University of Ottawa

Department of Mathematics and Statistics

MAT 1302A: Mathematical Methods II
Instructor: Hadi Salmasian

Final Exam
April 2018

Surname ________________________ First Name ________________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) This exam consists of two parts. Questions 1 through 13 are *answer only*. For these questions, only your final answer will be considered for marks. Questions 14 through 20 are *long answer*. For these questions, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.
(c) The number of points available for each question is indicated in square brackets.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) You may use the second-to-last page of the exam as extra space for solutions. If you do so, indicate clearly on the page of the relevant question that you have continued your solution on the extra page.
(g) You may use the last page of the exam as scrap paper.
(h) Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you are complying with statement (h) above:

Signature: ______________________________

Good luck!

Please do not write in the table below.

Question	1–13	14	15	16	17	18	19	20	Total
Maximum	23	4	4	8	6	4	4	5	58
Grade									

Part A: Answer Only Questions

For Questions 1–13, only your final answer will be considered for marks. Write your final answers in the spaces provided.

1. [**2 points**] Consider the matrices

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & -\frac{1}{3} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4 \\ 3 & -3 \end{bmatrix}$$

Compute $B^T A$.

Final Answer:____________

2. [**2 points**] Suppose that A is a 4×7 matrix and B is a 6×4 matrix. For each statement below, write '**T**' if the statement is true, and write '**F**' if the statement is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

____ The columns of B can never form a basis of $\mathbb{R}^6$.

____ $\mathrm{Nul}(BA)$ is a subspace of $\mathbb{R}^6$.

____ The reduced echelon form of A always has at least one non-pivot column.

____ $\mathrm{rank}(BA) = 7 - \dim \mathrm{Nul}\,(BA)$.

3. **[2 points]** Suppose that A, B, and C are 4×4 matrices such that $\det(A) = -\dfrac{1}{2}$, $\det(B) = 5$, and $\det(-A^3BC^T) = 15$. Calculate $\det(C)$.

Final Answer:_____________

4. **[1 point]** Determine all values of $t \in \mathbb{R}$ such that the columns of the matrix

$$\begin{bmatrix} 3 & -4 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 5 & t & -1 \\ 0 & -5 & 3 & \frac{1}{2} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

are linearly <u>dependent</u>.

Final Answer:_____________

5. **[2 points]** Let $z = 1 - 2i$ and $w = i + 2$. Write the complex number $\dfrac{w}{1+\overline{z}}$ in the form $a + bi$ where a and b are real numbers.

Final Answer:_____________

6. [**2 points**] Let $A = \begin{bmatrix} -3 & 0 & 0 \\ -2 & 1 & 0 \\ 3-2i & -1+i & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Write down the eigenvalues of $(A+I)^2$ and their multiplicities.

Final Answer:____________

7. [**1 point**] Are the vectors

$$\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

linearly independent? Write 'Yes' if these vectors are linearly <u>independent</u>, and write 'No' if they are linearly <u>dependent</u>.

Final Answer:____________

8. [**2 points**] Suppose we are given an $n \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$. Assume that the linear system $A\mathbf{x} = \mathbf{b}$ is <u>inconsistent</u>. For each statement below, write 'T' if it is true, and write 'F' if it is false. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

____ $\operatorname{Col} A$ is not equal to $\mathbb{R}^n$.

____ A is invertible.

____ A has an eigenvector with eigenvalue $\lambda = 0$.

____ $\mathbf{b} = \mathbf{0}_{n\times 1}$, where $\mathbf{0}_{n\times 1}$ denotes the zero vector in $\mathbb{R}^n$.

9. [**2 points**] Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in $\mathbb{R}^4$ such that $2\mathbf{a} - 3\mathbf{b} = \mathbf{b} + 5\mathbf{c}$. Justify that $\mathbf{a}$ belongs to $\mathrm{Span}\{\mathbf{b}, \mathbf{c}\}$ by expressing $\mathbf{a}$ explicitly as a linear combination of $\mathbf{b}$ and $\mathbf{c}$.

Final Answer:____________

10. [**1 point**] Determine the value of the parameter t such that $\underline{\lambda = i}$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & t \\ -2 & 2i+1 \end{bmatrix}.$$

Final Answer:____________

11. [**2 points**] For each of the following subsets of $\mathbb{R}^3$, write 'Y' if the set is a *subspace* of $\mathbb{R}^3$, and write 'N' if it is not. You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. You cannot receive a negative score on this question.

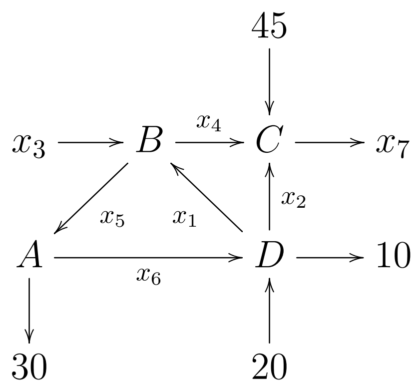
____ $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1 + x_2 + 1 = 0 \right\}$

____ $\left\{ x \begin{bmatrix} -3 \\ 1 \\ -\frac{1}{2} \end{bmatrix} + y \begin{bmatrix} -2 \\ 0 \\ -7 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$

____ The eigenspace of the matrix $A = \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ -3 & 3 & 0 \\ 2 & 0 & 5 \end{bmatrix}$ corresponding to its eigenvalue $\lambda = 6$.

____ $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \text{ and } y = x^2 \right\}$

12. [**2 points**] Consider the traffic flow described by the following diagram. The letters A through D label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.



Write down a linear system describing the traffic flow, i.e., all constraints on the variables $x_i, i = 1, \ldots, 7$. (Do not solve the linear system.)

Final Answer:____________

13. [**2 points**] Let A, B, and C be $n \times n$ invertible matrices. Solve the matrix equation $AX^TC - B = -BC$ for the matrix X.

Final Answer:____________

Part B: Long Answer Questions

For Questions 14–20, you must show your work and justify your answers to receive full marks. Partial marks may be awarded for making sufficient progress towards a solution.

14. **[4 points]** Is the following linear system consistent or inconsistent? If it is consistent, then write down the general solution in vector parametric form.

$$\begin{cases} x_1 & & -2x_3 & -x_4 & -2x_5 & = & 1 \\ & x_2 & +5x_3 & & +x_5 & = & -7 \\ 2x_1 & & -4x_3 & +x_4 & -7x_5 & = & 11 \end{cases}$$

15. [**4 points**] Calculate the determinant of the following matrix using the method of co-factor expansion.

$$M = \begin{bmatrix} 0 & 15 & 0 & -1 \\ 5 & -3 & 7 & -2 \\ 0 & 3 & 0 & 0 \\ 1 & 11 & -1 & 9 \end{bmatrix}.$$

16. (a) [**4 points**] Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & -2 & 0 \\ -4 & 0 & 3 \end{bmatrix}$. Calculate the eigenvalues of A.

(b) [**1 point**] Is A diagonalizable? You should justify your answer.

(c) [**3 points**] Let $B = \begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & -3 \\ 2 & 2 & 8 \end{bmatrix}$. Find a basis for the eigenspace of B corresponding to the eigenvalue $\lambda = 2$.

17. The city of Quicheton has two culinary arts institutes: *ChefAcademie* and *RoyalToque*. In the beginning of January, the two institutes had equal student enrolment levels. During January, *ChefAcademie* updated its classes. As a result, at the end of January $\frac{1}{10}$ of *ChefAcademie*'s students switched to *RoyalToque*, while $\frac{1}{4}$ of *RoyalToque*'s students switched to *ChefAcademie*.

(a) **[1 point]** Write down the migration matrix M and the initial state vector $\mathbf{x}_0$ for this problem.

(b) **[1 point]** What fraction of the total number of students will be enrolled at each of the institutes in the beginning of February?

(c) **[4 points]** If the same migration trend continues for several months, then in the long run what are the predicted enrolment ratios of each of the institutes? In your final answer, you should clearly indicate the fraction of students for each institute.

18. [**4 points**] Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 2 & 1 & -5 & 7 & -5 \\ -1 & 2 & -10 & 14 & 4 \end{bmatrix}.$$

Find a basis for $\operatorname{Nul} A$.

19. [**4 points**] Let

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & -1 & -3 \\ 2 & 0 & -6 \end{bmatrix}$$

Find the inverse of A.

20. An economy consists of two sectors: Agriculture and Service. In order to produce one unit, the Agriculture sector consumes $\frac{1}{3}$ units from Agriculture and $\frac{1}{3}$ units from Service. Also, in order to produce one unit, the Service sector consumes $\frac{2}{3}$ units from Agriculture and $\frac{1}{6}$ units from Service.

(a) **[1 point]** Write the consumption matrix for this economy.

(b) **[1 point]** Write the Leontief Input-Output Model production equation.

(c) **[3 points]** Determine the production levels needed to satisfy a final demand of 12 units from the Agriculture sector and 6 units from the Service sector.

Extra page for answers.

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