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Isolatedness of the minimal representation and minimal decay of exceptional groups

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Abstract. Using a new definition of rank for representations of semisimple groups sharp results are proved for the decay of matrix coefficients of unitary representations of two types of non-split p-adic simple algebraic groups of exceptional type. These sharp bounds are achieved by minimal representations. It is also shown that in one of the cases considered, the minimal representation is isolated in the unitary dual.

1. Introduction

Let $\mathbf{G}$ be an absolutely simple affine algebraic group defined over a local field $\mathbb{F}$ of characteristic zero. We assume G is $\mathbb{F}$-isotropic. Let G be a topological finite central extension of $\mathbf{G}_{\mathbb{F}}$, the group of $\mathbb{F}$-points of $\mathbf{G}$. Fix a maximal compact subgroup K of G. Let π be a unitary representation of G. Throughout this article we will denote the Hilbert space on which π acts by $\mathcal{H}_\pi$. For any two vectors $v, w \in \mathcal{H}_\pi$, the matrix coefficient $f_{v,w}$ is the complex-valued function defined on G as

$$f_{v,w}(g) = (\pi(g)v, w)$$

where $(\cdot, \cdot)$ is the inner product of $\mathcal{H}_\pi$.

Definition 1. *A unitary representation π of G is said to be strongly L^p if and only if for any vector v in the (clearly dense) set of K-finite vectors in $\mathcal{H}_\pi$, we have*

$$f_{v,v} \in L^p(G, dg)$$

where dg is the Haar measure on G.

We say that π is strongly $L^{p+\varepsilon}$ if and only if it is strongly L^q for any $q > p$.

Remark. Definition 1 is stronger than its analogue in [Li]. In fact working with this definition requires more care than Li's original one does. Based on the results of [Oh] and their applications, the author prefers Definition 1.

Cowling [Co] showed that G has property T of Kazhdan if and only if there exists a real number $p < \infty$, only dependent up on G, such that any non-trivial

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irreducible unitary representation of G is strongly L^p. The infimum of all such numbers p is called the *minimal decay* of G [LZ], [Oh]. We denote this infimum by $p(G)$. The exact value of $p(G)$ is known for classical real Lie groups [Li]. For exceptional groups some estimates are obtained in [Oh],[LZ]. It is known that $p(G) = 6$ when G is of absolute type $\mathbf{G}_2$. Recently, Loke and Savin [LS] used a notion of N-rank developed by Weissman [Ws] for simply-laced groups to find $p(G)$ when G is a split exceptional simply-laced group.

We use Tits' notation in [Ti] to identify simple groups over p-adic fields. The main point of this paper is to prove the following statement:

Theorem 1. *$p({}^2\mathbf{E}^2_{6,4}) = 8$, $p(\mathbf{E}^9_{7,4}) = \frac{26}{3}$, and the minimal representation of $\mathbf{E}^9_{7,4}$ is isolated in its unitary dual endowed with the Fell topology.*

Remark. 1. We will see that the decay bounds are achieved by minimal representations of these groups.

2. Although the notion of N-rank used by Savin is defined in a way quite different from the one introduced by the author, much of the methods used by Savin and the author are similar, and based on Howe's original ideas [Hw1]. The main result of [LS] follows from the method of this paper as well. On the other hand, the result of this paper does not follow from the work of Loke and Savin since there is no parabolic with an abelian unipotent radical when the relative root system is of type $\mathbf{F}_4$. However, for the sake of brevity, we do not include the calculations for split groups here. Besides, we address some issues about the minimal representations which are not addressed in full generality in [GS1]. Therefore, this paper is essentially a complement to the papers [GS1] and [LS].

2. Rank of a unitary representation

In this section we simply recall the definition of the notion of rank introduced in [Sa] and the main result about purity of rank proved there.

Let G be as in section 1. Throughout this section we assume that G satisfies properties $(H0)$ through $(H3)$ of [GS1, Section 3]. Take $G_1 = G$ and let P_1 be the Heisenberg parabolic of G, as defined in [GS1],[Ws] ,[Sa]. Let $P_1 = L_1N_1$ be the Levi decomposition of P_1. N_1 is a Heisenberg group unless G_1 is of type $\mathbf{A}_1$.

Note that in terms of relative root systems, the center of N_1 corresponds to the highest root β of the relative root system of G, and N_1 includes all unipotent subgroups U_γ correpsonding to relative roots $\gamma \in S$ where

$$S = \{\gamma : (\gamma, \beta) > 0\} \tag{2.1}$$

with $(\cdot, \cdot)$ being the Killing form. There exists a unique simple relative root $\alpha \in S$.

Suppose L_1 is isotropic. Then the relative root system of L_1 is either simple or of the form $\mathbf{A}_1 \times \mathbf{R}$ or $\mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{R}$. Let $G_2 = [L_1, L_1]$ when $[L_1, L_1]$ is simple, and otherwise let G_2 be equal to the simple factor of $[L_1, L_1]$ which corresponds to the system $\mathbf{R}$, and let $P_2 = L_2N_2$ be the Heisenberg parabolic of G_2. Again if

L_2 is isotropic we can consider its simple factor G_3 and the Heisenberg parabolic $P_3 = L_3 N_3$ of G_3, and so on. The unipotent subgroup

$$N_G = N_1 \cdot N_2 \cdot N_3 \cdots \tag{2.2}$$

of G is in fact the unipotent radical of a parabolic subgroup of G. N_G is a tower of extensions by Heisenberg groups; however, the last group in the sequence can be an abelian group.

One can also construct a family of representations of G in a similar fashion. Any irreducible unitary representation of a Heisenberg group over a local field belongs to one of the following classes. The first class consists of characters, i.e. representations which act trivially on the center of the group. The second class consists of those representations which have a non-trivial central character. By the Stone-von Neumann theorem, the elements of this class are uniquely determined by their central characters. See [Hw2] or [Ty] for more details.

Now let $\mathfrak{H}$ be a Heisenberg group, and let χ be an arbitrary non-trivial character of the center of $\mathfrak{H}$. Denote the unitary representation with this central character by ρ_χ.

Definition 2. ([Sa, Section 4]) *Let N_G be as in (2.2) with height $r = r_{N_G}$; i.e.*

$$N_G = N_1 \cdot N_2 \cdots N_r.$$

Let ρ_{χ_i} be the extension of a representation of N_i with central character χ_i to N_G [1]*. Any representation of the form $\rho_{\chi_1} \otimes \rho_{\chi_2} \otimes \cdots \otimes \rho_{\chi_k}$ is called a rankable representation of N_G of rank k.*

The following proposition is a special, more polished form of the main result in [Sa, Section 5].

Proposition 1. *Let G, N_G be as in Definition 2. Suppose $r_{N_G} > 2$. Let π be a unitary representation of G without a G-fixed vector. Then the spectral measure for the direct integral decomposition of $\pi_{|N_G}$ is supported on rankable unitary representations. Moreover, if π is irreducible, then exactly one of the following cases can occur:*

i. The spectral measure is supported on rankable representations of rank one.
ii. The spectral measure is supported on rankable representations of rank strictly larger than one.

Therefore, the following definition of rank, given in [Sa, Definition 5.3.3], distinguishes the representations of rank one from other representations.

Definition 3. *Let G be as in section 1, N_G be as in (2.2), and π be an irreducible unitary representation of G. π is said to have rank k if and only if the spectral measure in the direct integral decomposition of $\pi_{|N_G}$ is supported on rankable representations of G of rank k.*

[1] ρ_{χ_i} is first extended to $N_{i+1} \cdots N_r$ using the Weil representation [We] and then trivially on the normal subgroup $N_1 \cdots N_{i-1}$.

3. Matrix coefficients

Let G be as in section 1 with maximal compact subgroup K. Let B be the minimal parabolic of G and A the split torus of G. Since $G = BK$, for any $g \in G$ there exists an element $b = b(g) \in B$ (unique modulo $B \cap K$) such that $b^{-1}g \in K$. Let δ_G be the modular function of B. The Harish-Chandra function Ξ, which is the spherical matrix coefficient of $\mathrm{Ind}_B^G(1)$, is equal to

$$\Xi(g) = \int_K \delta_G^{\frac{1}{2}}(b(kg))dk. \tag{3.1}$$

Note that $\delta_G(a)$ assumes values in $\mathbb{R}^+$, so it is identically equal to 1 on compact subgroups of B; i.e. $\delta_G(b(g))$ is well defined.

Definition 4. *Let π be a unitary representation of G. Let K be the maximal compact subgroup of G, and let Ψ be a positive integer-valued function whose domain is equal to the set $\hat{K}$ of the equivalence classes of finite-dimensional irreducible representations of K.*
We say π is $(\Xi^{\frac{1}{k}}, \Psi)$-bounded if and only if for any K-finite vectors $v, w \in \mathcal{H}_\pi$ which belong to K-isotypic spaces of $\nu_v, \nu_w \in \hat{K}$, we have

$$|f_{v,w}(g)| \leq \Xi^{\frac{1}{k}}(g)\Psi(\nu_v)\Psi(\nu_w).$$

Let $t > 1$. It is known (e.g. [Wal, Prop. 4.5.3] and [Si, Lemma 4.1.1]) that if

$$A_t = \{a \in A : |\alpha(a)| \geq t \text{ for any positive root } \alpha\}$$

then for any $a \in A_t$ we have

$$C\delta_G(a)^{\frac{1}{2}} \leq \Xi(a) \leq C_t\delta_G(a)^{\frac{1}{2}}. \tag{3.2}$$

Proposition 2. *Let G be as in section 1 and K be the maximal compact of G. Let Ξ be defined as in (3.1), and $\Psi(\nu)$ be an arbitrary function from the K-types to $\mathbb{Z}^+$.*

1. *The subset of $(\Xi^{\frac{1}{k}}, \Psi)$-bounded representations of G is closed in the unitary dual of G. Any such representation π is strongly $L^{2k+\varepsilon}$.*
2. *Let π be a unitary representation of G which is strongly $L^{p+\varepsilon}$. Let k be a positive integer such that $p \leq 2k$. Then π is $(\Xi^{\frac{1}{k}}, \dim(\nu))$-bounded.*

Proof. These are restatements of Howe's results in [Hw1] in general form. Use (3.2) for the first part. For the second part see [Hw1, Corollary 7.2]. □

Let $H_1 \times \cdots \times H_m \subset G$ be a product of semisimple groups embedded inside G, such that if A_i denotes the maximal split torus of H_i, then $A_1 \times \cdots \times A_m$ is equal to the maximal split torus A of G. Choose positive Weyl chambers A^+, A_1^+, ..., A_m^+ such that $A_1^+ \times \cdots \times A_m^+ \subseteq A^+$. Let δ_{H_i} be the modular function of the minimal parabolic subgroup of H_i. Obviously we can assume δ_{H_i} is also defined on all of A so that it is the modular function of the minimal parabolic of the reductive group $A \cdot H_i$.

Proposition 3. *Assume the setting introduced above. Let π be a unitary representation of G such that $\pi_{|H_i}$ is strongly $L^{p_i+\varepsilon}$. Let k_i's be positive integers such that $2k_i \geq p_i$. Suppose*

$$\text{for any } a \in A, \qquad \prod_{i=1}^{m} \delta_{H_i}(a)^{\frac{1}{2k_i}} \geq \delta_G(a)^{\frac{1}{p}}.$$

Then π is strongly $L^{p+\varepsilon}$.

Proof. This follows from [Li, Theorem 3.1] or [Oh, Prop. 2.7].

4. Minimal representations of archimedean groups

In this section and the next one we address the classification of rank one and minimal representations. The classification of such representations in the classical case is already known by the work of Howe, J.-S. Li and Vogan. Here we study exceptional groups. Therefore, in this section we assume G is as in section 2, but also of exceptional type.

The traditional definition of minimal representations is different for the archimedean and non-archimedean case. A notion of a weakly minimal representation is introduced in [GS1, Definition 3.6] in the non-archimedean case. In fact, that definition simply means that the representation is of rank one in the sense of Definition 3 above. Therefore it is quite natural to generalize it to the archimedean case as well. In fact, as seen below, this new definition of the minimal representation agrees with the traditional one which is in terms of the Joseph ideal.

Let G be as in section 2, and let π be a unitary representation of G without a nontrivial G-fixed vector. Suppose P is the Heisenberg parabolic of G. By Mackey theory [Ma], $\pi_{|P}$ is a direct integral of representations of the form

$$\mathrm{Ind}_{[P,P]}^{P}(\nu_\chi \otimes \rho_\chi) \tag{4.1}$$

where ν_χ is a representation of $[P, P]$ which factors through $[P, P]/N$, and ρ_χ is an extension of the representation of N with central character χ to $[P, P]$. Note that χ is supposed to be a nontrivial unitary character. Since the action of the split torus of G on the center of N has only finitely many orbits, $\pi_{|P}$ will be a finite direct sum of representations of the form given above.

We now recall the definition of the minimal representation when the local field $\mathbb{F}$ is archimedean. For simplicity let us assume $\mathbb{F} = \mathbb{R}$. The complex case can be treated similarly. Let $\mathfrak{g}_\circ$ be the Lie algebra of G, with complexification $\mathfrak{g} = \mathfrak{g}_\circ \otimes \mathbb{C}$. $\mathfrak{g}$ has a complex Heisenberg parabolic subalgebra $\mathfrak{p}$ with Levi decomposition

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{h}. \tag{4.2}$$

Note that $\mathfrak{h}$ will be the complexification of the Lie algebra of N where $P = LN$. Let $\mathfrak{m} = [\mathfrak{l}, \mathfrak{l}]$. Assume $\mathfrak{g}$ is not of type $\mathbf{A}_l$, and let $\mathcal{J}_\circ$ be the Joseph ideal of $U(\mathfrak{g})$. For the definition of the Joseph ideal, see [Jo] or [GS2].

Let $\mathcal{U}(\mathfrak{h})$ be the universal enveloping algebra of $\mathfrak{h}$. Define the map

$$\Theta : \mathfrak{m} \mapsto \mathcal{U}(\mathfrak{h})$$

as follows [2]. Let

$$\mathfrak{h} = W \oplus \mathfrak{z}$$

where W is a symplectic space and $\mathfrak{z}$ is the center of $\mathfrak{h}$. Denote the symplectic bilinear form of W by $\omega(\cdot,\cdot)$. We can identify $\mathfrak{sp}_n$ with the symplectic Lie algebra $\mathfrak{sp}(W)$. There is a canonical map

$$\mathfrak{sp}(W) \mapsto (W^* \otimes W^*)^{S_2}$$

where S_2 is the symmetric group on two elements (i.e. $S_2 = \{\pm 1\}$), which is defined by sending an element $A \in \mathfrak{sp}(W)$ to the bilinear form

$$\omega_A(x, y) = \omega(Ax, y).$$

Using ω, we can identify W^* with W by sending any y to the linear form $y \mapsto \omega(x, y)$. Thus we have a map

$$\mathfrak{sp}(W) \mapsto (W \otimes W)^{S_2}.$$

Since there is a natural map $(W \otimes W)^{S_2} \mapsto \mathcal{U}(\mathfrak{h})$, we obtain a composition map $A \mapsto n_A$ from $\mathfrak{sp}_n$ to $\mathcal{U}(\mathfrak{h})$. Now set

$$\Theta(X) = \frac{1}{2} n_X.$$

We have

$$[\Theta(X), \Theta(Y)] = Z\Theta([X, Y])$$

where Z is a fixed nonzero element of $\mathfrak{z}$.

By [Sa, Prop. 3.1.1], we can consider $\mathfrak{m}$ as a Lie subalgebra of a complex symplectic Lie algebra $\mathfrak{sp}_n$ which acts in the usual way on the Heisenberg algebra $\mathfrak{h}$ (see [Hw1, Section 1]). Thus we can define $\Theta(X)$ for any $X \in \mathfrak{m}$.

Lemma 1. *Let $\mathcal{J}$ be a primitive ideal of infinite codimension in $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{l}$ be as in (4.2) and $\mathfrak{m} = [\mathfrak{l}, \mathfrak{l}]$. Suppose $\mathcal{J}$ contains $ZX - \Theta(X)$ for all $X \in \mathfrak{m}$. Then $\mathcal{J}$ is the Joseph ideal.*

Proof. This follows immediately from [GS1, Proposition 4.3] and the uniqueness of the Joseph ideal [GS2]. □

Definition 5. *Let G be as above (i.e. $\mathbb{F} = \mathbb{R}$). An irreducible unitary representation π of G is called minimal iff the annihilator of the Harish-Chandra module associated to π is the Joseph ideal.*

[2] This neat description is borrowed from [GS1].

Proposition 4. *Let π be an irreducible unitary representation of G. π is a minimal representation if and only if π is rank one (in the sense of Definition 3).*

Proof. Let π be a representation of G of rank one. We will show that π is a minimal representation. Intuitively, the proof is as follows. Let $d\pi$ denote the infinitesimal $\mathfrak{g}$-action. By Mackey theory, the restriction of π to P is a finite direct sum of representations given in (4.1). But the representations ν_χ will be of rank zero, i.e. they are trivial $[P, P]$ modules. Consequently, for some m, we have

$$\pi_{|P} = \bigoplus_{i=1}^{m} n_i \operatorname{Ind}_{[P,P]}^{P} \rho_{\chi_i} \tag{4.3}$$

where $n_i \in \{0, 1, 2, ..., \infty\}$ for each i. Note that in fact $m = 1$ except for the Hermitian case, where $m = 2$. Let

$$\varpi_i = \operatorname{Ind}_{[P,P]}^{P} \rho_{\chi_i}. \tag{4.4}$$

By [Sa, Lemma 4.3.2] or [GV, Theorem 5.19] it follows that for any nontrivial unitary character ψ,

$$d\rho_\psi(ZX - \Theta(X)) = 0 \quad \text{for any} \quad X \in \mathfrak{m} \tag{4.5}$$

and therefore a similar equality should be true for each $d\varpi_i$ and also for $d\pi$; i.e.

$$ZX - \Theta(X) \in \operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi). \tag{4.6}$$

The result now follows from Lemma 1.

We now have to give a rigorous proof of (4.6). Let $\mathcal{H}_\pi$ be the Hilbert space of the representation π, and $\mathcal{H}_\pi^\infty$ be its subset of G-smooth vectors.

Let $\varpi \in \{\varpi_1, ..., \varpi_m\}$. We have the following lemma.

Lemma 2. *Let $\mathcal{H}_\varpi$ denote the Hilbert space of the representation ϖ, and let $\mathcal{H}_\varpi^\infty$ be the set of P-smooth vectors in $\mathcal{H}_\varpi$. Then*

$$d\varpi(ZX - \Theta(X))v = 0 \quad \textit{for any } X \in \mathfrak{m}, v \in \mathcal{H}_\varpi^\infty.$$

Proof. Recall that $\mathcal{H}_\varpi^\infty$ is a space of smooth left-quasi-$[P, P]$-invariant functions on P where P acts on it by the right regular representation. Let $X \in \mathfrak{m}$. Take any $f \in \mathcal{H}_\varpi^\infty$. Let e be the identity element of P, and denote the right action of $p \in P$ on $\mathcal{H}_\varpi$ by R_p. Thus $\varpi(p)$ is equal to R_p.

Let $M_X = ZX - \Theta(X)$. For any $p \in P$ we have

$$\begin{aligned}(d\varpi(M_X)f)(p) &= (d\varpi(Ad(p)(M_X))(R_p f))(e) \\ &= d\rho_\psi(Ad(p)(M_X))(f(p)) = d\rho_{p^{-1}\cdot\psi}(M_X)(f(p)).\end{aligned}$$

Here $p \cdot \psi(x) = \psi(p^{-1}xp)$ as usual. An application of (4.5) completes the proof. □

By Lemma 2, $d\varpi(ZX - \Theta(X))v = 0$ for any $v \in \mathcal{H}_\varpi^\infty$. Since $\pi_{|P}$ is a (finite or infinite) direct sum of representations unitarily equivalent to ϖ, the algebraic sum of the spaces of P-smooth vectors will be a norm-dense P-invariant subspace of the space $\overline{\mathcal{H}}_\pi^\infty$ of P-smooth vectors in $\mathcal{H}_\pi$. However, by a result of Poulsen [Po], this space is dense in the Fréchet topology of $\overline{\mathcal{H}}_\pi^\infty$, and therefore $d\pi(ZX - \Theta(X))v = 0$ for any $v \in \overline{\mathcal{H}}_\pi^\infty$. But $\mathcal{H}_\pi^\infty \subseteq \overline{\mathcal{H}}_\pi^\infty$, which proves that π is a minimal representation in the sense of Definition 5.

Now we address the converse. Let π be a minimal representation in the sense of Definition 5. We think of $\pi_{|P}$ as a representation of $\tilde{P}^\circ$, the universal cover of the connected component of identity of P. Again by Mackey theory, we can express $\pi_{|\tilde{P}^\circ}$ as

$$\pi_{|\tilde{P}^\circ} = \bigoplus_{i=1}^{m} \nu_{\chi_i} \otimes \mathrm{Ind}_{[\tilde{P}^\circ, \tilde{P}^\circ]}^{\tilde{P}^\circ} \rho_{\chi_i}$$

where now ν_{χ_i}'s are indeed representations of $\tilde{L}^\circ$ which are extended trivially on N to $\tilde{P}^\circ$. If π is minimal, then Lemma 2 implies that the actions of $d\nu_{\chi_i}$'s have to be zero; i.e. ν_{χ_i}'s have to be trivial modules. This proves that π is a rank one representation. □

5. Classification of rank one representations

Let G be as in section 2 and also assume $\mathbf{G}$ is of exceptional type. Let $P = LN$ be the Heisenberg parabolic of G. Let π be an irreducible representation of G of rank one. Then we have a decomposition as in (4.3) with ν_{χ_i}'s being trivial modules. Let s_α be the reflection inside the relative Weyl group of G which corresponds to the simple root $\alpha \in S$ (see (2.1)). By Mackey's subgroup theorem [Ma] one can see that

Lemma 3. *If ϖ_i and ϖ_j correspond to different orbits of the split torus on the center of N, then $\mathrm{Res}^P_{P\cap s_\alpha P s_\alpha^{-1}} \varpi_i$ and $\mathrm{Res}^P_{P\cap s_\alpha P s_\alpha^{-1}} \varpi_j$ are non-isomorphic irreducible representations of the group $P \cap s_\alpha P s_\alpha^{-1}$.*

Let π be an irreducible representation of G of rank one. From the irreducibility claimed in Lemma 3 it follows that if T is a bounded operator from the space of π to itself and it commutes with all $\pi(g)$ for $g \in P \cap s_\alpha P s_\alpha^{-1}$, then it commutes with all $\pi(g)$ for $g \in P$ as well. For a similar reason, T should commute with all $\pi(g)$ for $g \in s_\alpha P s_\alpha^{-1}$ as well. However, P and $s_\alpha P s_\alpha^{-1}$ generate G; therefore T should be an intertwining operator of the irreducible representation π, which means that T is a scalar. Consequently, we have proved that the only intertwining operators of $\pi_{|P\cap s_\alpha P s_\alpha^{-1}}$ are scalars. This means that $\pi_{|P}$ should be irreducible; i.e. $\pi_{|P} = \varpi$ for some $\varpi = \varpi_i$. Moreover, $\pi(s_\alpha)$ will be the intertwining operator between two irreducible representations of $P \cap s_\alpha P s_\alpha^{-1}$, and therefore $\pi(s_\alpha)$ will be determined uniquely. Therefore once we know the restriction of π to P, there is at most one way to extend it to a representation of G.

Proposition 5. *Let G be as above. Suppose the absolute root system of G is of exceptional type and G has split rank at least 3. Then there is a unique representation of rank one of G, unless G is a Hermitian form of E_7 (i.e. $\mathbb{F} = \mathbb{R}$), in which case there are exactly two such representations.*

Proof. From the discussion above, and the fact that representations ϖ_i correspond to distinct orbits of the center of N, the proposition follows from the number of orbits of the split torus on the center of N. Consider the smallest subspace of the Lie algebra of G which contains the one-dimensional space of the highest root and is invariant under the action of the rank-one subgroup corresponding to α. It is either the two-dimensional representation of $SL_2(\mathbb{F})$ or a 10-dimensional representation of $SO(9, 1)$ (in the real Hermitian case). In the former case, the split torus of $SL_2(\mathbb{F})$ has one orbit, so there can be at most one minimal representation in each case. The representations are constructed in [Tor]. In the latter case, there are two orbits. The two minimal representations are constructed in [Sah]. □

6. Decay of minimal representations

As we mentioned in the introduction, the method of this paper can be used to show that the minimal representation is isolated for all simply-laced exceptional groups as well as two non-split exceptional p-adic groups. However, the details for the case of split groups is not included here since the author noticed that it was done simultaneously in [LS].

The non-split p-adic groups under consideration will be the $\mathbb{F}$-points of simply connected algebraic groups of absolute types $\mathbf{E}_6$ and $\mathbf{E}_7$ which have Tits index ${}^2\mathbf{E}^2_{6,4}$ and $\mathbf{E}^9_{7,4}$. They are of relative type $\mathbf{F}_4$, and the dimension of short root spaces are 2 and 4 respectively. By Proposition 5 the minimal representation is unique in each case, and all other nontrivial irreducible unitary representations are of rank two or larger. The minimal representations have Iwahori-fixed vectors and the exponents of these representations are calculated in [GS1, Section 8]. By a standard result relating the matrix coefficients to the exponents (e.g. see [Cs, Section 4]), one can calculate the precise L^p decay of the minimal representations. Let the restricted Dynkin diagram of G be labelled by $\alpha_1, ..., \alpha_4$ such that the highest root β is

$$\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

Note that this is different from the labelling chosen by Lusztig and also used in [GS1]. Let us choose a minimal parabolic B of G, and therefore identify the positive Weyl chamber of the split torus A.

Let $\pi_{\min}$ denote the (unique) minimal representation of G, as constructed in [GS1]. We use the calculation of exponents of $\pi_{\min}$ given in [GS1, Section 8]. See [GS1, Section 6] for the definition of and basic results on exponents. The following statement follows from standard facts (see [Cs, Section 4] or [Hw3]).

◇ $\pi_{\min}$ is strongly $L^{p+\varepsilon}$ if and only if for every exponent μ of $\pi_{\min}$, the function $\delta_G^{\frac{1}{2}-\frac{1}{p}}(a)\mu(a)$ is bounded on the negative Weyl chamber.

We treat each case separately below.

- ${}^2\mathbf{E}_{6,4}^2$:

  By [GS1, Section 8.4], the logarithms of absolute values of exponents of $\pi_{\min}$ are

$$\begin{cases} \log|\overline{\chi_4}| = -8\alpha_1 - 15\alpha_2 - 22\alpha_3 - 12\alpha_4 \\ \log|\chi_4| = -8\alpha_1 - 15\alpha_2 - 22\alpha_3 - 12\alpha_4 \\ \log|\chi_2| = -7\alpha_1 - 15\alpha_2 - 22\alpha_3 - 12\alpha_4 \end{cases} \tag{6.1}$$

  Moreover,

$$\log\delta_G = 22\alpha_1 + 42\alpha_2 + 60\alpha_3 + 32\alpha_4.$$

  $\pi_{\min}$ is strongly $L^{p+\varepsilon}$ if and only if for any χ in 6.1, the coefficients of α_i's in

$$\chi + (\frac{1}{2} - \frac{1}{p})\log\delta_G$$

  are nonnegative. This is equivalent to $p \geq 8$.
- $\mathbf{E}_{7,4}^9$:

  The logarithms of absolute values of exponents are equal to

$$\begin{cases} \log|\chi_4| = -13\alpha_1 - 24\alpha_2 - 36\alpha_3 - 20\alpha_4 \\ \log|\chi_2| = -11\alpha_1 - 24\alpha_2 - 36\alpha_3 - 20\alpha_4 \end{cases} \tag{6.2}$$

  Moreover,

$$\log\delta_G = 34\alpha_1 + 66\alpha_2 + 96\alpha_3 + 52\alpha_4.$$

  Just as before, it follows that $\pi_{\min}$ is strongly $L^{p+\varepsilon}$ if and only if $p \geq \frac{26}{3}$.

7. Minimal decay and isolatedness of $\pi_{\min}$

In this section we complete the proof of Theorem 1. Let G be one of the two groups introduced in the previous section. Let H_i, $i \in \{1, 2\}$ be defined as follows.

$\diamond H_1$ is the rank one subgroup corresponding to the highest root β.
$\diamond H_2$ is the subgroup corresponding to all roots perpendicular to the highest root (i.e. its relative diagram is the subdiagram $\mathbf{C}_3$ of $\mathbf{F}_4$).

Lemma 4. *Let G, H_i $(i \in \{1, 2\})$ be as above. Let π be an irreducible unitary representation of G which is not the trivial or the minimal representation. Then $\pi_{|H_1}$ is strongly $L^{2+\varepsilon}$ and $\pi_{|H_2}$ is strongly $L^{4+\varepsilon}$.*

Proof. For $i = 1$, this follows from [Oh, Theorem 4.1]. We prove the lemma for $i = 2$.

As before, let $P = LN$ be the Heisenberg parabolic of G. Using Mackey theory as in (4.1), $\pi_{|[P,P]}$ can be expressed as a direct integral of representations of the form

$$\mathrm{Res}^{P}_{[P,P]}\mathrm{Ind}^{P}_{[P,P]}(\nu \otimes \rho).$$

Let $M = [L, L]$. Since π is not minimal, it follows from Proposition 5 that π has to be of rank at least two.Therefore $\pi_{|M}$ is a direct integral of representations of the form

$$\nu_\chi \otimes \rho'_\chi \tag{7.1}$$

where ν_χ is a representation of M of rank at least one, i.e. it does not have a nonzero M-fixed vector, and ρ'_χ is given as

$$\rho'_\chi = \mathrm{Res}^{[P,P]}_{M} \rho_\chi$$

where ρ_χ is the extension of the representation of N with central character χ to M using the construction of the oscillator representation [We]. Although the central character χ can be arbitrary, there will be only finitely many representations ρ'_χ (at most equal to the number of elements of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$). Therefore $\pi_{|M}$ is a finite direct sum of representations of the form given in (7.1), and moreover since π is of rank at least two, none of the ν_χ's can have trivial subrepresentations.

Suppose ν_χ is strongly $L^{p+\varepsilon}$ and ρ'_χ is strongly $L^{q+\varepsilon}$. Then by an application of Hölder's inequality, one can see that $\nu_\chi \otimes \rho'_\chi$ is strongly $L^{r+\varepsilon}$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and if this is true for all the direct summands of π then $\pi_{|M}$ will be strongly $L^{r+\varepsilon}$ as well. Therefore what remains to be done is to find suitable values of p and q such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{4}$. To this end we use [LZ, Prop. 2.2] followed by [Oh, Lemma 7.3].

Let $\Phi(a)$ denote the function introduced in [LZ, Prop. 2.2]; i.e. Φ is an upper bound on the matix coefficients of the set of K-finite vectors in the restriction of the oscillator representation to M.

Let $\alpha_1, ..., \alpha_4$ be simple roots of the Dynkin diagram of G numbered such that the highest root is

$$\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

Let $a \in A_{H_2}$ and let $y_i = |e^{\alpha_i}(a)|$. Since $|e^{\beta}(a)| = 1$, we have

$$|e^{\alpha_1}(a)| = y_2^{-\frac{3}{2}} y_3^{-2} y_4^{-1}.$$

Therefore a simple calculation shows that

$$\Phi(a) \leq y_2^{\frac{-3r-5}{4}} y_3^{-r-2} y_4^{\frac{-r-3}{2}}$$

where r is the dimension of a short root space.

Note that by [Oh, lemma 7.3], we have:

◇ ρ'_χ is strongly $L^{p+\varepsilon}$ if and only if the exponent of any y_i in $\Phi^p(a)\delta_{H_2}(a)$ is nonpositive.

On the other hand,

$$\delta_{H_2}(a) = y_2^{3r+3} y_3^{6r+4} y_4^{4r+2}.$$

Therefore

$$\delta_{H_2}(a) = \begin{cases} y_2^9 y_3^{16} y_4^{10} & \text{if } G \text{ is } {}^2\mathbf{E}_{6,4}^2 \\ \\ y_2^{15} y_3^{28} y_4^{18} & \text{if } G \text{ is } \mathbf{E}_{7,4}^9 \end{cases} \tag{7.2}$$

and consequently,

$$q = \begin{cases} 4 & \text{if } G \ \text{ is } {}^2\mathbf{E}_{6,4}^2 \\ \frac{36}{7} & \text{if } G \ \text{ is } \mathbf{E}_{7,4}^9 \end{cases}$$

For $G = {}^2\mathbf{E}_{6,4}^2$ the lemma is already proved, since one can definitely take $p < \infty$ (because H_2 has property T). For $\mathbf{E}_{7,4}^9$, it follows from [Oh, Theorem 7.4] that $p \leq 18$. Note that although Oh's result is stated for irreducible unitary representations, it is easily seen to hold for *any* unitary representation without a nonzero fixed vector. It is not hard to check that $\frac{7}{36} + \frac{1}{18} = \frac{1}{4}$! This proves the lemma for $i = 2$. □

We will prove Theorem 1 now. We use Proposition 3.

Let $a \in A_G$ and let $y_i = |e^{\alpha_i}(a)|$.

• ${}^2\mathbf{E}_{6,4}^2$: We have

$$\delta_G(a) = y_1^{22} y_2^{42} y_3^{60} y_4^{32} \text{ and } \delta_{H_1}^{\frac{1}{2}}(a)\delta_{H_2}^{\frac{1}{4}}(a) = y_1 y_2^{\frac{15}{4}} y_3^6 y_4^{\frac{7}{2}}.$$

Consider the element $w = s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1}$ of the Weyl group of the relative root system of G. Let H_i^w ($i \in \{1, 2\}$) denote the conjugate of H_i by w, i.e. its root system consists of all roots $w \cdot \gamma$ where γ is in the root system of H_i. Then

$$\delta_{H_1^w}^{\frac{1}{2}}(a)\delta_{H_2^w}^{\frac{1}{4}}(a) = y_1^{\frac{11}{4}} y_2^{\frac{21}{4}} y_3^{\frac{15}{2}} y_4^4.$$

Proposition 3 implies that any nontrivial and non-minimal irreducible representation of G is strongly $L^{8+\varepsilon}$; i.e. $p(G) \leq 8$. By the result of section 6 we should have $p(G) = 8$.

• $\mathbf{E}_{7,4}^9$: We have

$$\delta_G(a) = y_1^{34} y_2^{66} y_3^{96} y_4^{52} \quad \text{and} \quad \delta_{H_1}^{\frac{1}{2}} \delta_{H_2}^{\frac{1}{4}}(a) \ = y_1 y_2^{\frac{21}{4}} y_3^9 y_4^{\frac{11}{2}}.$$

Again we conjugate the groups by $w = s_{\alpha_4} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1}$ and we get

$$\delta_{H_1^w}^{\frac{1}{2}}(a)\delta_{H_2^w}^{\frac{1}{4}}(a) = y_1^{\frac{9}{2}} y_2^{\frac{35}{4}} y_3^{\frac{25}{2}} y_4^7$$

which gives $r \leq \frac{192}{25} < 8$. Therefore, any non-minimal irreducible representation of G is strongly $L^{8+\varepsilon}$, and therefore $(\Xi^{\frac{1}{4}}, \dim(\nu))$-bounded.

However, by Proposition 2, the minimal representation of G can not be $(\Xi^{\frac{1}{4}}, \dim(\nu))$-bounded since it is not strongly $L^{8+\varepsilon}$. Therefore the complement of the set {trivial, $\pi_{\min}$} is a closed set in the unitary dual of G; i.e. $\pi_{\min}$ is isolated as well. Moreover, by section 6, $p(G) = \frac{26}{3}$.

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