University of Ottawa

Department of Mathematics and Statistics

MAT 2355 : Introduction to Geometry

Professor: Hadi Salmasian

Final Exam

December 10, 2012

Surname _____ First Name _____

Student # _____

Instructions:

(a) You have 3 hours to complete this exam.

(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(c) Write your student number at the top of each page in the space provided.

(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(e) You are strongly recommended to write in **pen**, not pencil.

(f) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1–3	4	5–6	7–8	9	10	11	12	13	14	Total
Maximum	6	5	7	3	1	3	4	2	3	3	37
Grade											

1. **[2 points]** Write the statement of Euclid's parallel postulate.

Solution: For every line ℓ in $\mathbb{E}^2$ and a point P not on ℓ, we can draw a unique line parallel to ℓ through P.

2. **[1 point]** Write the statement of the Triangle Inequality for two vectors in $\mathbb{R}^n$.

Solution: For every $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

3. **[3 points]** For each of the following statements, determine if the statement is true or false.

Circle **T** for the true statements and **F** for the false statements. For each correct answer you will receive .5 points. For each incorrect answer you will receive -.5 points. If you do not circle any of the letters next to a statement, you will receive 0 points for that statement.

T **F** The only isometries of $\mathbb{E}^2$ are translations, rotations, and reflections.

T **F** If two triangles $\triangle ABC$ and $\triangle A'B'C'$ in $\mathbb{E}^2$ satisfy $\angle A = \angle A'$ and $\angle B = \angle B'$ then $\triangle ABC$ and $\triangle A'B'C'$ are similar triangles.

T **F** Every conformal map $f : \mathbb{E}^2 \to \mathbb{E}^2$ is an isometry.

T **F** In a parallelogram $ABCD$, the intersection point M of the diagonals AC and BD is the midpoint of both AC and BD.

T **F** The Mercator projection is conformal.

T **F** Every two distinct great circles of a sphere intersect in two antipodal points.

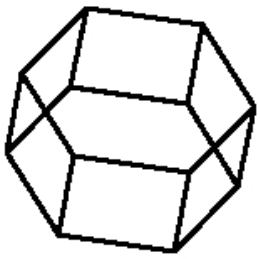
Solution: F,T,F,T,T,T

4. (a) **[2 points]** Write the definition of a Platonic solid.

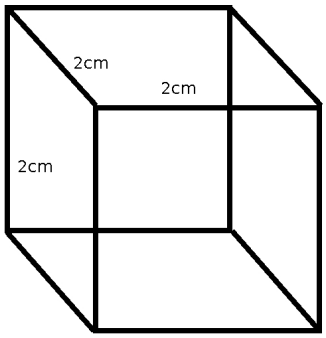
Solution: A convex polyhedron is called a Platonic solid if it satisfies the following two conditions:

- All faces of P are congruent regular polygons.
- The same number of faces meet at each vertex.

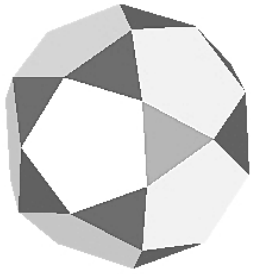
(b) **[3 points]** Based on your definition, determine if each of the following polyhedra is a Platonic solid or not. In each case, if you believe that the polyhedron is not a Platonic solid, then you should justify your answer.



Solution: It is not because the faces are not congruent.



Solution: It is a Platonic solid.



Solution: It is not because its faces are not congruent.

5. **[5 points]** Complete the following sentences:

(a) The Cartan-Dieudonné Theorem states that every isometry of $\mathbb{E}^2$ can be expressed as . . . . . .

(b) By the Isoperimetric Inequality, a closed non-intersecting curve of length L in the

plane encloses an area of . . . . . and equality occurs when the curve is . . . . . .

(c) The sum of angles of a heptagon is equal to . . . . . .

A *heptagon* is a convex polygon with seven sides.

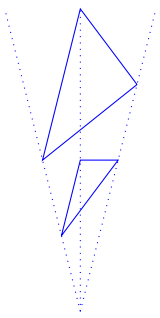
(d) By Euler's Theorem, any convex polyhedron which has 11 vertices and 15 edges will have . . . . . faces.

(e) In the upper half-plane model of hyperbolic geometry, the geodesics are . . . . . .

Solution: (a) a composition of at most three reflections.
(b) $\frac{L^2}{4\pi}$ – a circle.
(c) 5π.
(d) 6.
(e) half-lines perpendicular to the x-axis and semicircles with centre on the x-axis.

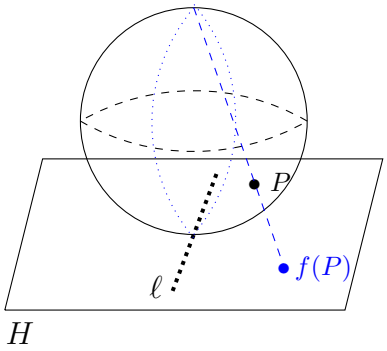
6. [**2 points**] Write the meaning of the statement "two triangles $\triangle ABC$ and $\triangle A'B'C'$ are in perspective". Then draw a picture of two triangles in perspective.

Solution:
Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are in perspective if and only if the lines AA', BB', and CC' are concurrent.



7. (a) [**1 point**] Complete the picture below by finding the stereographic projection of the given point P on the plane H. (The point P lies on the sphere.)

(b) [**1 point**] Describe (both in words and in the picture) the set of points on the sphere which are mapped by the stereographic projection to the given line ℓ on the plane H. (The line ℓ is tangent to the sphere at the south pole.)



Solution: See picture for part (a). For part (b) the answer is a great circle passing through the north and south poles.

8. [**1 point**] Compute the product $(\mathbf{i}-\mathbf{j})(\mathbf{j}+2\mathbf{k})$ where $\mathbf{i}-\mathbf{j}$ and $\mathbf{j}+2\mathbf{k}$ are quaternions.

Solution: $1-2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

9. [**1 point**] Using complex numbers write a formula for the counterclockwise 90-degree rotation in $\mathbb{E}^2$ with centre $(1, \frac{1}{2})$. Your answer should be in the form $R(z) = az + b$ for $a, b \in \mathbb{C}$.

Solution: The centre of the rotation is $z_\circ = 1 + \frac{1}{2}i$. By the formula in the notes we can write

$$\begin{aligned} R(z) &= e^{i\frac{\pi}{2}}z + (1 + \frac{1}{2}i) - e^{\frac{\pi}{2}}(1 + \frac{1}{2}i) \\ &= iz + (1 + \frac{1}{2}i) - i(1 + \frac{1}{2}i) = iz - \frac{i}{2} + \frac{3}{2}. \end{aligned}$$

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10. **[3 points]** Let $A = [(1,1,1)]$, $B = [(1,2,0)]$, and $C = [(0,t,1)]$ be three points of the projective space $\mathbb{P}^2$. Determine all values of $t \in \mathbb{R}$ such that A, B, and C lie on a projective line.

Solution: We need to determine all values of t such that the radial lines corresponding to A, B, C lie on a radial plane. The equation of such a plane is of the form $ax+by+cz = 0$ where at least one of a, b, c is nonzero. To determine if such a, b, c exist, we should solve the following linear system to see if there are nonzero solutions:

$$\begin{cases} 1 \cdot a + 1 \cdot b + 1 \cdot c = 0 \\ 1 \cdot a + 2 \cdot b + 0 \cdot c = 0 \\ 0 \cdot a + tb + 1 \cdot c = 0 \end{cases}$$

The augmented matrix of the system can be row reduced as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & t & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & t & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - tR_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1+t & 0 \end{array}\right]$$

The matrix is in echelon form. Since the first and the second columns are pivot columns, the variables a and b are basic (i.e., dependent). The system has nontrivial solutions if and only if there exists at least one free variable (i.e., independent variable), that is, if c is free, or equivalently, if the third column is not a pivot column (which happens for $t = -1$). Therefore the points A, B, C lie on a projective line if and only if $t = -1$.

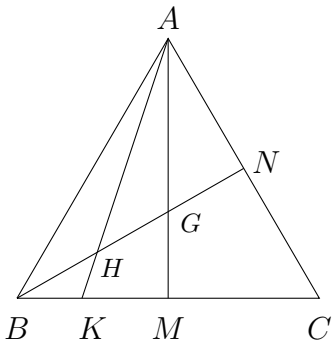
11. (a) **[1 point]** Let A, B, C, D be collinear points. Write the definition of the cross-ratio $[A, B, C, D]$.

Solution:

$$[A, B, C, D] = \frac{AC \times BD}{AD \times BC}$$

(b) **[3 points]** Consider an equilateral triangle $\triangle ABC$. Let M be the midpoint of BC, and N be the midpoint of AC. Let G be the intersection of AM and BN, and H be the midpoint of BG. Finally, let K be the intersection of AH and BC. (See the picture.) Compute the ratio $\frac{BK}{BC}$.

Hint: You can use the fact that $\frac{BG}{GN} = 2$, which you proved in the first assignment.



Solution:
Let $a = BC$. Then:

$$[B, K, M, C] = \frac{BM \cdot KC}{BC \cdot KM} = \frac{\frac{a}{2}(KM + \frac{a}{2})}{a \cdot KM} = \frac{1}{2}\left(1 + \frac{a}{2KM}\right).$$

By considering central projection with respect to A, we have:

$$[B, K, M, C] = [B, H, G, N] = \frac{BG \cdot HN}{BN \cdot HG} = \frac{4}{3}.$$

It follows that

$$\frac{1}{2}\left(1 + \frac{a}{2KM}\right) = \frac{4}{3} \;\Rightarrow\; KM = \frac{3}{10}a \;\Rightarrow\; BK = \frac{a}{2} - \frac{3a}{10} = \frac{a}{5} \;\Rightarrow\; \frac{BK}{BC} = \frac{1}{5}.$$

12. [**2 points**] Recall that $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$. By definition, the *first octant* on S^2 is the set

$$\mathcal{P} = \{(x, y, z) \in S^2 \ : \ x > 0 \text{ and } y > 0 \text{ and } z > 0\}.$$

Let $\triangle ABC$ be a spherical triangle on S^2 whose vertices lie in the first octant, that is, $A, B, C \in \mathcal{P}$. Prove that

$$\angle A + \angle B + \angle C < \frac{3}{2}\pi.$$

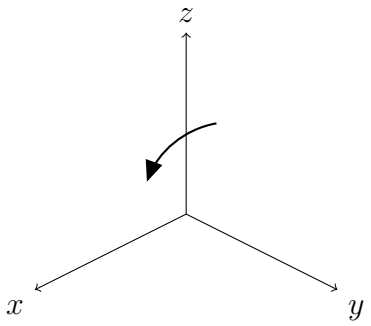
Solution: The area of $\triangle ABC$ is strictly smaller than the area of $\mathcal{P}$. We have $\text{area}(\mathcal{P}) = \frac{4\pi}{8} = \frac{\pi}{2}$. Therefore

$$\angle A + \angle B + \angle C < \text{area}(\triangle ABC) + \pi = \frac{\pi}{2} + \pi = \frac{3}{2}\pi.$$

13. **[3 points]** Let $R : \mathbb{E}^3 \to \mathbb{E}^3$ be the counterclockwise rotation about the y-axis with angle $\frac{\pi}{4}$ (see the picture below). Find a 3×3 matrix A such that $R(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for every $\vec{\mathbf{x}} \in \mathbb{E}^3$.

You might need one or more of the following:

θ	$\cos\theta$
0	1
$\pm\frac{\pi}{3}$	$\frac{1}{2}$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\pm\frac{\pi}{2}$	0



Solution: The rotation does not change the y-coordinate of points, and in the xz-plane it is the clockwise rotation with angle $\frac{\pi}{4}$. The matrix of the latter rotation is

$$\begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and therefore

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

14. **[3 points]** Consider the unit circle S^1 in the xy-plane parametrized as

$$S^1 = \{\, (\cos t, \sin t, 0) \in \mathbb{R}^3 \ : \ 0 \le t \le 2\pi \,\}.$$

Find the parametric description of the reflection of S^1 by the plane $x + 2y - z = 1$. Your solution should be a curve of the form

$$\gamma(t) = (x(t), y(t), z(t))$$

where $x(t), y(t)$, and $z(t)$ are functions of t.

Solution: The given plane can be expressed as $H = \{\vec{\mathbf{v}} \in \mathbb{R}^3 \ : \ \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = b\}$ where

$$\vec{\mathbf{a}} = (1, 2, -1) \text{ and } b = 1.$$

An arbitrary point on the circle S^1 is $\vec{\mathbf{v}} = (\cos(t), \sin(t), 0)$. Its projection on the plane H is equal to

$$(\cos(t), \sin(t), 0) + 2\left(\frac{1 - \cos(t) - 2\sin(t)}{6}\right)(1, 2, -1)$$

$$= (\cos(t), \sin(t), 0) + \left(\frac{1}{3}(1 - \cos(t) - 2\sin(t)), \frac{2}{3}(1 - \cos(t) - 2\sin(t)), -\frac{1}{3}(1 - \cos(t) - 2\sin(t))\right)$$

$$= \left(\frac{1}{3} + \frac{2}{3}\cos(t) - \frac{2}{3}\sin(t), \frac{2}{3} - \frac{1}{3}\sin(t) - \frac{2}{3}\cos(t), -\frac{1}{3} + \frac{2}{3}\sin(t) + \frac{1}{3}\cos(t)\right)$$

Therefore

$$\begin{cases} x(t) = \frac{1}{3} + \frac{2}{3}\cos(t) - \frac{2}{3}\sin(t) \\ y(t) = \frac{2}{3} - \frac{2}{3}\cos(t) - \frac{1}{3}\sin(t) \\ z(t) = -\frac{1}{3} + \frac{1}{3}\cos(t) + \frac{2}{3}\sin(t) \end{cases}$$