rangle\}. \tag{2.6}$$

We can write $\mathfrak{sp}_\infty(R) = \varinjlim \mathfrak{sp}_m(R)$. The group $\mathfrak{G}_m$ acts by conjugation as R-linear automorphisms of $\mathfrak{sp}_m(R)$, and there is an obvious embedding of $\mathfrak{G}_m$ into $\mathfrak{G}$.

2.5. The Lie algebras $\mathfrak{so}_\infty$ and $\mathfrak{so}_\infty(R)$. These two Lie algebras can be described as in section 2.4. Let $\mathcal{B}, \mathcal{B}_m, V, V_m$ and V_R be defined as in section 2.4, and suppose $\langle \cdot, \cdot \rangle$ is the symmetric k-bilinear (respectively, R-bilinear) form on V (respectively, V_R) which extends the relations in (2.3). Again $V_R \otimes_R V_R$ can be made an ascociative algebra with multiplication (2.4). We also have an R-linear isomorphism of Lie algebras

$$\begin{array}{rcl} \iota : V_R \otimes_R V_R & \to & \mathfrak{gl}(V_R) \\ v \otimes w & \mapsto & vw^T S \end{array}$$

with

$$S = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}$$

and one can check that

$$\iota^{-1}(\mathfrak{so}_\infty(R)) = \{v \otimes w - w \otimes v \mid v, w \in V_R\}.$$

The group $\mathfrak{G}$ of automorphisms of $\mathfrak{so}_\infty(R)$ is defined as in section 2.4. The Lie algebra $\mathfrak{so}_{2m}$ is defined as in (2.5), and if we set $\mathfrak{so}_{2m}(R) = \mathfrak{so}_{2m} \otimes_k R$, then one can also write $\mathfrak{so}_\infty(R) = \varinjlim \mathfrak{so}_{2m}(R)$. The groups $\mathfrak{G}$ and $\mathfrak{G}_m$ are defined exactly as in section 2.4, and $\mathfrak{G}_m$ acts by conjugation on $\mathfrak{so}_{2m}(R)$ as its R-linear automorphisms. There exists an obvious embedding of $\mathfrak{G}_m$ into $\mathfrak{G}$.

2.6. Maximal toral subalgebras. Henceforth $\mathfrak{g}$ represents one of $\mathfrak{sl}_\infty, \mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$, $\mathfrak{g}(R)$ represents the corresponding Lie algebra $\mathfrak{g} \otimes_k R$, and $\mathcal{B}, \mathcal{B}_m, V_R$ and V_R^* are the objects defined accordingly.

A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}(R)$ is called a *maximal toral subalgebra*[1] whenever the following two conditions hold.

(a) There exists a k-basis $\mathcal{C}$ of $\mathfrak{g}(R)$ such that for every $X \in \mathcal{C}$, there exists a k-linear functional $\lambda_X : \mathfrak{h} \to k$ satisfying $[H, X] = \lambda_X(H)X$ for every $H \in \mathfrak{h}$.

(b) If $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{g}(R)$ containing $\mathfrak{h}$ and satisfying property (a) above, then $\mathfrak{h}' = \mathfrak{h}$.

Remark 2.1. It is easily seen that a maximal toral subalgebra is commutative. From property (a) above it immediately follows that $\mathfrak{g}(R)$ has a root space decomposition. Indeed, let

$$\mathfrak{h}^* = \{\lambda : \mathfrak{h} \to k \mid \lambda \text{ is } k\text{-linear}\}.$$

For any $\alpha \in \mathfrak{h}^*$, set

$$\mathfrak{g}(R)_\alpha = \{X \in \mathfrak{g}(R) \mid \text{for every } H \in \mathfrak{h},\ [H, X] = \alpha(H)X\}$$

and let $\mathfrak{z}(\mathfrak{h})$ denote the commutant of $\mathfrak{h}$ in $\mathfrak{g}(R)$. Then we can write

$$\mathfrak{g}(R) = \mathfrak{z}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}(R)_\alpha \tag{2.7}$$

where $\Delta \subset \mathfrak{h}^* - \{0\}$.

A simple but important example of a maximal toral subalgebra is given below.

Example 2.1. Let $\mathfrak{t}$ be a splitting Cartan subalgebra of $\mathfrak{g}$ (in the sense of [**NeSt1**]). For example, one can take $\mathfrak{t}$ to be the Lie subalgebra consisting of elements which act diagonally on the basis $\mathcal{B}$. Then $\mathfrak{h} = \mathfrak{t} \otimes_k 1 \subset \mathfrak{g} \otimes_k R$ is a maximal toral subalgebra of $\mathfrak{g}(R)$. (Maximality follows from the existence of root decomposition.)

3. Standard representations as weight modules

From now on, $\mathfrak{h}$ will be an arbitrary maximal toral subalgebra of $\mathfrak{g}(R)$. Recall that $\mathfrak{g}(R)$ is one of $\mathfrak{sl}_\infty(R), \mathfrak{sp}_\infty(R)$ and $\mathfrak{so}_\infty(R)$, and for any positive integer m, the Lie algebra $\mathfrak{g}_m(R)$ is equal to $\mathfrak{sl}_m(R)$, $\mathfrak{sp}_m(R)$, or $\mathfrak{so}_{2m}(R)$ accordingly. Also, we recall that $\mathfrak{G}$ and $\mathfrak{G}_m$ represent the groups corresponding to $\mathfrak{g}(R)$ and $\mathfrak{g}_m(R)$ defined in sections 2.3-2.5.

Our first lemma is standard. See [**MoPi**, Proposition 2.1.1] for a proof.

Lemma 3.1. *Let $\mathfrak{a}$ be an arbitrary Lie algebra over k, and W be a vector space over k which is also an $\mathfrak{a}$-module such that*

$$W = \bigoplus_{\lambda \in \mathfrak{a}^*} W_\lambda$$

[1] It is worth noting that our maximal toral subalgebras are closely related to Neeb and Penkov's split Cartan subalgebras.

where $\mathfrak{a}^*$ *is the vector space of* k*-linear functionals* $\lambda : \mathfrak{a} \to k$ *and*

$$W_\lambda = \{w \in W \mid \text{ for every } X \in \mathfrak{a}, \quad X \cdot w = \lambda(X)w\}.$$

Suppose $W' \subseteq W$ *is an* $\mathfrak{a}$*-submodule of* W*. Then* $W' = \bigoplus_{\lambda \in \mathfrak{a}^*} W'_\lambda$ *where* $W'_\lambda = W_\lambda \cap W'$*.*

LEMMA 3.2. *Let* W *be an arbitrary vector space over* k*. Suppose* $T \in \mathrm{End}_k(W)$ *and* $\mathcal{A} \subset \mathrm{End}_k(W)$ *are such that*

(a) *There exists a* k*-basis* $\mathcal{C}_1$ *of* W *consisting of eigenvectors of* T *with eigenvalues in* k*.*
(b) *There exists a* k*-basis* $\mathcal{C}_2$ *of* W *consisting of common eigenvectors of elements of* $\mathcal{A}$ *with eigenvalues in* k*.*
(c) T *commutes with elements of* $\mathcal{A}$*.*

Then there exists a basis $\mathcal{C}$ *of* W *consisting of common eigenvectors of elements of* $\{T\} \cup \mathcal{A}$ *with eigenvalues in* k*.*

PROOF. Apply Lemma 3.1 twice setting $\mathfrak{a} = \mathrm{Span}_k(\mathcal{A})$ and $\mathfrak{a} = \mathrm{Span}_k(T)$. $\square$

For any positive integer m, the set $\mathfrak{t}_m$ of elements of $\mathfrak{g}_m$ which act diagonally on $\mathcal{B}_m$ is a split Cartan subalgebra of $\mathfrak{g}_m$. The following proposition follows from [**Pi1**, Theorem 1].

PROPOSITION 3.1. *Suppose* $\mathfrak{h}_m$ *is a Lie subalgebra of* $\mathfrak{g}_m(R)$ *such that there exists a basis* $\mathcal{C}$ *of* $\mathfrak{g}_m(R)$ *whose elements are common eigenvectors of* ad_H *for all* $H \in \mathfrak{h}_m$*. Then* $\mathfrak{h}_m$ *and* $\mathfrak{t}_m \otimes_k 1$ *are conjugate by an element of* $\mathfrak{G}_m$*.*

REMARK 3.1. When $\mathfrak{g}_m = \mathfrak{so}_{2m}$, the group $\mathfrak{G}_m$ is not equal to R-points of a simply connected Chevalley group. Although results in [**Pi1**] are stated for simply connected Chevalley groups, the adjoint action of $\mathrm{Spin}_{2m}(R)$ on $\mathfrak{g}_m(R)$ factors through $\mathfrak{G}_m$.

Let $\mathrm{M}_{d\times d}(R)$ denote the associative algebra of $d \times d$ matrices with entries in R, and $\mathrm{SL}_d(R)$ be the multiplicative group of elements of $\mathrm{M}_{d\times d}(R)$ with determinant one. For the next lemma, we consider

$$R^{\oplus d} = \underbrace{R \oplus \cdots \oplus R}_{d \text{ times}}$$

as a left $\mathrm{M}_{d\times d}(R)$-module in the obvious way. We can think of $R^{\oplus d}$ as an infinite dimensional vector space over k, and the action of an element of $\mathrm{M}_{d\times d}(R)$ on $R^{\oplus d}$ yields a k-linear transformation of this vector space.

LEMMA 3.3. *Let* $\mathcal{A} \subset \mathrm{M}_{d\times d}(R)$ *be a set of commuting elements such that for any* $T \in \mathcal{A}$*, there exists a* k*-basis of* $R^{\oplus d}$ *consisting of eigenvectors of* T*, with eigenvalues in* k*.*

(a) *There exists a* k*-basis* $\{E_\nu\}$ *of* $\mathrm{M}_{d\times d}(R)$ *such that*

$$\text{for any } T \in \mathcal{A}, \text{ we have } [T, E_\nu] = t_\nu E_\nu \text{ where } t_\nu \in k.$$

(b) *There exists an element* $P \in \mathrm{SL}_d(R)$ *such that* $P\mathcal{A}P^{-1}$ *consists of diagonal matrices with entries in* k*.*

PROOF. (a) Since $\mathrm{M}_{d\times d}(R) \subseteq \mathrm{M}_{d\times d}(K)$, elements of $\mathcal{A}$ can be considered as K-linear maps $K^{\oplus d} \to K^{\oplus d}$ as well.

Fix $T \in \mathcal{A}$. From the assumptions of the lemma it follows that every vector in $K^{\oplus d}$ is a sum of eigenvectors of T with eigenvalues in k. Therefore we can find a K-basis of $K^{\oplus d}$ which consists of such eigenvectors of T. In other words, T is a diagonalizable element of $\mathrm{M}_{d\times d}(K)$, with eigenvalues in k.

Since $\mathcal{A}$ is a commuting family, it is simultaneously diagonalizable; i.e., there exists a matrix $P \in GL_d(K)$ such that the elements of $P\mathcal{A}P^{-1}$ are diagonal matrices, with entries in k. Let the $\mathrm{E}_{i,j}$'s $(1 \leq i, j \leq d)$ be the standard matrix units, and set $F_{i,j} = P^{-1}\mathrm{E}_{i,j}P$. Then the $F_{i,j}$'s form a K-basis of $\mathrm{M}_{d\times d}(K)$. Moreover, for every $1 \leq i, j \leq d$ and every $T \in \mathcal{A}$, there exists a $c_{i,j,T} \in k$ such that $[T, F_{i,j}] = c_{i,j,T}F_{i,j}$.

Using the latter K-basis of $\mathrm{M}_{d\times d}(K)$ and a k-basis of K, one can construct a k-basis for $M_{d\times d}(K)$ of common eigenvectors of $\mathrm{ad}_X, X \in \mathcal{A}$. If we consider $\mathrm{M}_{d\times d}(K)$ as a vector space over k, then $\mathrm{M}_{d\times d}(R)$ is a subspace which satisfies $[\mathcal{A}, \mathrm{M}_{d\times d}(R)] \subseteq \mathrm{M}_{d\times d}(R)$. Thus to finish the proof of part (a) of the lemma, it suffices to use Lemma 3.1.

(b) We saw in the proof of part (a) that as an element of $M_{d\times d}(K)$, any $X \in \mathcal{A}$ is diagonalizable with eigenvalues in k. Therefore the trace of X lies inside k. Let I_d denote the identity element in $\mathrm{M}_{d\times d}(R)$ and set $\mathcal{A}' = (\mathcal{A} + k\mathrm{I}_d) \cap \mathfrak{sl}_d(R)$. Part (a) applies to $\mathcal{A}'$. From $\mathfrak{sl}_d(R) \subset M_{d\times d}(R)$, $[\mathcal{A}, \mathfrak{sl}_d(R)] \subseteq \mathfrak{sl}_d(R)$ and Lemma 3.1, it follows that one can also find a k-basis $\mathcal{C}$ of $\mathfrak{sl}_d(R)$ consisting of common eigenvectors of ad_X, for all $X \in \mathcal{A}'$, with eigenvalues in k. Proposition 3.1 implies that there exists an element $P \in \mathrm{SL}_d(R)$ such that $P\mathcal{A}'P^{-1}$ consists of diagonal matrices with entries in k. It follows immediately that $P\mathcal{A}P^{-1}$ consists of diagonal matrices in $M_{d\times d}(R)$ with entries in k. (Here we use $\mathrm{char}(k) = 0$.) □

LEMMA 3.4. *Any finite subset S of $\mathfrak{g}(R)$ lies inside a finite-dimensional k-subspace W_S of $\mathfrak{g}(R)$ such that $[\mathfrak{h}, W_S] \subseteq W_S$.*

PROOF. It suffices to prove the lemma for a single element. Choose an element $X \in \mathfrak{g}(R)$. By (2.7) we can write X as a finite sum $X = X_{\mathfrak{z}} + \sum_{\alpha\in\Delta} X_\alpha$, and it suffices to take $W_{\{X\}} = \mathrm{Span}_k\{X_{\mathfrak{z}}\} \cup \{X_\alpha\}_{\alpha\in\mathfrak{h}^*}$. □

The main purpose of the next lemmas in this section is to show that V_R is an $\mathfrak{h}$-weight module.

LEMMA 3.5. *Let $H \in \mathfrak{h}$. Consider the action of H on V_R. Then we have*

$$V_R = \bigoplus_{s\in k} V_{R,s} \quad \textit{where} \quad V_{R,s} = \{v \in V_R \mid H \cdot v = sv\}. \tag{3.1}$$

Each $V_{R,s}$ is a free R-module. If $s \in k - \{0\}$, then $\mathrm{rk}_R(V_{R,s}) < \infty$.

PROOF. Let m be a positive integer such that $H \in \mathfrak{g}_m(R)$. It follows that $[H, \mathfrak{g}_m(R)] \subseteq \mathfrak{g}_m(R)$. Therefore Lemma 3.1 implies that there exists a k-basis of $\mathfrak{g}_m(R)$ which consists of eigenvectors of ad_H with eigenvalues in k.

From Proposition 3.1 it follows that there exists an element $D \in \mathfrak{G}_m$ such that $DHD^{-1} \in \mathfrak{t}_m \otimes_k 1 \subset \mathfrak{g}_m(R)$. The embedding $\mathfrak{G}_m \to \mathfrak{G}$ maps D to an element of $\mathfrak{G}$ which we denote by D as well. The action of DHD^{-1} on V_R is diagonal with respect to the R-basis $\mathcal{B}$, and its eigenvalues lie in k. Therefore we can decompose V_R into a direct sum

$$V_R = \bigoplus_{s\in k} V_{R,D,s}$$

of R-modules where

$$V_{R,D,s} = \{v \in V_R \mid \ DHD^{-1} \cdot v = sv\}.$$

There are only finitely many nonzero summands, and each nonzero $V_{R,D,s}$ is a free R-module, spanned by a subset of $\mathcal{B}$. Moreover, from $DHD^{-1} \in \mathfrak{g}_m(R)$ it follows that when $s \neq 0$, we have $\mathrm{rk}_R(V_{R,D,s}) < \infty$.

The map $\sigma_D : V_R \to V_R$ defined by $\sigma_D(v) = D\cdot v$ is an R-module automorphism of V_R. To obtain the decomposition (3.1), we set $V_{R,s} = \sigma_D^{-1}(V_{R,D,s})$. ☐

Remark 3.2. When $\mathfrak{g} = \mathfrak{sl}_\infty$, a similar argument shows that for every $H \in \mathfrak{h} \subset \mathfrak{g}(R)$, there exists a decomposition of V_R^* as a direct sum of R-modules

$$V_R^* = \bigoplus_{s\in k} V_{R,s}^*$$

where $V_{R,s}^* = \{v \in V_R^* \mid v \cdot H = -sv\}$. Each $V_{R,s}^*$ is a free R-module, and if $s \neq 0$ then $\mathrm{rk}_R(V_{R,s}^*) < \infty$.

Lemma 3.6. *There exists a nonzero element $v \in V_R$ and a $\lambda \in \mathfrak{h}^*$ such that for any $H \in \mathfrak{h}$, we have $H \cdot v = \lambda(H)v$.*

Proof. Fix $H \in \mathfrak{h}$ and choose $s \in k - \{0\}$ such that $V_{R,s} \neq \{0\}$. Since $\mathfrak{h}$ is commutative, we have $\mathfrak{h} \cdot V_{R,s} \subseteq V_{R,s}$. Therefore each element $X \in \mathfrak{h}$ gives rise to an R-linear map $T_X : V_{R,s} \to V_{R,s}$. The set $\mathcal{L} = \{T_X\}_{X\in\mathfrak{h}}$ forms a commutative Lie subalgebra of $\mathrm{End}_R(V_{R,s})$. Note that since $V_{R,s}$ is a free R-module of finite rank, we have $\mathrm{End}_R(V_{R,s}) \simeq \mathrm{M}_{d\times d}(R)$ for some d.

Lemma 3.1 implies that for any element $X \in \mathcal{L}$, there exists a k-basis of $V_{R,s}$ which consists of eigenvectors of T_X with eigenvalues in k. Using the standard embedding $\mathrm{End}_R(V_{R,s}) \to \mathrm{End}_K(V_{R,s} \otimes_R K)$ one can extend T_X to a K-linear map

$$\hat{T}_X : V_{R,s} \otimes_R K \to V_{R,s} \otimes_R K.$$

It follows that there exists a K-basis of $V_{R,s} \otimes_R K$ consisting of eigenvectors of $\hat{T}_X$, with eigenvalues in k. Therefore $\mathcal{L}$ is a commuting and diagonalizable family of elements in $\mathrm{End}_K(V_s \otimes_R K)$. Since $\mathrm{End}_K(V_{R,s} \otimes_R K) \simeq \mathrm{M}_{d\times d}(K)$, it is possible to diagonalize elements of $\mathcal{L}$ simultaneously. Therefore elements of $\mathcal{L}$ must have a common eigenvector in $V_{R,s} \otimes_R K$ with eigenvalues in k. After rescaling, we can assume that this common eigenvector is in fact in $V_{R,s}$. ☐

Lemma 3.7. *If $U(\mathfrak{h})$ denotes the universal enveloping algebra of $\mathfrak{h}$, then for every $v \in V_R$ we have $\dim_k(U(\mathfrak{h}) \cdot v) < \infty$.*

Proof. Set $W = \{v \in V_R \mid \dim_k(U(\mathfrak{h}) \cdot v) < \infty\}$. From Lemma 3.6 it follows that $W \neq \{0\}$. Next we show that W is invariant under the action of $\mathfrak{g}(R)$. Choose any $w \in W$. For any $X \in \mathfrak{g}(R)$ we have

$$U(\mathfrak{h})X \cdot w \subseteq XU(\mathfrak{h}) \cdot w + [U(\mathfrak{h}), X] \cdot w$$

and from Lemma 3.4 we know that $\dim_k([U(\mathfrak{h}), X]) < \infty$. It follows that $X\cdot w \in W$.

To complete the proof of the lemma, it suffices to show that $W = V_R$. We will give a proof of this fact only in the case $\mathfrak{g} = \mathfrak{sl}_\infty$, and leave the remaining cases to the reader. Recall that in this case $\mathcal{B} = \{e_1, e_2, e_3, \dots\}$. Let $v = r_1e_{p_1} + \cdots + r_le_{p_l}$ be a nonzero vector in W, where $p_1 < \cdots < p_l$ and $r_1, \dots, r_l \in R$. From $\mathfrak{g}(R)$-invariance of W it follows that $r_1e_1 = \mathrm{E}_{p_1+1,1}\mathrm{E}_{p_1,p_1+1} \cdot v \in W$. Next choose a k-basis $\{v_1, \dots, v_q\}$ for the finite-dimensional vector space $U(\mathfrak{h}) \cdot r_1e_1$. For any

$1 \leq j \leq q$, if $X_j \in U(\mathfrak{h})$ is such that $X_j \cdot r_1 e_1 = v_j$, then $r^{-1} \cdot v_j = X_j \cdot e_1 \in V_R$. Therefore $\{r^{-1} \cdot v_1, \ldots, r^{-1} \cdot v_q\}$ is a basis for $U(\mathfrak{h}) \cdot e_1$, which implies that $e_1 \in W$.

From $e_1 \in W$ it immediately follows that $re_j \in W$ for any $j \in \{1, 2, 3, \ldots\}$ and $r \in R$, i.e., $W = V_R$, which completes the proof of the lemma. □

LEMMA 3.8. *We can decompose V_R into a direct sum of its R-submodules $V_{R,\alpha}$, where $\alpha \in \mathfrak{h}^*$; i.e.,*

$$V_R = \bigoplus_{\alpha \in \mathfrak{h}^*} V_{R,\alpha} \tag{3.2}$$

where

$$V_{R,\alpha} = \{w \in V_R \mid \text{ for every } H \in \mathfrak{h},\ H \cdot w = \alpha(H)w\}. \tag{3.3}$$

If $\alpha \neq 0$ then $V_{R,\alpha}$ is a free R-module of finite rank.

PROOF. Pick any $v \in V_R$ and set $W_v = U(\mathfrak{h}) \cdot v$. We can (and will) consider H as an element of $\mathrm{End}_k(W_v)$. Recall the following facts.

(a) By Lemma 3.7 we have $\dim_k(W_v) < \infty$.
(b) From Lemma 3.1 it follows that for every element $H \in \mathfrak{h}$, there exists a k-basis of W_v consisting of eigenvectors of H, with eigenvalues in k.

Each element of $\mathfrak{h}$ gives rise to an element of $\mathrm{End}_k(W_v)$. These elements commute and are diagonalizable. Therefore they can be diagonalized simultaneously. It follows that every element of W_v (including v itself!) is a sum of $\mathfrak{h}$-weight vectors. This amounts to a direct sum decomposition

$$V_R = \bigoplus_{\alpha \in \mathfrak{h}^*} V_{R,\alpha}$$

where $V_{R,\alpha}$ is defined as in (3.3). Clearly every $V_{R,\alpha}$ is an R-module.

For any $\alpha \neq 0$, there exists an $H \in \mathfrak{h}$ such that $\alpha(H) \neq 0$. Consider the direct sum decomposition of V_R given by (3.1) with respect to H. We have $V_{R,\alpha} \subseteq V_{R,s}$ for some $s \neq 0$. In Lemma 3.5 it was proved that $V_{R,s}$ is a free R-module of finite rank. Therefore there exists a positive integer d such that $\mathrm{End}_R(V_{R,s}) \simeq \mathrm{M}_{d\times d}(R)$. Elements of $\mathfrak{h}$ give rise to a commuting set $\mathcal{L}$ of elements of $\mathrm{End}_R(V_{R,s})$ which satisfy the assumptions of Lemma 3.3. Part (b) of Lemma 3.3 implies that $\mathcal{L}$ is simultaneously diagonalizable by an element of $\mathrm{SL}_d(R)$. Hence there also exists an R-basis of $V_{R,s}$ whose elements are indeed $\mathfrak{h}$-weight vectors. In particular, for every $\alpha \in \mathfrak{h}^*$ such that $\alpha(H) = s$, $V_{R,\alpha}$ is an R-submodule of $V_{R,s}$ generated by a subset of this R-basis. Consequently, $V_{R,\alpha}$ is a free R-module of finite rank. □

REMARK 3.3. Again, when $\mathfrak{g} = \mathfrak{sl}_\infty$, a similar argument shows that there exists a decomposition

$$V_R^* = \bigoplus_{\alpha \in \mathfrak{h}^*} V_{R,\alpha}^*$$

with properties similar to those stated in Lemma 3.8, where

$$V_{R,\alpha}^* = \{v \in V_R^* \mid \text{for every } H \in \mathfrak{h},\ v \cdot H = -\alpha(H)v\}.$$

4. The trivial $\mathfrak{h}$-submodule in V_R

Let $V_{R,0}$ denote the summand in the decomposition (3.2) of Lemma 3.8 corresponding to the zero functional $0 \in \mathfrak{h}^*$. (When $\mathfrak{g} = \mathfrak{sl}_\infty$, $V^*_{R,0}$ is defined similarly.) From now on, when $\mathfrak{g} \neq \mathfrak{sl}_\infty$ we set $V^*_R = V_R$ and $V^*_{R,\alpha} = V_{R,\alpha}$. Also, for any $v \in V_{R,\alpha}$ and $w \in V^*_{R,\beta}$, we introduce an element $v \star w \in V_R \otimes_R V^*_R$ by

$$v \star w = \begin{cases} v \otimes w & \text{if } \mathfrak{g} = \mathfrak{sl}_\infty \\ v \otimes w + w \otimes v & \text{if } \mathfrak{g} = \mathfrak{sp}_\infty \\ v \otimes w - w \otimes v & \text{if } \mathfrak{g} = \mathfrak{so}_\infty \end{cases}$$

LEMMA 4.1. *Let $x \in V_{R,\alpha}$ and $y \in V^*_{R,\beta}$. For every $H \in \mathfrak{h}$ we have*

$$[H, \iota(x \star y)] = (\alpha(H) + \beta(H))\iota(x \star y).$$

PROOF. When $\mathfrak{g} = \mathfrak{sl}_\infty$, the lemma follows from

$$[H, \iota(x \otimes y)] = H\iota(x \otimes y) - \iota(x \otimes y)H = \alpha(H)xy + \beta(H)xy = (\alpha(H) + \beta(H))xy.$$

Next assume $\mathfrak{g} = \mathfrak{sp}_\infty$. (The argument for $\mathfrak{g} = \mathfrak{so}_\infty$ is similar.) Since $H \in \mathfrak{g}(R)$, it is a sum of elements of the form $vw^TS + wv^TS$ and therefore we have $H^TS = -SH$. Thus

$$\begin{aligned} [H, \iota(x \star y)] &= Hxy^TS + Hyx^TS - xy^TSH - yx^TSH \\ &= \alpha(H)xy^TS + \beta(H)yx^TS + xy^TH^TS + yx^TH^TS \\ &= \alpha(H)xy^TS + \beta(H)yx^TS + \beta(H)xy^TS + \alpha(H)yx^TS \\ &= (\alpha(H) + \beta(H))\iota(x \star y) \end{aligned}$$

□

LEMMA 4.2. *If $x \in V_{R,\alpha}$, $y \in V^*_{R,\beta}$, and $\alpha + \beta \neq 0$ then $\langle x, y\rangle = 0$.*

PROOF. When $\mathfrak{g} = \mathfrak{sl}_\infty$, the lemma easily follows from

$$\alpha(H)\langle x, y\rangle = \langle H \cdot x, y\rangle = \langle x, y \cdot H\rangle = -\beta(H)\langle x, y\rangle.$$

When $\mathfrak{g} \neq \mathfrak{sl}_\infty$, we have

$$\alpha(H)\langle x, y\rangle = \langle H \cdot x, y\rangle = -\langle x, H \cdot y\rangle = -\beta(H)\langle x, y\rangle$$

and the argument is similar. □

The next lemma follows readily from Lemma 4.2. We leave the details to the reader.

LEMMA 4.3. *When $\mathfrak{g} = \mathfrak{sl}_\infty$, we have*

$$\mathfrak{z}(\mathfrak{h}) = \mathfrak{sl}_\infty(R) \cap \bigoplus_{\alpha \in \mathfrak{h}^*} \iota(V_{R,\alpha} \otimes_R V^*_{R,-\alpha}).$$

When $\mathfrak{g} \neq \mathfrak{sl}_\infty$, we have $\mathfrak{z}(\mathfrak{h}) = \bigoplus_{\alpha \in \mathfrak{h}^*} V_{R,\alpha} \star V_{R,-\alpha}$ where

$$V_{R,\alpha} \star V_{R,-\alpha} = \{\iota(v \star w) \mid v \in V_{R,\alpha} \text{ and } w \in V^*_{R,-\alpha}\}.$$

LEMMA 4.4. *If $\alpha \in \mathfrak{h}^*$ satisfies $V_{R,\alpha} \neq \{0\}$, then there exist $x \in V_{R,\alpha}$ and $y \in V^*_{R,-\alpha}$ such that $\langle x, y\rangle = 1$.*

Proof. For simplicity we assume $\mathfrak{g} = \mathfrak{sl}_\infty$. For the remaining cases the argument is the same.

Since $\langle \cdot, \cdot \rangle$ is nondegenerate, from Lemma 4.2 it follows that $\langle V_{R,\alpha}, V^*_{R,-\alpha} \rangle \neq 0$. Fix $v \in V_{R,\alpha}$ such that $\langle v, V^*_{R,-\alpha} \rangle \neq 0$. We can express v in terms of elements of $\mathcal{B}$, i.e.,

$$v = c_1 e_{p_1} + \cdots + e_l e_{p_l} \quad \text{where} \quad c_1, \ldots, c_l \in R.$$

Since R is a principal ideal domain, the elements $c_1, \ldots, c_l$ have a greatest common divisor $d \in R$, and moreover there exist $c'_1, \ldots, c'_l$ such that $c_1 c'_1 + \cdots + c_l c'_l = d$. Now set $x = d^{-1} v$ and $w = c'_1 e^*_{p_1} + \cdots + c'_l e^*_{p_l}$. Clearly, we have $\langle x, w \rangle = 1$. Now by Lemma 3.8 we can write $w = \sum_{\beta \in \mathfrak{h}^*} w_\beta$, where for each β we have $w_\beta \in V^*_{R,\beta}$. Lemma 4.2 implies that $\langle x, w_{-\alpha} \rangle = 1$.

□

Proposition 4.1. *When $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\mathfrak{sp}_\infty$, we have $V_{R,0} = \{0\}$.*

Proof. First assume $\mathfrak{g} = \mathfrak{sl}_\infty$. Suppose $V_{R,0} \neq \{0\}$. Since $V_R \neq V_{R,0}$, we can choose $\alpha \neq 0$ such that $V_{R,\alpha} \neq \{0\}$, and (by Lemma 4.4) select elements $x \in V_{R,0}$, $y \in V^*_{R,0}$, $x_1 \in V_{R,\alpha}$ and $y_1 \in V^*_{R,-\alpha}$ such that $\langle x, y \rangle = \langle x_1, y_1 \rangle = 1$. Clearly, xy and $x_1 y_1$ are elements of $\mathfrak{gl}(V_R)$. Note that

$$xyxy = \langle x, y \rangle xy = xy \quad \text{and} \quad x_1 y_1 x_1 y_1 = \langle x_1, y_1 \rangle x_1 y_1 = x_1 y_1.$$

By Lemma 4.2, $\langle x_1, y \rangle = \langle x, y_1 \rangle = 0$. Therefore $xyx_1y_1 = x_1y_1xy = 0$, i.e., xy and x_1y_1 are commuting idempotents of $\mathfrak{gl}(V_R)$. Now $xy - x_1y_1 \in \mathfrak{sl}_\infty(R)$, and by Lemma 4.3, $xy - x_1y_1 \in \mathfrak{z}(\mathfrak{h})$.

Our next goal is to prove that in fact $xy - x_1y_1 \in \mathfrak{h}$. To this end, we first prove that there exists a k-basis $\mathcal{C}$ (respectively, $\mathcal{C}_1$) of $\mathfrak{sl}_\infty(R)$ consisting of eigenvectors of ad_{xy} (respectively, $\mathrm{ad}_{x_1y_1}$) with eigenvalues in k.

Set $T = xy$ and note that for some integer m, $T = xy$ is an element of $\mathfrak{gl}_m(R) \subset \mathfrak{gl}_m(K)$ with the property that $T^2 = T$. Therefore as an element of $\mathfrak{gl}_m(K)$, T is a diagonalizable matrix with eigenvalues in k; i.e., there exists a $P \in GL_m(K)$ such that PTP^{-1} is a diagonal matrix with entries in k. It follows that there exists a K-basis of $\mathfrak{sl}_\infty(K) = \mathfrak{sl}_\infty(R) \otimes_R K$ which consists of eigenvectors of ad_T with eigenvalues in k (for example, we can use elements of the form $P^{-1}\mathrm{E}_{i,j}P$ and $P^{-1}(\mathrm{E}_{i,i} - \mathrm{E}_{j,j})P$ where $i \neq j$). From this K-basis, and a k-basis of K, we can build a k-basis of $\mathfrak{sl}_\infty(K)$ which consists of eigenvectors of ad_T with eigenvalues in k. Since $\mathfrak{sl}_\infty(R)$ is a subspace invariant under ad_T, we can find a k-basis $\mathcal{C}$ of $\mathfrak{sl}_\infty(R)$ consisting of eigenvectors of ad_T with eigenvalues in k. The same proof applies to $\mathrm{ad}_{x_1y_1}$ and yields the required k-basis $\mathcal{C}_1$.

Since ad_{xy} and $\mathrm{ad}_{x_1y_1}$ commute, Lemma 3.2 implies that there exists a k-basis of $\mathfrak{sl}_\infty(R)$ consisting of eigenvectors of $\mathrm{ad}_{xy-x_1y_1}$ with eigenvalues in k. From $xy - x_1y_1 \in \mathfrak{z}(\mathfrak{h})$, the root space decomposition (2.7), Lemma 3.2, and maximality of $\mathfrak{h}$, it follows that $xy - x_1y_1 \in \mathfrak{h}$. Next we note that

$$(xy - x_1y_1)x = xyx - x_1y_1x = \langle x, y \rangle x - \langle x, y_1 \rangle x_1 = x - 0 = x$$

and at the same time since $x \in V_{R,0}$ and $xy - x_1y_1 \in \mathfrak{h}$, we should have

$$(xy - x_1y_1)x = 0.$$

This contradicts $x \neq 0$. The proof of the Lemma for $\mathfrak{g} = \mathfrak{sl}_\infty$ is complete.

Next assume $\mathfrak{g} = \mathfrak{sp}_\infty$. Let $\alpha \in \mathfrak{h}^*$ be such that $V_{R,\alpha} \neq \{0\}$. By Lemma 4.4 we can find $v \in V_{R,\alpha}$ and $w \in V^*_{R,-\alpha} = V_{R,-\alpha}$ such that $\langle v, w\rangle = 1$. Then

$$\iota(v \otimes w)\iota(v \otimes w) = vw^T Svw^T S = \langle w, v\rangle vw^T S = -\iota(v \otimes w)$$

and similarly $\iota(w \otimes v)\iota(w \otimes v) = \iota(w \otimes v)$. Set $A = vw^T S$ and $B = wv^T S$. Thus $A^2 = -A$ and $B^2 = B$, and as in the case $\mathfrak{g} = \mathfrak{sl}_\infty$ we conclude that there exists a k-basis $\mathcal{C}_A$ (respectively, $\mathcal{C}_B$) of $\mathfrak{gl}(V_R)$ consisting of eigenvectors of ad_A (respectively, ad_B) with eigenvalues in k. Moreover, $AB = vw^T Swv^T S = \langle w, w\rangle vv^T S = 0$. With a similar reasoning, one can see that $BA = 0$. Thus ad_A and ad_B are commuting and diagonalizable; hence Lemma 3.2 implies that there exists a k-basis of $\mathfrak{gl}(V_R)$ consisting of common eigenvectors of ad_A and ad_B, with eigenvalues in k. Obviously, elements of the latter basis are eigenvectors of ad_{A+B}, with eigenvalues in k. But $A + B \in \mathfrak{sp}_\infty(R)$, hence $[A + B, \mathfrak{sp}_\infty(R)] \subseteq \mathfrak{sp}_\infty(R)$. Lemma 3.1 implies that there exists a k-basis of $\mathfrak{sp}_\infty(R)$ consisting of eigenvectors of $A+B$. By Lemma 4.3, $A+B \in \mathfrak{z}(\mathfrak{h})$. From Lemma 3.2 and maximality of $\mathfrak{h}$, it follows that $A+B \in \mathfrak{h}$. However,

$$(A + B)v = vw^T Sv + wv^T Sv = \langle w, v\rangle v + \langle v, v\rangle w = -v$$

and since $v \in V_{R,\alpha}$, it follows that $\alpha(A + B) = -1$. This in particular shows that $\alpha \neq 0$, i.e., $V_{R,0} \neq \{0\}$ is impossible.

□

The proof of the case $\mathfrak{g} = \mathfrak{sp}_\infty$ in Proposition 4.1 can be readily adapted to a proof of the following lemma.

Lemma 4.5. *If $\mathfrak{g} = \mathfrak{so}_\infty$, then it is impossible to find vectors $v, w \in V_{R,0}$ satisfying*

$$\langle v, v\rangle = \langle w, w\rangle = 0 \quad \text{and} \quad \langle v, w\rangle = 1. \tag{4.1}$$

The next lemma is probably standard, but we would like to give a proof as we are unable to find a proper reference.

Lemma 4.6. *Let $\mathfrak{g} = \mathfrak{sp}_\infty$ or $\mathfrak{so}_\infty$. If $\lambda \in \mathfrak{h}^* - \{0\}$ then $\mathrm{rk}_R(V_{R,\lambda}) = \mathrm{rk}_R(V_{R,-\lambda})$ and there exist R-bases $\{a_1, \dots, a_r\}$ for $V_{R,\lambda}$ and $\{b_1, \dots, b_r\}$ for $V_{R,-\lambda}$ such that*

$$\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0 \quad \text{and} \quad \langle a_i, b_j\rangle = \delta_{i,j} \quad \text{for every } 1 \leq i, j \leq r.$$

Proof. By Lemma 4.4 we can choose $x \in V_{R,\lambda}$ and $y \in V_{R,-\lambda}$ such that $\langle x, y\rangle = 1$. Set $a_1 = x$ and $b_1 = y$.

Let $V_x = \{v \in V_{R,-\lambda} \mid \langle v, x\rangle = 0\}$ and $V_y = \{v \in V_{R,\lambda} \mid \langle v, y\rangle = 0\}$. For any $w \in V_{R,\lambda}$ we have $w = (w - \langle w, y\rangle x) + \langle w, y\rangle x$ which shows that $V_{R,\lambda} = Rx \oplus V_y$. Similarly, we can show that $V_{R,-\lambda} = Ry \oplus V_x$. The above argument can indeed be repeated for $V_x \oplus V_y$, and a simple induction on rank completes the proof of the lemma. □

Proposition 4.2. *If $\mathfrak{g} = \mathfrak{so}_\infty$, then $V_{0,R}$ is a free R-module of rank at most two.*

Proof. Since R is a principal ideal domain, from [**Lam**, §2E, Corollary (2.27)] it follows that $V_{R,0}$ is a free R-module. Suppose $\mathrm{rk}_R(V_{R,0}) > 2$. Our strategy is to find vectors $v, w \in V_{R,0}$ that satisfy (4.1), contradicting Lemma 4.5. To this end we need to use some results about quadratic forms over rings. For a comprehensive reference on this subject, see [**Knus**].

First assume $\mathrm{rk}_R(V_{R,0}) < \infty$. Choose an integer n large enough such that $V_{R,0} \subseteq W_{n,R}$ where $W_{n,R} = \mathrm{Span}_R\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. From decomposition (3.2) and Lemma 4.2 it follows that

$$W_{n,R} = V_{R,0} \oplus V_{R,0}^{\perp}$$

where $V_{R,0}^{\perp} = \{x \in W_{n,R} \mid \langle x, V_{R,0}\rangle = 0\}$. By [**Lam**, §2E, Corollary (2.27)] we know that $V_{R,0}^{\perp}$ is a free module as well. Now the discriminant of $(W_{n,R}, \langle \cdot, \cdot \rangle)$ is $1 \in R^{\times}/R^{\bullet\bullet}$ (where $R^{\bullet\bullet} = \{r^2 \mid r \in R^{\bullet}\}$) and is also equal (modulo $R^{\bullet\bullet}$) to the product of discriminants of $(V_{n,R}, \langle \cdot, \cdot \rangle)$ and $(V_{n,R}^{\perp}, \langle \cdot, \cdot \rangle)$. (See [**Knus**, (3.2.1), page 12] for a definition of the discriminant of a quadratic module.) Therefore the discriminant of $(V_{R,0}, \langle \cdot, \cdot \rangle)$ is also a unit in $R^{\times}/R^{\bullet\bullet}$, which means that $(V_{R,0}, \langle \cdot, \cdot \rangle)$ is indeed a quadratic space (see [**Knus**, (5.3.5), page 25]). Now from the main result of [**Par**] and the fact that k is algebraically closed the existence of v, w which satisfy (4.1) follows easily.

Next assume $\mathrm{rk}_R(V_{R,0}) = \infty$. The tensor product $V_{R,0} \otimes_R K$ is a vector space over K. By [**Ser**, §3, Exemple 3.3b] we know that K is a C_1 field, and therefore there exists a nonzero vector $v_1 \in V_{R,0} \otimes_R K$ such that $\langle v_1, v_1 \rangle = 0$. After rescaling, we can indeed assume that $v_1 \in V_{R,0}$. We can express v_1 in terms of elements of $\mathcal{B}$ as

$$v_1 = c_1 e_1 + \cdots + c_l e_l + c_1' f_1 + \cdots + c_l' f_l$$

where the coefficients belong to R. Without loss of generality, we assume that the greatest common divisor of these coefficients is 1. An argument similar to the proof of Lemma 4.4 shows that there exists a vector $w_1 \in V_{R,0}$ such that $\langle v_1, w_1 \rangle = 1$. Now consider the R-submodule M_{v_1,w_1} of $V_{R,0}$ spanned by v_1, w_1. It is free in generators v_1, w_1, and its discriminant is $1 \in R^{\times}/R^{\bullet\bullet}$, i.e., $(M_{v_1,w_1}, \langle \cdot, \cdot \rangle)$ is a quadratic space. Choose an integer $m' > 5$ large enough such that $v_1, w_1 \in W_{m',R}$ where

$$W_{m',R} = \mathrm{Span}_R\{e_1, \ldots, e_{m'}, f_1, \ldots, f_{m'}\}.$$

From [**Knus**, Lemma (5.4.1), page 26] it follows that M_{v_1,w_1} has an orthognal complement $M_{v,w}^{\perp}$ in $W_{m',R}$, i.e.,

$$W_{m',R} = M_{v_1,w_1} \oplus M_{v_1,w_1}^{\perp}.$$

Note that again from [**Lam**, §2E, Corollary (2.27)] it follows that $M_{v_1,w_1}^{\perp}$ is a free R-module. Moreover, we have $\mathrm{rk}_R(M_{v_1,w_1}^{\perp}) > 2$. The argument used to find $v_1, w_1 \in V_{R,0}$ can be modified slightly and applied to $M_{v_1,w_1}^{\perp}$, so that we can find vectors $v_2, w_2 \in M_{v_1,w_1}^{\perp}$ such that $\langle v_2, w_2 \rangle = 1$ and $\langle v_2, v_2 \rangle = 0$. Obviously, the discriminant of $(Rv_1 \oplus Rw_1 \oplus Rv_2 \oplus Rw_2, \langle \cdot, \cdot \rangle)$ is a unit in $R^{\times}/R^{\bullet\bullet}$, and therefore the latter quadratic module is indeed a quadratic space. Thus we can apply the main result of [**Par**] to this module, and a simple argument proves the existence of vectors v, w satisfying (4.1). □

5. Main theorem

THEOREM 5.1. *When $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\mathfrak{sp}_\infty$, any two maximal toral subalgebras of $\mathfrak{g}(R)$ are conjuagate under the action of $\mathfrak{G}$. When $\mathfrak{g} = \mathfrak{so}_\infty$, under the action of $\mathfrak{G}$ there are at most five conjugacy classes of maximal toral subalgebras.*

REMARK 5.1. It will be seen in the proof of Theorem 5.1 below that in the orthogonal case the number of conjugacy classes is related to the classification of quadratic spaces of small rank over the ring R.

PROOF. We explain the proof in detail in the cases $\mathfrak{g} = \mathfrak{sl}_\infty$ and $\mathfrak{sp}_\infty$, and then briefly mention the minor modifications required for the case $\mathfrak{g} = \mathfrak{so}_\infty$.

From Lemma 4.1 it follows that

$$V_R = \bigoplus_{\lambda \in \mathfrak{h}^* - \{0\}} V_{R,\lambda}$$

where each $V_{R,\lambda}$ is a free R-module of finite rank. Pick an R-basis $\mathcal{B}' = \{f'_1, f'_2, f'_3, ...\}$ of V_R which is compatible with this decomposition, i.e., it is a union of R-bases for $V_{R,\lambda}$'s and if $\mathfrak{g} = \mathfrak{sp}_\infty$, then it is chosen according to Lemma 4.6.

Recall that $\mathcal{B}$ denotes the standard R-basis of V_R. When $\mathfrak{g} = \mathfrak{sl}_\infty$ and $\mathcal{B} = \{e_1, e_2, e_3, \ldots\}$, let $T : V_R \to V_R$ be the R-linear transformation which maps e_i to f'_i. When $\mathfrak{g} = \mathfrak{sp}_\infty$ and $\mathcal{B} = \{e_1, e_2, e_3, \ldots, f_1, f_2, f_3, \ldots\}$, let $T : V_R \to V_R$ be the R-linear transformation which maps elements of $\mathcal{B}$ to elements of $\mathcal{B}'$ bijectively such that for every $v, w \in V_R$ we have $\langle v, w\rangle = \langle Tv, Tw\rangle$. (It suffices to partition $\mathcal{B}$ suitably into subsets of the form $\{e_{s_1}, \ldots, e_{s_l}, f_{s_1}, \ldots, f_{s_l}\}$ and map them to the chosen R-basis of $V_{R,\lambda} \oplus V_{R,-\lambda}$.) The matrix of T in the R-basis $\mathcal{B}$ of V_R has only finitely many nonzero entries in each column. Moreover, $T^{-1}\mathfrak{h}T$ consists of elements of $\mathrm{End}_R(V_R)$ such that $\mathcal{B}$ becomes an R-basis consisting of their common eigenvectors, with eigenvalues in k. However, a priori it seems to be possible to obtain elements in $T^{-1}\mathfrak{g}(R)T$ (and even in $T^{-1}\mathfrak{h}T$) which are not $\mathcal{B}$-finite. To show that this is not the case, it suffices to prove that T and T^{-1} are both $\mathcal{B}$-finitary.

From now on we assume $\mathfrak{g} = \mathfrak{sl}_\infty$. The proof for the case $\mathfrak{g} = \mathfrak{sp}_\infty$ is similar. We start with T^{-1}. Suppose $e_i = \sum a_{i,j} f'_j$. Without loss of generality we will assume $j = 1$ and prove that only finitely many $a_{i,1}$'s are nonzero. Suppose $f'_1 \in V_{R,\lambda}$. Since $\lambda \neq 0$, there exists an $H \in \mathfrak{h}$ such that $\lambda(H) \neq 0$. Therefore if $a_{i,1} \neq 0$ then $H \cdot e_i \neq 0$. But since H is $\mathcal{B}$-finite, the latter statement can only be true for finitely many e_i's.

Conversely, suppose $f'_i = \sum b_{i,j} e_j$, and we want to show that only finitely many of the $b_{i,1}$'s are nonzero. Set $X = \mathrm{E}_{1,1} - \mathrm{E}_{2,2}$. Then $X \cdot f'_i = b_{i,1}e_1 - b_{i,2}e_2$. So if $b_{i,1} \neq 0$ then $X \cdot f'_i \neq 0$. However, by (2.7) we know that

$$X = X_0 + \sum X_\alpha \tag{5.1}$$

where $X_0 \in \mathfrak{z}(\mathfrak{h})$ and $X_\alpha \in \mathfrak{g}(R)_\alpha$. To complete the proof, it suffices to show that if Y is any of the summands on the right hand side of the expression 5.1, then

$$Y \cdot f'_i \neq 0 \text{ happens for only finitely many } f'_i\text{'s.} \tag{5.2}$$

Our next goal is to prove the latter claim.

If $Y \in \mathfrak{z}(\mathfrak{h})$, then $Y \cdot f'_i \neq 0$ happens only for finitely many i. This is because Y is $\mathcal{B}$-finite, and hence $\mathrm{Im}(Y)$ lies inside the R-submodule of V_R generated by $e_1, ..., e_{n_0}$ for a fixed integer n_0. Since Y commutes with $\mathfrak{h}$, we should have $Y \cdot V_{R,\lambda} \subseteq V_{R,\lambda}$. Since for every λ we have $\mathrm{rk}_R(V_{R,\lambda}) < \infty$, it suffices to show that the restriction of Y to $V_{R,\lambda}$ can be nonzero only for finitely many λ's. Suppose, on the contrary, that one can find an infinite set

$$\Lambda = \{\lambda_1, \lambda_2, ...\} \subseteq \mathfrak{h}^*$$

such that for every $\lambda \in \Lambda$, there exists a vector $v_\lambda \in V_{R,\lambda}$ such that $Y \cdot v_\lambda \neq 0$. The vectors $Y \cdot v_{\lambda_p}$ belong to $\mathrm{Im}(Y)$ and are linearly independent over R, because they belong to distinct weight spaces of $\mathfrak{h}$. This contradicts the fact that $\mathrm{rk}_R(\mathrm{Im}(Y)) < \infty$, which completes the proof of (5.2) for the case $Y \in \mathfrak{z}(\mathfrak{h})$.

Next we prove that if $Y \in \mathfrak{g}(R)_\alpha$, then $Y \cdot f'_i \neq 0$ happens for only finitely many i. Note that $Y \cdot V_{R,\lambda} \subseteq V_{R,\lambda+\alpha}$. If there are infinitely many λ's such that $Y \cdot V_{R,\lambda} \neq \{0\}$, then linear independence (over R) of $V_{R,\lambda+\alpha}$'s contradicts the fact that $\mathrm{rk}_R(\mathrm{Im}(Y)) < \infty$.

Finally, when $\mathfrak{g} = \mathfrak{so}_\infty$, the proof is essentially the same as the the case $\mathfrak{g} = \mathfrak{sp}_\infty$. The only difference is that $V_{R,0}$ may be nonzero. In this case, by Proposition 4.2 we have $\mathrm{rk}_R(V_{R,0}) \leq 2$. Since $V_{R,0}$ has an orthogonal complement in V_R (see (3.2)), one can repeat an argument given in the proof of Proposition 4.2 to prove that $(V_{R,0}, \langle\cdot,\cdot\rangle)$ is indeed a quadratic space. The main result of [**Par**] shows that this quadatic space is diagonalizable, and there are five possibilities: $V_{R,0} = \{0\}$, $V_{R,0} \simeq R$ with diagonal quadratic forms of types $\{1\}$ and $\{t\}$, and $V_{R,0} \simeq R \oplus R$ with diagonal quadratic forms of types $\{1,1\}$ and $\{1,t\}$ (note that $\{t,t\}$ and $\{1,1\}$ are the same). In any case, an argument similar to the case $\mathfrak{g} = \mathfrak{sp}_\infty$ proves that maximal toral subalgebras with the same type of quadratic space on $V_{R,0}$ are conjugate. □

The definition of a maximal toral subalgebra given in section 2.6 applies mutatis mutandis to the universal central extension of $\mathfrak{g}(R)$. Using Theorem 5.1, general remarks about the universal central extension of $\mathfrak{g}(R)$ recorded in Section 6 below, and the argument given in [**Pi2**], one obtains the following corollary, whose proof is left to the reader.

COROLLARY 5.1. *Let $\hat{\mathfrak{g}}(R)$ denote the universal central extension of $\mathfrak{g}(R)$. If $\mathfrak{g} \neq \mathfrak{so}_\infty$, then any two maximal toral subalgebras of $\hat{\mathfrak{g}}(R)$ are conjugate under $\mathrm{Aut}_k(\hat{\mathfrak{g}}(R))$. If $\mathfrak{g} = \mathfrak{so}_\infty$, there are at most five conjugacy classes of maximal toral subalgebras in $\hat{\mathfrak{g}}(R)$.*

REMARK 5.2. A detailed description of conjugacy classes of maximal toral subalgebras and corresponding root systems in the orthogonal case will appear in another article.

6. Universal central extensions

In this section we record certain facts about universal central extensions of Lie algebras for which we are unable to find a reference in the literature. Recall that every perfect Lie algebra has a universal central extension (u.c.e.).

LEMMA 6.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be arbitrary perfect Lie algebras. Suppose $p : \hat{\mathfrak{a}} \to \mathfrak{a}$ is the u.c.e. of $\mathfrak{a}$ and $q : \tilde{\mathfrak{b}} \to \mathfrak{b}$ is an arbitrary central extension of $\mathfrak{b}$. If $i : \mathfrak{a} \to \mathfrak{b}$ is a Lie algebra monomorphism, then there exists a unique Lie algebra homomorphism $i' : \hat{\mathfrak{a}} \to \tilde{\mathfrak{b}}$ which makes the diagram*

$$\begin{tikzcd} \hat{\mathfrak{a}} \arrow[r, "i'"] \arrow[d, "p"'] & \tilde{\mathfrak{b}} \arrow[d, "q"] \\ \mathfrak{a} \arrow[r, "i"'] & \mathfrak{b} \end{tikzcd}$$

commutative.

PROOF. For any such map i', we should have $i'(\hat{\mathfrak{a}}) \subseteq q^{-1}(i(\mathfrak{a}))$. Since i is one to one,

$$q_{|q^{-1}(i(\mathfrak{a}))} : q^{-1}(i(\mathfrak{a})) \to i(\mathfrak{a})$$

is a central extension of $i(\mathfrak{a}) \simeq \mathfrak{a}$. Universality of $\hat{\mathfrak{a}}$ implies that there exists a unique map $\hat{\mathfrak{a}} \to q^{-1}(i(\mathfrak{a}))$ which makes the above diagram commutative. $\square$

Consider a direct system

$$\mathfrak{a}_1 \xrightarrow{i_1} \mathfrak{a}_2 \xrightarrow{i_2} \mathfrak{a}_3 \xrightarrow{i_3} \cdots$$

of perfect Lie algebras where all the i_r's are monomorphisms, and let $i_{p\to q} : \mathfrak{a}_p \to \mathfrak{a}_q$ be the map induced by the direct system. (Thus $i_{p\to p+1}$ is just i_p.) Suppose $\pi_i : \hat{\mathfrak{a}}_i \to \mathfrak{a}_i$ is the u.c.e. of $\mathfrak{a}_i$. For any positive integers $p < q$, Lemma 6.1 implies that there exists a unique map $\hat{i}_{p\to q} : \hat{\mathfrak{a}_p} \to \hat{\mathfrak{a}_q}$ making

$$\begin{array}{ccc} \hat{\mathfrak{a}}_p & \xrightarrow{\hat{i}_{p\to q}} & \hat{\mathfrak{a}}_q \\ {\scriptstyle \pi_p}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_q} \\ \mathfrak{a}_p & \xrightarrow[i_{p\to q}]{} & \mathfrak{a}_q \end{array}$$

commutative. From $i_{q\to r} \circ i_{p\to q} = i_{p\to r}$ and uniqueness of $\hat{i}_{p\to r}$ it follows that $\hat{i}_{q\to r} \circ \hat{i}_{p\to q} = \hat{i}_{p\to r}$. Therefore we have a direct system

$$\hat{\mathfrak{a}}_1 \xrightarrow{\hat{i}_1} \hat{\mathfrak{a}}_2 \xrightarrow{\hat{i}_2} \hat{\mathfrak{a}}_3 \xrightarrow{\hat{i}_3} \cdots$$

where $\hat{i}_r = \hat{i}_{r\to r+1}$. Set $\mathfrak{a} = \varinjlim \mathfrak{a}_n$ and $\hat{\mathfrak{a}} = \varinjlim \hat{\mathfrak{a}}_n$. The direct system provides a map $\pi : \hat{\mathfrak{a}} \to \mathfrak{a}$. Let $\mathfrak{z}_i = \ker(\pi_i)$. The above remarks lead to a proof of the following proposition.

PROPOSITION 6.1. *Assume $i_r(\mathfrak{z}_r) \subseteq \mathfrak{z}_{r+1}$ for every r. Then $\hat{\mathfrak{a}}$ is the u.c.e. of $\mathfrak{a}$.*

REMARK 6.1. Suppose $m > 1$ and $\mathfrak{a}_m = \mathfrak{g}_m(R)$, with the maps $\mathfrak{a}_m \to \mathfrak{a}_{m+1}$ as given in sections 2.3-2.5. Using Cassel's construction of the u.c.e. of $\mathfrak{g}_m(R)$, one can see that as a vector space $\hat{\mathfrak{a}}_m = \mathfrak{a}_m \oplus k$, and moreover $\mathfrak{a}_{m-1} \oplus k$ is a Lie subalgebra of $\hat{\mathfrak{a}}_m$ isomorphic to the u.c.e. of $\mathfrak{g}_{m-1}(R)$. At this point it is not difficult to see that Proposition 6.1 can be applied to this direct system.

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