Mat 2355 Final Exam.

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Please write your name and student number **on each booklet**. Could you also please list on the cover of the first booklet the questions in the order you answered them.

You must give a complete solution with details. Calculators are not allowed.

1. (10) Let $ABCD$ be a parallelogram and P a point on the segment $\overline{AB}$ other than A or B. Show that if $Q = \overleftrightarrow{PC} \cap \overleftrightarrow{AD}$ then

$$\frac{AQ}{QD} = \frac{PA}{AB}.$$

2. (10) Let R_θ be the rotation of angle θ around the origin. Show that R_θ is the composition of two reflections.

3. (20) Let $\triangle ABC$ be a spherical triangle on a sphere S of radius R around its centre O.

a) To $\triangle ABC$ six angles $\angle A$, $\angle B$ $\angle C$, $\angle a$, $\angle b$ $\angle c$ are associated. Explain how these angles are measured?

b) What are the minimal subsets of these six angles which determine $\triangle ABC$ (up to isometry). State explicitly the formulas you are using.

4. (15) Let S be a sphere of radius R about its centre O.
a)) Let H be a plane tangent to S at a point A and f the central projection on H of the open hemisphere containing A (open means without the boundary, that is, the equator). So if P belongs to the open hemisphere, $f(P) = \overleftrightarrow{OP} \cap H$. Show that the image of a curve C on the hemisphere is a line in H if and only if C is an arc of a great circle.

b) Does f preserve distances?

5. (15) a) State Desargues' Theorem for the projective plane $\mathbf{P}^2$.

b) State the dual of Desargues' Theorem in $\mathbf{P}^{2*}$.

Please include labelled diagrams.

c) Given a triangle $\triangle ABC$ in $\mathbf{E}^2$ and two distinct point A', B' not equal to A, B or C, such that $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ intersect in a point P, is it always possible to choose a point C' such that $\triangle ABC$ and $\triangle A'B'C'$ are in perspective?

6. (15) a) Let A, B, C, D be distinct points on a Euclidean line and k the cross ratio $[A, B, C, D]$. ($[A, B, C, D] = k$). Show that D is uniquely determined by A, B, C and k. Hint, use coordinates.

b) Let L and L' be distinct Euclidean lines which intersect in a point P and $f : L \to L'$ a map of L on L' which fixes P ($f(P) = P$) and preserves the cross ratio ($[f(A), f(B), f(C), f(D)] = [A, B, C, D]$ for all A, B, C, $D \in L$). Show that f is a central projection.

7. (15) Here is a geometrical proof that the foci of an ellipse are equidistant from the ellipse, that is, $F_1A_1 = F_2A_2$. (The first diagram on page 4 is that of a cone, an ellipse and the Dandelin spheres which determine the foci. The second diagram is that of the plane containing V, the vertex of the cone, O_1, O_2, the centres of the spheres S_1 et S_2 and the foci F_1 and F_2 in the first diagram.) You are asked to briefly justify each line. Of course if there is a line which you can't justify, you should just keep on going.

$\angle O_2E_2A_2 = 90°$

$\angle O_2F_2A_2 = 90°$

$O_2E_2 = O_2F_2$

$A_2E_2 = A_2F_2$

$\triangle O_2E_2A_2 \cong \triangle O_2F_2A_2$

$\angle O_2A_2E_2 = \angle O_2A_2F_2$

Similarly, $\angle O_1A_2F_1 = \angle O_1A_2G_1$

$\angle O_2A_2O_1 = 90°$

$\angle O_1F_1A_2 = 90°$

$\triangle O_2F_2A_2 \sim \triangle A_2F_1O_1$
($\sim$ means the triangles are similar)

$\frac{F_2A_2}{F_2O_2} = \frac{F_1O_1}{F_1A_2}$

Similarly, $\frac{F_1A_1}{F_1O_1} = \frac{F_2O_2}{F_2A_1}$

$F_2A_2 \cdot F_1A_2 = F_1O_1 \cdot F_2O_2$
($\cdot$: here this just means the product of two numbers)

$F_1A_1 \cdot F_2A_1 = F_1O_1 \cdot F_2O_2$

$F_1A_1(F_2F_1 + F_1A_1) = F_2A_2(F_1F_2 + F_2A_2)$

$F_1A_1 = F_2A_2.$