UNIVERSITÉ D'OTTAWA • UNIVERSITY OF OTTAWA

Sciences • Science

Mathématiques et statistique
Mathematics and Statistics

MAT 2355 Final Exam

April 17, 2001. Duration: 3 hours

Family Name:______________________

First Name:______________________

Student number:______________________

1	
2	
3	
4	
5	
6	
(Bonus) 7	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 3 hours to complete this exam.

2. This is a closed book exam, and no notes of any kind are permitted. Calculators are allowed, but may only be used for computations and not for storage of data or text. The use of cell phones is not permitted.

3. The correct answer requires reasonable justification written legibly and logically –you must convince me that you know why your solution is correct.

4. Each question is worth an equal number of points. Questions 1 to 6 should be attempted by everyone. Question 7 is a bonus question, and should not be tried until all parts of questions 1-6 have been completed and checked.

5. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.

1. a) Express the translation of $\mathbf{R}^3$ defined by $f(x, y, z) = (x+4, y-2, z-2)$ as a product of two reflections, and check your answer.

b) Express the rotation by $\frac{3\pi}{2}$ about $(1, -1)$ as a product of two reflections.

c) (i) Show that if $P = (0, 0)$, $Q = (1, 1)$, $R = (1, -1)$ and $S = (1, -3)$, then there is an orientation preserving similarity $f : \mathbf{R}^2 \to \mathbf{R}^2$ such that $f(\overline{PQ}) = \overline{RS}$.

(ii) Find an orientation *reversing* similarity g, such that $g(\overline{PQ}) = \overline{RS}$.

2. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x, y) = \frac{1}{5}(10 + 4x - 3y, -3x - 4y).$$

a) Show that f is an orientation reversing isometry of $\mathbf{R}^2$.

b) Find a line $L \subset \mathbf{R}^2$, and a vector w which is parallel to L, so that f is reflection in L followed by translation by w.

c) Check that f is the same as the isometry which is translation by w followed by reflection in L.

3. Let $C_0 = \{v \in \mathbf{S}^2 \mid (1, -1, 0) \cdot v = 0\}$ and $C_1 = \{v \in \mathbf{S}^2 \mid (-1, 0, 1) \cdot v = 0\}$ be great circles on $\mathbf{S}^2$.

a) Find $C_0 \cap C_1$.

b) Find formulae for the reflections R_{C_0} and R_{C_1} in C_0 and C_1 respectively.

c) Use your answers from (b) to find $A \in O(3)$ so that $R_{C_1} \circ R_{C_0} = Av$.

d) Use the formula $R_{a,\theta}(v) = (v \cdot a)(1 - \cos\theta)a + (\cos\theta)v + (\sin\theta)(a \times v)$, (valid only when $||a|| = 1$) to verify that the $R_{C_1} \circ R_{C_0}$ is the rotation by $\frac{2\pi}{3}$ about the point $\frac{\sqrt{3}}{3}(1, 1, 1) \in \mathbf{S}^2$.

4. Let $g : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $g(v) = Av$ where $A = \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ 1 & -2 & -2 \\ -2 & 1 & -2 \end{bmatrix}$.

a) Show that g is an orientation reversing isometry of $\mathbf{R}^3$.

b) Find a vector $a \in \mathbf{S}^2$, and $\theta \in [0, 2\pi)$ such that g is rotation by θ about a, followed by reflection in the plane through the origin perpendicular to a.

c) Check your answer in (b) by an explicit computation.

5. The latitude and longitude of Ottawa and Madrid are given below:

	Latitude	Longitude
Ottawa	$45°24'\,N$	$75°43'\,W$
Madrid	$40°26'\,N$	$3°42'\,W$

(Round all answers to 3 decimal places.)

a) Find the (ϕ, θ) coordinates of Ottawa and Madrid from the given data. Express your answer in radians.

b) Find the shortest distance from Ottawa to Madrid. Assume that the earth is a sphere of radius 6380 km.

c) Find the bearing of Madrid from Ottawa. Express your answer in radians.

d) If you were to start at Ottawa and maintain the bearing found in (c), where would you end up?

6. Let $N = (0,0,1)$ and f and g denote the stereographic and the central projections

$$f : \mathbf{S}^2 - \{(0,0,1)\} \to H_0 = \{(x,y,0) \mid x, y \in \mathbf{R}^2\}$$

and

$$g : \mathbf{S}^2_- = \{(x,y,z) \in \mathbf{S}^2 \mid z < 0\} \to H_{-1} = \{(x,y,-1) \mid x, y \in \mathbf{R}^2\}$$

and let

$$C = \{(x,y,z) \in \mathbf{S}^2 \mid x + y = 0 \text{ and } z < 0\}$$

be part of a great circle on $\mathbf{S}^2$.

a) Show that $f(C)$ lies in a line in H_0 and that $g(C)$ lies in a line in H_{-1}.

b) Show that the lines in (a) are parallel.

c) Will $f(C')$ lie in a line in H_0 for all great circles C' in $\mathbf{S}^2$?

d) Explain why there can be no isometry from any region of $\mathbf{S}^2$ (with non-zero area) into the plane $\mathbf{R}^2$.

7. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by $f(v) = Ev + a$, where E is an invertible 2×2 matrix and $a \in \mathbf{R}^2$. (We do *not* assume that E is orthogonal.)

a) Show that f preserves geodesics (i.e. takes lines to lines).

b) Suppose that f is conformal, i.e. f preserves angles. Show that f is a similarity.

c) Suppose that f is conformal, and that f also preserves area. Show that f is an isometry.