On the equation $aX^4 - bY^2 = 2$

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Abstract

Using the Thue-Siegel method, it is effectively proved that for an odd positive integer $t$, there are at most two positive integer solutions to the quartic Diophantine equation $(t + 2)X^4 - tY^2 = 2$.

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1 Introduction

The problem of determining upper bounds for the number of integer points on elliptic curves has received considerable attention, and is a notoriously difficult problem. In a series of papers [10], [11], [12], [13], [14], Ljunggren proved absolute upper bounds for the number of positive integer solutions to equations of the form

$$aX^4 - bY^2 = c$$

In the case $c = 1$, Ljunggren [11] proved that the equation $X^4 - bY^2 = 1$ has at most two solutions in positive integers $X, Y$. This result was extended by Bennett and the third author in [3], wherein it was proved that all equations of the form $a^2X^4 - bY^2 = 1$, with $a > 1$, have at most one solution in positive integers. In [5], Chen and Voutier proved that the equation $aX^4 - Y^2 = 1$ ($a > 2$) has at most one solution in positive integers. These results have recently been extended by Akhtari [1], wherein it was proved that any equation of the form $aX^4 - bY^2 = 1$ has at most two solutions in positive integers. A key fact in proving this result is that one needs only to prove the result for the subfamily of equations $(t + 1)X^4 - tY^2 = 1$.

Noticeably absent from the above list of values for $c$ is the particular value $c = 2$. The third author [17] has recently proved, under stringent conditions on $a, b$, that the equation $aX^4 - bY^2 = 2$ has at most one solution in positive integers. Proving a similar result for the general equation $aX^4 - bY^2 = 2$ remains elusive. However, the methods of [1] can be employed for the particular subfamily of equations $(t + 2)X^4 - tY^2 = 2$. The purpose of the present paper is to prove an upper bound in the case that $c = 2$. In particular, we prove

**Theorem 1.1.** For all odd positive integers $t > 40,000$, the equation

$$(1.1) \quad (t + 2)X^4 - tY^2 = 2$$

has at most two solutions in positive integers $X, Y$. For the remaining odd positive integers $t$, equation (1.1) has at most three solutions in positive integers $X, Y$.  

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Theorem 1.1 is likely not best possible. We conjecture that the only positive integer solution to equation (1.1) is \((X, Y) = (1, 1)\), and more generally, that any equation of the form \(aX^4 - bY^2 = 2\), with \(a\) and \(b\) odd, has at most one solution in positive integers, and that such a solution must arise from the fundamental solution to the quadratic equation \(aX^2 - bY^2 = 2\).

2 Linear Recurrences

For \(t \geq 1\) and odd, let \(\alpha = \frac{\sqrt{t+2} + \sqrt{t}}{\sqrt{2}}\), and for \(k \geq 0\), define sequences \(\{V_k\}, \{U_k\}\) by

\[\alpha^{2k+1} = \frac{V_{2k+1} \sqrt{t+2} + U_{2k+1} \sqrt{t}}{\sqrt{2}},\]

\[\alpha^{2k} = V_{2k} + U_{2k} \sqrt{t(t+2)}\].

All positive integer solutions \((X, Y)\) to the quadratic equation

\[(t+2)X^2 - tY^2 = 2\]

are given by \((X, Y) = (V_{2k+1}, U_{2k+1})\).

Thus, a positive integer solution \((X, Y)\), with \(X > 1\), to equation (1.1) is equivalent to the existence of positive integers \((t, k)\) satisfying

\[X^2 = V_{2k+1}(t)\].

Our strategy to prove Theorem 1.1 is to first prove that for fixed \(t\), the equations

(2.1) \[X^2 = V_{4k+1}(t)\]

and

(2.2) \[X^2 = V_{4k+3}(t)\]

are solvable for at most one integer \(k > 0\). This results in an upper bound of three solutions for equation (1.1). We then show that for \(t\) large enough, equation (2.1) has no solution with \(k > 0\).

3 Reduction to Thue equations

We show here that a solution to equation (1.1) gives rise to a solution to a Thue equation.

It is easily proved by induction that the following relation holds for \(\{V_n\}\)

(3.1) \[V_{4k+1} = V_{2k+1}^2 + 2tU_{2k}^2\].

Therefore, if \(V_{4k+1} = X^2\), then

\[X^2 = V_{2k+1}^2 + 2tU_{2k}^2\],

and hence, \((X - V_{2k+1})(X + V_{2k+1}) = 2tU_{2k}^2\). It follows that there are positive integers \(r, s, A, B\), with \(rs = 2t\) and \(U_{2k} = 2AB\), for which

\[X - V_{2k+1} = 2rA^2, X + V_{2k+1} = 2sB^2\].
Consequently, $V_{2k+1} = sB^2 - rA^2$, and from the easily seen identity $V_{2k+1} = V_{2k} + tU_{2k}$, one has that $V_{2k} = sB^2 - rA^2 - 2tAB$. Substituting these expressions for $V_{2k}$ and $U_{2k}$ into the equation $V_{2k}^2 - t(t+2)U_{2k}^2 = 1$ results in the equation

\[ s^2B^4 - 4tsAB^3 - 12tA^2B^2 + 4rtBA^3 + r^2A^4 = 1. \]

Multiplying this equation through by $s^2$, and setting

\[ x = -sB, y = A \]

shows that $x$ and $y$ satisfy the Thue equation

\[ x^4 + 4tx^3y - 12tx^2y^2 - 8t^2xy^3 + 4t^2y^4 = s^2. \]

Similar to (3.1), one has the relation

\[ V_{4k+3} = V_{2k+1}^2 + 2tU_{2k+2}^2, \]

and so if $V_{4k+3} = X^2$, then

\[ X^2 = V_{2k+1}^2 + 2tU_{2k+2}^2. \]

Therefore, $(X - V_{2k+1})(X + V_{2k+1}) = 2tU_{2k+2}^2$, and so there are positive integers $r, s, A, B$, with $rs = 2t$ and $U_{2k+2} = 2AB$, for which

\[ X - V_{2k+1} = 2rA^2, X + V_{2k+1} = 2sB^2. \]

It follows that $V_{2k+1} = sB^2 - rA^2$, and from identity $V_{2k+2} = V_{2k+1} + tU_{2k+2}$, one has that $V_{2k+2} = sB^2 - rA^2 + 2tAB$. Substituting these expressions into the equation $V_{2k+2}^2 - t(t+2)U_{2k+2}^2 = 1$ gives

\[ s^2B^4 + 4tsAB^3 - 12tA^2B^2 - 4rtBA^3 + r^2A^4 = 1. \]

Multiplying through by $s^2$ and letting

\[ x = sB, y = A \]

shows that $x$ and $y$ satisfy equation (3.3).

Asymptotically, the roots of the polynomial

\[ p_t(x) = x^4 + 4tx^3 - 12tx^2 - 8t^2x + 4t^2 \]

are

\[
\beta^{(1)} = \sqrt{2t} + 1 + \frac{1}{2\sqrt{2t}} - \frac{1}{2t} - \frac{9}{16t\sqrt{2t}} + O\left(\frac{1}{2t^2}\right)
\]
\[
\beta^{(2)} = -\sqrt{2t} + 1 - \frac{1}{2\sqrt{2t}} - \frac{1}{2t} + \frac{9}{16t\sqrt{2t}} + O\left(\frac{1}{2t^2}\right)
\]
\[
\beta^{(3)} = \frac{1}{2} - \frac{5}{16t} + \frac{23}{64t^2} + O\left(\frac{509}{1024t^3}\right)
\]
\[
\beta^{(4)} = -4t - \frac{5}{2} + \frac{21}{16t} - \frac{84}{64t^2} + O\left(\frac{1789}{1024t^3}\right).
\]

Carefully analyzing the construction of the Thue equation (3.3), it is not difficult to verify that if $X^2 = V_{4k+3}$, with $k > 0$, then the corresponding positive integer solution $(x, y)$, given by (3.2), to equation (3.3), satisfies the property that the closest root to $x/y$ is $\beta^{(4)}$, whereas if $X^2 = V_{4k+1}$, then the corresponding positive integer solution $(x, y)$, given by (3.4), to equation (3.3), satisfies the property that the closest root to $x/y$ is $\beta^{(1)}$. We make this comment more concrete in the following.
Lemma 3.1. If $X^2 = V_{4k+1}$ is solvable with $X$ an integer and $k > 0$, and if $(x, y)$, given by (3.2), is the corresponding solution to the Thue equation (3.3), then

$$|x/y - \beta| < \frac{1}{16t|y|^4}.$$ 

If $X^2 = V_{4k+3}$ is solvable with $X$ an integer and $k > 0$, and if $(x, y)$, given by (3.4), is the corresponding solution to the Thue equation (3.3), then

$$|x/y - \beta| < \frac{1}{4t|y|^4}.$$ 

proof. We will prove the result for the equation $X^2 = V_{4k+1}$, as the proof for the second case is essentially identical. We will use the fact that $V_{2k}/U_{2k}$ is a close approximation to $\sqrt{t(t+2)}$. The definition of $(x, y)$ gives

$$\frac{x}{y} = \frac{-sB}{A} = \frac{-2sB^2}{2AB} = \frac{X + V_{2k+1}}{U_{2k}} = \frac{-\sqrt{V_{4k+1}} + V_{2k+1}}{U_{2k}}.$$ 

Using the relations $V_{4k+1} = V_{2k+1}^2 + 2tU_{2k}^2$ and $V_{2k+1} = V_{2k} + tU_{2k}$, we find that

$$x/y = -(\sqrt{(V_{2k}/U_{2k})^2 + 2t(V_{2k}/U_{2k}) + (t^2 + t) + (V_{2k}/U_{2k}) + t}).$$ 

Using the fact that $V_{2k}/U_{2k}$ is a close approximation to $\sqrt{t(t+2)}$, we see that the above expression is closely approximated by $-(4t+5)/2$, which shows that the closest root to $x/y$ is $\beta^{(4)}$. As

$$|p_t(x, y)| = |y|^4 \prod_{i=1}^{4} |x/y - \beta^{(i)}| \leq 4t^2,$$

we see that

$$|x/y - \beta^{(4)}| \leq \frac{4t^2}{|y|^4 \prod_{i \neq 4} |x/y - \beta^{(i)}|}.$$ 

Using the crude estimate $4t$ for each factor $|x/y - \beta^{(i)}|$, we see that

$$|x/y - \beta^{(4)}| \leq \frac{1}{16t|y|^4}.$$ 

4 Lower bounds for $k$, $t$ and $|y|$

In order to prove Theorem 1, we first need to verify that equation (1.1) has only the positive integer solution $(X, Y) = (1, 1)$ for all $t$ up to a certain bound. Two independent computations, using PARI and MAGMA, were run in order to verify that equation (1.1) has no positive integer solutions other than $(X, Y) = (1, 1)$ for all $1 \leq t \leq 1200$. This was achieved by computing all integer solutions to each Thue equation of the form $p_t(x, y) = s^2$, where $p_t(x, y)$ is given in (3.5), and $s$ is a divisor of $2t$.

We will also need a lower bound for $k$. The polynomial $V_{4k+1}(t)$ is monic and of even degree. Therefore, Runge’s method can be applied directly to equation (2.1). Fortunately, Runge’s method has been implemented in MAGMA by Beukers and Tengely (see [4]). This rather short MAGMA computation verified that no positive integer solutions $(X, t)$ to equation (2.1) exist for each $1 \leq k \leq 24$.

We now describe how to obtain the lower bound $k > 6$ in the case of solving $X^2 = V_{4k+3}$. Firstly, it is trivial to see that $V_{4k+3} \equiv 3 \pmod 4$ for $k$ even. If $k = 1$, then the equation is simply $X^2 = 8t^3 + 20t^2 + 12t + 1$, 

$$4^4$$
which is easily seen to have no solutions in positive integers $X, t$ using MAGMA. For the case $k = 3$, the equation $V_{15} = X^2$ implies that $V_5 = 4t^2 + 6t + 1$ is either a square or three times a square by elementary divisibility properties of terms in the sequence $\{V_n\}$ (see [8] for details). Evidently, neither of these possibilities is possible. Finally, if $k = 5$, we use the fact that for each $i \geq 0$, the Jacobi symbol $(V_{16i+7}/V_{16i+3}) = -1$, which is easily proved by induction.

We use the above lower bounds for $k$ to obtain lower bounds for $|y|$, where $(x, y)$ is a solution to the Thue equation (3.3) arising from a solution to equation (1.1). In the case of a solution to (1.1) with $X^2 = V_{4k+3}$, we see from the above construction that

$$2ry^2 = \sqrt{V_{4k+3} - V_{2k+1}} = \sqrt{V_{2k+1}^2 + 2tU_{2k+2}^2 - V_{2k+1}}.$$ 

It follows that

$$2ry^2 = \frac{2tU_{2k+2}}{\sqrt{V_{2k+1}^2 + 2tU_{2k+2}^2 + V_{2k+1}}}.$$ 

Dividing the numerator and denominator of this equation by $\sqrt{U_{2k+2}}$ gives

$$2ry^2 = \frac{2\sqrt{U_{2k+2}}}{\sqrt{(V_{2k+1}/tU_{2k+2})^2 + 2 + (V_{2k+1}/\sqrt{U_{2k+2}})}}.$$ 

Using the fact that $V_{2k+1} < U_{2k+2}$, it follows that

$$2ry^2 > \sqrt{U_{2k+2}},$$

and from the lower bound

$$U_{2k+2} > (2t)^k > (2t)^6,$$

it follows that

$$|y| > 2\sqrt{2t^{9/4}}.$$ 

In the case of a solution to $X^2 = V_{4k+1}$, with $k > 0$, we obtain a much larger lower bound since the equation $X^2 = V_{4k+1}$ was solved using Runge's method for $1 \leq k \leq 24$. In this case, an analysis similar to the one given above shows that $|y| > 2^{10t^{11}}$.

5 Associated Fourth Roots of Unity

Let

$$p_t(x, y) = x^4 + 4tx^3y - 12tx^2y^2 - 8t^2xy^3 + 4t^2y^4$$

and $t$ be a positive integer. Our goal is to find, for fixed $t$, an upper bound upon the number of coprime nonzero integer solutions to the inequality

$$(5.1) \quad 0 < p_t(x, y) \leq 4t^2.$$ 

To proceed, let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ be linear functions of $(x, y)$ so that

$$\xi^4 = 4(-\sqrt{\frac{t}{2} + 1})(x + \sqrt{2ty})^4 \quad \text{and} \quad \eta^4 = 4(-\sqrt{\frac{t}{2} - 1})(x - \sqrt{2ty})^4.$$ 

We refer to $(\xi, \eta)$ as a pair of resolvent forms. Note that $\xi^4 = -\eta^4$ and that

$$p_t(x, y) = \frac{1}{8}(\xi^4 - \eta^4)$$
and if \((\xi, \eta)\) is a pair of resolvent forms then there are precisely three others with distinct ratios, say \((-\xi, \eta)\), \((i\xi, \eta)\) and \((-i\xi, \eta)\). Let \(\omega\) be a fourth root of unity, \((\xi, \eta)\) a fixed pair of resolvent forms and set

\[ z = 1 - \left( \frac{\eta(x, y)}{\xi(x, y)} \right)^4. \]

We say that the integer pair \((x, y)\) is related to \(\omega\) if

\[ \left| \omega - \frac{\eta(x, y)}{\xi(x, y)} \right| < \frac{1}{4} \left| z \right|. \]

It turns out that each nontrivial solution \((x, y)\) to (3.3) is related to a fourth root of unity:

Lemma 5.1. Suppose that \((x, y)\) is a positive integral solution to (5.1), with \(\omega_j\) a fixed pair of resolvent forms and set

\[ z = \frac{4}{\xi^4(x, y)}. \]

Then

\[ |\omega_j - \frac{\eta(x, y)}{\xi(x, y)}| < \frac{1}{4} |z(x, y)|. \]

Proof. We begin by noting that

\[ |z| = \left| \frac{\xi^4 - \eta^4}{\xi^4} \right| = \frac{8 p_1(x, y)}{\left| \xi^4 \right|}, \]

and, from \(xy \neq 0\),

\[ |\xi^4(x, y)| \geq 8 \sqrt{2}(\sqrt{t})^5, \]

whereby

\[ |z| \leq \frac{4t^2}{\sqrt{2}(\sqrt{t})^5} < 1. \]

Since \(\eta = -\xi\), it follows that

\[ \left| \frac{\eta}{\xi} \right| = 1, \quad |1 - z| = 1. \]

Now let \(4\theta = \arg \left( \frac{\eta(x, y)^4}{\xi(x, y)^4} \right)\). We have

\[ \sqrt{2 - 2 \cos(4\theta)} = |z| < 1, \]

and so \(|\theta| < \frac{\pi}{12}\). Since

\[ \left| \omega_j - \frac{\eta(x, y)}{\xi(x, y)} \right| \leq |\theta|, \]

it follows that

\[ \left| \omega_j - \frac{\eta(x, y)}{\xi(x, y)} \right| \leq \frac{1}{4} \sqrt{2 - 2 \cos(4\theta)} \left| 1 - \frac{\eta(x, y)^4}{\xi(x, y)^4} \right|. \]

From the fact that \(\frac{4|\theta|}{\sqrt{2 - 2 \cos(4\theta)}} < 1\) whenever \(0 < |\theta| < \frac{\pi}{12}\), we obtain the desired inequality.

We now put \(\omega_i = \frac{\eta(x, y)^4}{\xi(x, y)^4} \) for \(1 \leq i \leq 4\). The \(\omega_i\) are the distinct fourth roots of unity. The following lemma represents a key step towards the proof of Theorem 1.1.
Proof. Assume that $X^2 = V_{4k+3}$, as the other case is proved in the same way. The goal is to prove that

$$\left| \frac{\eta(x,y) - \eta(x,y)}{\xi(x,y)} \right| < 1/4 \left| 1 - \left( \frac{\eta(x,y)}{\xi(x,y)} \right)^4 \right|.$$ 

Obtaining a common divisor for the left side, expanding and simplifying, shows that the above inequality is the same as

$$\frac{|2\sqrt{-2t}|x - \beta(1)y|}{|\beta(1) + \sqrt{-2t}|x + y\sqrt{-2t}|} < \frac{|2P_t(x,y)|}{|\xi(x,y)^4|}.$$ 

Cross multiplying the above and dividing through by $|y|^4$, we reduce the problem to proving the inequality

$$(5.3) \quad \frac{|2\sqrt{-2t}|2 - \sqrt{-2t}|x/y + \sqrt{-2t}|^3}{|\beta(1) + \sqrt{-2t}|} < \frac{|x/y - \beta(2)||x/y - \beta(3)||x/y - \beta(4)|}{|\xi(x,y)^4|}.$$ 

Lemma 3.1 shows that $x/y$ is very close to $\beta(1)$. Indeed, Lemma 3.1, together with the lower bound for $|y|$ determined in section 4, show that $|x/y - \beta(1)| < 1/(2^410)$. This difference is sufficiently small that we will replace $x/y$ in (5.3) by $\beta(1)$, which entails that we need to prove the inequality

$$(5.4) \quad 2\sqrt{-2t}|2 - \sqrt{-2t}|\beta(1) + \sqrt{-2t}|^2 < |\beta(1) - \beta(2)||\beta(1) - \beta(3)||\beta(1) - \beta(4)|.$$ 

Expanding the left hand side of (5.4) gives an estimate with leading terms $16t^2 + 20t$, while that for the right hand side has leading terms $16t^2 + 24t$. \qed

6 The Main Lemma

The following represents the most crucial lemma of this paper, as it provides for an absolute bound for the number of integer solutions to equation (1.1).

Lemma 6.1. There is at most one solution of (5.1) related to each fourth root of unity.

Because of the lower bound for $k$ obtained in section 4, we may assume that $k > 6$. Furthermore, since $\xi_4 = 4(-\sqrt{-2} + 1)(x_i + \sqrt{-2}y_i)^4$, via calculus one may conclude that

$$(6.1) \quad |\xi_4| > t^{14}.$$ 

6.1 Approximating polynomials

The following Lemma gives a family of dense approximations to $\xi/\eta$ from rational function approximations to the binomial function $(1 - z)^{1/4}$.

Lemma 6.2. Let $r$ be a positive integer and $g \in \{0,1\}$. Put

$$A_{r,g}(z) = \sum_{m=0}^{r} \binom{r - g + \frac{1}{4}}{m} \binom{2r - g - m}{r - g} (-z)^m,$$

$$B_{r,g}(z) = \sum_{m=0}^{r-g} \binom{r - \frac{1}{4}}{m} \binom{2r - g - m}{r} (-z)^m.$$
Lemma 6.3. For sense, increase rapidly in height. will induce a strong gap principle”, guaranteeing that solutions to inequality (5.1) must, in a certain sense, increase rapidly in height.

\[ A_{r,g}(z) = (1-z)^{1/4} B_{r,g}(z) = z^{2r+1-g} F_{r,g}(z) \]

and

\[ |F_{r,g}(z)| \leq \frac{(r+1/4)(r-1/4)}{(2r+1-g)} (1-|z|)^{-\frac{1}{4}(2r+1-g)}. \]

(ii) For all complex numbers \( z \) with \( |1-z| \leq 1 \) we have

\[ |A_{r,g}(z)| \leq \left( \frac{2r-g}{r} \right). \]

(iii) For all complex numbers \( z \neq 0 \) and for \( h \in \{1, 0\} \) we have

\[ A_{r,0}(z)B_{r+h,1,1}(z) \neq A_{r+h,1}(z)B_{r,0}(z). \]

Proof. See the proof of Lemma 4.1 of [1].

Combining the polynomials of Lemma 6.2 with the resolvent forms \( \xi(x, y) \) and \( \eta(x, y) \), we will consider the complex sequences \( \Sigma_{r,g} \) given by

\[ \Sigma_{r,g} = \frac{\eta_2}{\xi_2} A_{r,g}(z_1) - (-1)^{r+1} \frac{\eta_1}{\xi_1} B_{r,g}(z_1) \]

where \( z_1 = 1 - \eta_1^2/\xi_1^2 \). Define

\[ \Lambda_{r,g} = \frac{\xi_2^{4r+1-g}}{(\frac{1}{2} - 1)^{1/4}} \Sigma_{r,g}. \]

We will show that \( \Lambda_{r,g} \) is either an integer in \( \mathbb{Q}(\sqrt{-2t}) \) or a fourth root of such an integer. If \( \Lambda_{r,g} \neq 0 \), this provides a lower bound upon \( |\Lambda_{r,g}| \). In conjunction with the inequalities derived in Lemma 6.2, this will induce a strong “gap principle”, guaranteeing that solutions to inequality (5.1) must, in a certain sense, increase rapidly in height.

Lemma 6.3. If \( (x_1, y_1) \) and \( (x_2, y_2) \) are two pairs of rational integers then

\[ \frac{\xi(x_1, y_1)\eta(x_2, y_2)}{(-\frac{1}{2} - 1)^{1/4}}, \quad \xi(x_1, y_1)^3 \xi(x_2, y_2) \quad \text{and} \quad \eta(x_1, y_1)^3 \eta(x_2, y_2) \]

are integers in \( \mathbb{Q}(\sqrt{-2t}) \).

Proof. This is an immediate consequence of the definition of resolvent forms.

For a polynomial \( P(z) \) of degree \( n \), we will denote by \( P^*(x, y) = x^n P(y/x) \) an associated binary form. Let \( A_{r,g} \) and \( B_{r,g} \) be as in Lemma 6.2, and set

\[ C_{r,g}(z) = A_{r,g}(1-z) \quad \text{and} \quad D_{r,g}(z) = B_{r,g}(1-z). \]

For \( z \neq 0 \), we have \( D_{r,0}(z) = z^r C_{r,0}(z^{-1}) \), hence

\[ A^*_{r,0}(z, z + \bar{z}) = z^r A_{r,0} \left( 1 + \frac{\bar{z}}{z} \right) = z^r C_{r,0} \left( \frac{-\bar{z}}{z} \right) \]

\[ = (-1)^r \bar{z}^r D_{r,0} \left( \frac{-\bar{z}}{z} \right) = (-1)^r \bar{z}^r B_{r,0} \left( 1 + \frac{\bar{z}}{z} \right) \]

\[ = (-1)^r B^*_{r,0}(\bar{z}, \bar{z} + z) = (-1)^r B^*_{r,0}(z, z + \bar{z}). \]
Lemma 6.4. For any pair of integers \((x, y)\), both \(A_{r,g}(\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))\) and 
\(B_{r,g}(\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))\) are algebraic integers in \(\mathbb{Q}(\sqrt{-2t})\).

Proof. It is clear that \(A_{r,g}(\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))\) and 
\(B_{r,g}(\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))\) belong to 
\(\mathbb{Q}(\sqrt{-2t})\); we need only show that they are algebraic integers. From the definitions of 
\(A_{r,g}(x, y)\), \(B_{r,g}(x, y)\), 
\(\xi(x, y)\) and \(\eta(x, y)\) (in particular, since \(\xi^4(x, y) - \eta^4(x, y) = 8p_t(x, y)\)), this is an immediate consequence 
of Lemma 4.1 of [6], which, in this case, implies that

\[
\left(\frac{a/4}{n}\right) = 0
\]

is, for fixed nonnegative integers \(a\) and \(n\), a rational integer. 

\(\square\)

Proposition 6.1. Let \(\Lambda_{r,g}\) be the complex number defined in (6.6). Then \(\Lambda_{r,0}\) belongs to \(\mathbb{Z}[\sqrt{-2t}]\) and 
\(\Lambda_{r,1}\) is a fourth root of an integer in \(\mathbb{Q}(\sqrt{-2t})\).

Proof. We have

\[
\Lambda_{r,g} = \frac{\xi_1^{-g}m}{\left(-\frac{t}{2} - 1\right)^{1/4}} A_{r,g}(\xi_1^4, \xi_1^4 - \eta_1^4) - \frac{\eta_2^{-g}m}{\xi_1^{-g}m} B_{r,g}(\xi_1^4, \xi_1^4 - \eta_1^4).
\]

By Lemmas 6.3 and 6.4, \(\Lambda_{r,0} \in \mathbb{Z}[\sqrt{-2t}]\). Similarly, Lemmas 6.3 and 6.4 imply that \(\Lambda_{r,1}^4\) is an algebraic 
integer in \(\mathbb{Q}(\sqrt{-2t})\). We claim that it is not a rational integer. To see this, let us start by noting that

\[
\frac{\sum_{r,g}}{(\frac{-t}{2} - 1)^{1/4}} = \frac{\eta_2}{\xi_2} A_{r,g}(z_1) - (\frac{-t}{2} - 1)^{1/4} \frac{\eta_2}{\xi_2} A_{r,g}(z_1) = \frac{\eta_2}{\xi_2} A_{r,g}(z_1) - (\frac{-t}{2} - 1)^{1/4} \frac{\eta_2}{\xi_2} A_{r,g}(z_1)
\]

where \(\eta = (\sqrt{-t} - 1)^{1/4}\) and \(\xi = (\sqrt{-t} + 1)^{1/4}\). By Lemma 6.4,

\[
\frac{\eta_2}{\xi_2} A_{r,g}(z_1) - (\frac{-t}{2} - 1)^{1/4} \frac{\eta_2}{\xi_2} A_{r,g}(z_1) \in \mathbb{Q}(\sqrt{-2t})
\]

and so

\[
f = \mathbb{Q}(\sqrt{-2t}, \sum_{r,g}) = \mathbb{Q}(\sqrt{-2t}, (-\frac{t}{2} - 1)^{1/4} \eta) = \mathbb{Q}(\sqrt{-2t}, (-\frac{t}{2} + 1 - \sqrt{-2t})^{1/4}).
\]

If we choose a complex number \(X\) so that \(\xi(X, 1) = \eta(X, 1)\) then \(X \in f\) and

\[
p_t(X, 1) = \frac{1}{8}(\xi^4(X, 1) - \eta^4(X, 1)) = 0.
\]

Since \(p_t\) is irreducible, \(X\) and \(\Sigma_{r,g}\) both have degree 4 over \(\mathbb{Q}(\sqrt{-2t})\).

Suppose that \(\Lambda_{r,1}^2 \in \mathbb{Z}\). Then we have for some \(\rho, \rho_1 \in \{\pm 1, \pm i\}\), that \(\Lambda_{r,1} = \rho \Lambda_{r,1}\) and \((-\frac{t}{2} - 1)^{1/4} = \rho_1 (-\frac{t}{2} - 1)^{1/4}\), whence, from Lemma 6.3,

\[
\sum_{r,1} = \left(-\frac{t}{2} - 1\right)^{1/4} \xi_1^{-4r} \xi_2^{-1} \rho \Lambda_{r,1}
\]

\[
= \xi_1^{-4r} \xi_2^{-1} \eta_1^4 \rho_1 \left(\frac{\xi_1}{\eta_2} A_{r,1} \left(1 - \frac{\xi_4}{\eta_1^4}\right) - (\frac{-t}{2} - 1)^{1/4} \frac{\xi_1}{\eta_1} B_{r,1} \left(1 - \frac{\xi_4}{\eta_1^4}\right)\right)
\]

\[
= \rho \rho_1 \eta_1^4 \left(\left(A_{r,1} \left(1 - \frac{\xi_4}{\eta_1^4}\right) - (\frac{-t}{2} - 1)^{1/4} \frac{\xi_1}{\eta_2} B_{r,1} \left(1 - \frac{\xi_4}{\eta_1^4}\right)\right)\right).\]
This together with Lemmas 6.3 and 6.4 imply that \( \Sigma_r,1 \in \mathbb{Q}(\sqrt{-2t}, \rho \rho_1) \), which contradicts the fact that \( \Sigma_r,1 \) has degree 4 over \( \mathbb{Q}(\sqrt{-2t}) \). We conclude that \( \Lambda_r,1 \) can not be a rational integer. \( \square \)

From the well-known characterization of algebraic integers in quadratic fields, we know that for any square-free integer \( d \), the ring of integer \( \mathbb{Q}(\sqrt{d}) \) is

\[
\{ m + n\sqrt{d} \mid m, n \in \mathbb{Z} \} \text{ if } d \equiv 2, 3 \pmod{4},
\]

and

\[
\{ m + n \frac{1 + \sqrt{d}}{2} \mid m, n \in \mathbb{Z} \} \text{ if } d \equiv 1 \pmod{4}.
\]

We may therefore conclude that, if \( \Lambda_{r,g} \neq 0 \), \( g \in \{0, 1\} \), then

\[
|\Lambda_{r,g}| \geq 2^{-\frac{3}{4}} (2t)^{1 - \frac{3}{4}}.
\]

### 6.2 Gap Principles

Lemma 5.1 shows that each integer pair \( (x, y) \) is related to precisely one fourth root of unity. Let us fix such a fourth root, say \( \omega \), and suppose that we have distinct coprime positive solutions \( (x_1, y_1) \) and \( (x_2, y_2) \) to inequality (5.1), each related to \( \omega \). We will assume that \( |\xi(x_2, y_2)| \geq |\xi(x_1, y_1)| \). Let us write \( \eta_1 = \eta(x_1, y_1) \) and \( \xi_1 = \xi(x_1, y_1) \). We will use the following results to prove that \( (x_1, y_1) \) and \( (x_2, y_2) \) are far apart in height.

Since

\[
|z| = \frac{8p(x, y)}{|\xi|^4} \leq \frac{32t^2}{|\xi|^4},
\]

it follows from (5.2) that

\[
|\xi_1 \eta_2 - \xi_2 \eta_1| = |\xi_1 (\eta_2 - \omega \xi_2) - \xi_2 (\eta_1 - \omega \xi_1)| \leq 8t^2 \left( \frac{|\xi_1|}{|\xi_2|} + \frac{|\xi_2|}{|\xi_1|} \right) \leq \frac{16t^2 |\xi_1|}{|\xi_2|}.
\]

On the other hand, choosing our fourth root appropriately, we have

\[
\begin{pmatrix}
\sqrt{2}(-\sqrt{-\frac{t}{2}} + 1)^{1/4} & \sqrt{2}(-\sqrt{-\frac{t}{2}} + 1)^{1/4} \sqrt{-2t} \\
\sqrt{2}(-\sqrt{-\frac{t}{2}} - 1)^{1/4} & -\sqrt{2}(-\sqrt{-\frac{t}{2}} - 1)^{1/4} \sqrt{-2t}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix}
= \begin{pmatrix}
\xi_1 \\
\eta_1
\end{pmatrix}
\begin{pmatrix}
x_2 \\
y_2
\end{pmatrix}
\]

and so

\[
|\xi_1 \eta_2 - \xi_2 \eta_1| = \left| 4(t + 1)^{1/4} \sqrt{2t} (x_1y_2 - x_2y_1) \right|.
\]

Since \( x_1y_2 - x_2y_1 \) is a nonzero integer (recall that we assumed \( \gcd(x_1, y_1) = 1 \)), we have

\[
|\xi_1 \eta_2 - \xi_2 \eta_1| \geq 4 \sqrt{2t} (t + 1)^{1/4},
\]

and thus, combining (6.9) and (6.10), we conclude that if \( (x_1, y_1) \) and \( (x_2, y_2) \) are distinct solutions to (5.1), related to \( \omega \), with \( |\xi(x_2, y_2)| \geq |\xi(x_1, y_1)| \) then

\[
|\xi_2| > \frac{t^{-5/4}}{4} |\xi_1|^3.
\]

We will now combine inequality (6.7) with upper bounds from Lemma 6.2 to show that solutions to (5.1) are widely spaced :
Lemma 6.5. If $\Sigma_{r,g} \neq 0$, then

$$c_1(r, g) |\xi_1|^{4r+1-g} |\xi_2|^{-3} + c_2(r, g) |\xi_1|^{-4r-3(1-g)} |\xi_2| > 1,$$

where we may take

$$c_1(r, g) = \frac{2^{2r+11/4+5g/8}}{\sqrt{\pi r}} r^{5/4+3g/8}$$

and

$$c_2(r, g) = \frac{2^{1/4+5g/8-2r} 3^{2r+1-g}}{\pi \sqrt{r}} r^{4r+5/4-3g/8}.$$

Proof. By (6.2), we can write

$$\left| (\frac{t}{2} + 1)^{1/4} \Lambda_{r,g} \right| = |\xi_1|^{4r+1-g} |\xi_2| \left( \frac{\eta_2}{\xi_2} - \omega \right) A_{r,g}(z_1) + \omega A_{r,g}(z_1) + \omega A_{r,g}(z_1).$$

Since $|1 - z_1| = 1$ and $|z_1| \leq 1$, from (5.2), (6.8), (6.3), (6.4), and the inequality $|\xi_1|^4 > 8\sqrt{2} (r)^{5/2}$, we have

$$\left| (\frac{t}{2} + 1)^{1/4} \Lambda_{r,g} \right| \leq |\xi_1|^{4r+1-g} |\xi_2| \left( \frac{2r - g}{r} \right) 2t^2 |\xi_2| + \frac{(r-g+1/4)(r-1/4)}{(2r+1-g)} \frac{(9t^2)^{2r+1-g}}{|\xi_1|^4}.$$

Comparing this with (6.7), we obtain

$$c_1(r, g) |\xi_1|^{4r+1-g} |\xi_2|^{-3} + c_2(r, g) |\xi_1|^{-4r-3(1-g)} |\xi_2| > 1,$$

where we may take $c_1$ and $c_2$ so that

$$c_1(r, g) \geq 2^{11/4+5g/8} r^{5/4+3g/8} \left( \frac{2r}{r} \right)$$

and

$$c_2(r, g) \geq 2^{5g/8-1/4} 3^{2r+1-g} r^{4r+5/4-13g/8} \frac{(r-g+1/4)(r-1/4)}{(2r+1-g)}. $$

Applying the following version of Stirling’s formula (see Theorem (5.44) of [15])

$$\frac{1}{2\sqrt{k}4^k} \leq \frac{2k}{k} < \frac{1}{\sqrt{\pi k}4^k},$$

(valid for $k \in \mathbb{N}$) leads immediately to the stated choice of $c_1$. One can also show via Stirling’s formula that for $r \in \mathbb{N}$ and $g \in \{0, 1\}$, we have

$$\frac{(r-g+1/4)(r-1/4)}{(2r+1-g)} \frac{(r+1/4)}{(2r+1-g)} \frac{(r-1/4)}{(2r+1-g)} < \frac{\sqrt{2}}{\sqrt{\pi}4^r}$$

(see the proof of Lemma 6.1 of [1] for more details). This gives the desired value for $c_2(r, g)$.
6.3 The Proof of Lemma 6.1

We will start this section with the statements of two Lemmas from [1]. These Lemmas allow us to apply the strong gap principle provided by Lemma 6.5. We note here that although \( \xi(x, y) \) and \( \eta(x, y) \) are defined differently in [1], those properties of \( \xi \) and \( \eta \) used in the proofs of Lemmas 6.2 and 6.3 of [1], hold for our choice of \( \xi \) and \( \eta \) in the present paper.

Lemma 6.6. If \( r \in \mathbb{N} \) and \( h \in \{0, 1\} \), then at most one of \( \{\Sigma_{r,0}, \Sigma_{r+1,1}\} \) can vanish.

Lemma 6.7. Suppose that \( t > 1200 \). For \( r \in \{1, 2, 3, 4, 5\} \), we have \( \Sigma_{r,0} \neq 0 \).

Assume that there are two distinct coprime solutions \((x_1, y_1)\) and \((x_2, y_2)\) to inequality (5.1) with \( |\xi_2| > |\xi_1| \). We will show that \( |\xi_2| \) is arbitrary large in relation to \( |\xi_1| \). In particular, we will demonstrate via induction that

\[
|\xi_2| > \frac{\sqrt{r}}{t^{4r+7/4}} \left( \frac{4}{33^2} \right)^r |\xi_1|^{4r+3},
\]

for each positive integer \( r \). So by (6.1),

\[
|\xi_2| > t^{13r}
\]

for arbitrary \( r \), a contradiction.

To prove inequality (6.12) for \( r = 1 \), we use (6.11) and (6.1) to get

\[
c_1(1, 0)|\xi_1|^5|\xi_2|^{-3} < \frac{943/4}{\sqrt{\pi} t^5} |\xi_1|^{-4} < 0.01,
\]

and hence, since \( \Sigma_{1,0} \neq 0 \), Lemma 6.5 yields

\[
c_2(1, 0)|\xi_1|^{-7}|\xi_2| > 0.09,
\]

which, immediately, implies (6.12). We now proceed by induction. Suppose that (6.12) holds for some \( r \geq 1 \). Then

\[
c_1(r + 1, 0)|\xi_1|^{4r+5}|\xi_2|^{-3} < \frac{1}{\sqrt{\pi} t^{4r+26/4}} \left( \frac{33^6}{2^6} \right)^r |\xi_1|^{-8r-4},
\]

and hence, from (6.1), and the fact that \( t > 1200 \),

\[
c_1(r + 1, 0)|\xi_1|^{4r+5}|\xi_2|^{-3} < 0.1.
\]

If \( \Sigma_{r+1,0} \neq 0 \), then by Lemma 6.5,

\[
c_2(r + 1, 0)|\xi_1|^{-4(r+1)+3}|\xi_2| > 0.9,
\]

which again leads to (6.12). If, however, \( \Sigma_{r+1,0} = 0 \), then by Lemmas 6.6 and 6.7, both \( \Sigma_{r+1,1} \) and \( \Sigma_{r+2,1} \) are nonzero, and \( r \geq 2 \). Using the induction hypothesis, we find as previously that

\[
c_1(r + 1, 1)|\xi_1|^{4r+4}|\xi_2|^{-3} < 0.001,
\]

and thus by Lemma 6.5 conclude that

\[
c_2(r + 1, 1)|\xi_1|^{-4r-4}|\xi_2| > 0.999.
\]

It follows that

\[
|\xi_2| > \frac{\sqrt{r+1}}{2^{7/8} t^{4r+29/8}} \left( \frac{4}{33^2} \right)^{r+1} |\xi_1|^{4r+4}.
\]
Consequently, from (6.1), \( r \geq 6 \) and \( t > 1200 \),
\[
c_1(r + 2, 1)|\xi_1|^{4r+8}|\xi_2|^{-3} < 0.001.
\]
Therefore Lemma 6.5 implies that
\[
c_2(r + 2, 1)|\xi_1|^{-4r-8}|\xi_2| > 0.999,
\]
and so
\[
|\xi_2| > \frac{\pi}{2r+2} \sqrt{r+2} \left( \frac{4}{33^2} \right)^{r+2} t^{-4r-61/8} |\xi_1|^{4r+8}.
\]
From (6.1), it follows that
\[
|\xi_2| > \sqrt{r+1} \frac{4}{r^{4r+4+7/4}} \left( \frac{4}{33^2} \right)^{r+1} |\xi_1|^{4r+7},
\]
as desired. This completes the proof of inequality (6.12), and hence we conclude that there is at most
one solution to (5.1) related to each fourth root of unity.

We conclude from the above, in conjunction with Lemma 5.2, that there are at most three solutions in
positive integers to equation (1.1). In particular, there is the solution \((X, Y) = (1, 1)\), and for both \( i = 1 \) and \( i = 3 \), at most one integer \( k \) for which
\[
X^2 = V_{4k+1}^4 k + i
\]
is solvable. We now proceed to the proof that for
\( t \) large enough, the equation \( X^2 = V_{4k+1}^4 \) is not solvable for all \( k > 0 \).

7 An effective measure of approximation

In this section we will apply the hypergeometric method to obtain effective measures of approximation
to the two roots \( \beta^{(3)} \) and \( \beta^{(4)} \). Because of the relation \( \beta^{(3)} / \beta^{(4)} = -2t \), we will only need to deal with one
of the roots, say \( \beta^{(3)} \).

Our first lemma is Thue’s “Fundamental Theorem” [16] together with its relation to the hypergeometric
function, as discovered by Siegel.

Lemma 7.1. Let \( \alpha_1, \alpha_2, c_1 \) and \( c_2 \) be complex numbers with \( \alpha_1 \neq \alpha_2 \). For \( n \geq 2 \), we define the following polynomials
\[
a(X) = \frac{n^2 - 1}{6} (\alpha_1 - \alpha_2)(X - \alpha_2), \quad c(X) = \frac{n^2 - 1}{6} \alpha_1 (\alpha_1 - \alpha_2)(X - \alpha_2),
\]
\[
b(X) = \frac{n^2 - 1}{6} (\alpha_2 - \alpha_1)(X - \alpha_1), \quad d(X) = \frac{n^2 - 1}{6} \alpha_2 (\alpha_2 - \alpha_1)(X - \alpha_1),
\]
\[
u(X) = -c_2 (X - \alpha_2)^n \quad \text{and} \quad z(X) = c_1 (X - \alpha_1)^n.
\]

Putting \( \lambda = (\alpha_1 - \alpha_2)^2 / 4 \), for any positive integer \( r \), we define
\[
(\sqrt{\lambda})^r A_r(X) = a(X)X^{n_r^*}(z, u) + b(X)X^{n_r^*}(u, z) \quad \text{and} \quad
(\sqrt{\lambda})^r B_r(X) = c(X)X^{n_r^*}(z, u) + d(X)X^{n_r^*}(u, z).
\]

Then, for any root \( \beta \) of \( P(X) = z(X) - u(X) \), the polynomial
\[
C_r(X) = \beta A_r(X) - B_r(X)
\]
is divisible by \((X - \beta)^{2r+1}\).
proof This is a simplified version Lemma 2.1 from [5], obtained by noting that if $P(X)$ satisfies the differential equation given there, with $U(X) = (X - \alpha_1)(X - \alpha_2)$, then $P(X)$ must be of the form given here, which allows us to determine the above expressions.

**Lemma 7.2.** With the above notation, put $w(x) = z(x)/u(x)$ and write $w(x) = \mu e^{i\varphi}$ with $\mu \geq 0$ and $-\pi < \varphi \leq \pi$. Put $w(x)^{1/n} = \mu^{1/n} e^{i\varphi/n}$.

(i) For any $x \in C$ such that $w = w(x)$ is not a negative real number or zero,

\[
\left(\sqrt{x}\right)^r C_r(x) = \left\{ \beta \left(a(x)w(x)^{1/n} + b(x)\right) - \left(c(x)w(x)^{1/n} + d(x)\right) \right\} X_{n,r}(u, z)
\]

where the integration path is the straight line from $z$ to $z$.

(ii) Let $w = e^{i\varphi}$, $0 < \varphi < \pi$ and put $\sqrt{w} = e^{i\varphi/2}$. Then

\[
|R_{n,r}(w)| \leq \frac{n\Gamma(r + 1 + 1/n)}{r!\Gamma(1/n)} |\varphi| 1 - \sqrt{|w|}^{2r}.
\]

**proof** This is Lemma 2.5 of [5].

**Lemma 7.3.** Let $u, w$ and $z$ be as above. Then

\[
|X_{n,r}^*(u, z)| \leq 4|u|^r \frac{\Gamma(1 - 1/n)r!}{\Gamma(r + 1 - 1/n)} \left|1 + \sqrt{|w|}\right|^{2r-2}.
\]

**proof** This is Lemma 2.6 of [5].

**Lemma 7.4.** Let $N_{4,r}$ be the greatest common divisor of the numerators of the coefficients of $X_{4,r}(1 - 2x)$ and let $D_{4,r}$ be the least common multiple of the denominators of the coefficients of $X_{4,r}(x)$. Then the polynomial $(D_{4,r}/N_{4,r}) X_{4,r}(1 - 2x)$ has integral coefficients.

Moreover, $N_{4,r} = 2^r$ and

\[
D_{4,r} \frac{\Gamma(3/4)r!}{\Gamma(r + 3/4)} < 0.8397 \cdot 5.342^r \quad \text{and} \quad D_{4,r} \frac{\Gamma(r + 5/4)}{\Gamma(1/4)r!} < 0.1924 \cdot 5.342^r.
\]

**proof** Using the so-called Kummer transformation, we can write

\[
X_{4,r}(1 - 2x) = \frac{r(r + 1) \cdots (2r)}{(3/4)(7/4) \cdots (r - 1/4)}
\]

\[
= 2 F_1 (-r, -r - 1/4; -2r; 2x).
\]

Expanding the right-hand side, we find that

\[
X_{4,r}(1 - 2x) = \sum_{i=0}^{r} (-1)^i \frac{(r + 1) \cdots (2r - i)}{3 \cdot 7 \cdots (4r - 1)} \left( \begin{array}{c} r \\ i \end{array} \right) (4r - 4i + 1) \cdots (4r + 1) 2^{2r-i} x^i.
\]
Therefore, $2^r$ divides $N_{4,r}$ and by examining the coefficient of $x^r$, we see that $N_{4,r} = 2^r$. We know turn to the inequalities.

From the arguments in the proof of Proposition 2(c) from [9], we obtain

$$D_{4,r} < \exp(1.6708r + 3.43 \sqrt{r}) < 5.341227^r$$

for $r \geq 20000$. Since $\exp(0.000073r) > \exp(1.46) > 2$ for such values of $r$, the upper bound for $D_{4,r}$ holds for $r \geq 20000$.

For $r \geq 2$,

$$\frac{\Gamma(r + 5/4)}{\Gamma(1/4)r!} = \frac{5}{16} \prod_{i=2}^{r} \frac{i + 1/4}{i} < \frac{5}{16} \exp \left( \int_{1}^{r} \log \left( \frac{x + 1/4}{x} \right) dx \right)$$

$$< \frac{5}{16} \exp \left( \int_{1}^{r} \frac{dx}{4x} \right) \leq \frac{5}{16} r^{1/4}.$$

As a consequence, the inequalities in the statement of the lemma hold for $r \geq 20000$. A computation, similar to those described in the proof of Proposition 2 from [9], shows that the same inequalities holds for all smaller values of $r$.

**Lemma 7.5.** Let $\alpha_1, \alpha_2, A_r(X), B_r(X)$ and $P(X)$ be defined as in Lemma 3.1 and let $a, b, c$ and $d$ be complex numbers satisfying $ad - bc \neq 0$. Define

$$K_r(X) = aA_r(X) + bB_r(X) \quad \text{and} \quad L_r(X) = cA_r(X) + dB_r(X).$$

If $(x - \alpha_1)(x - \alpha_2) P(x) \neq 0$, then

$$K_{r+1}(x) L_r(x) \neq K_r(x) L_{r+1}(x),$$

for all $r \geq 0$.

**proof** This is Lemma 2.7 of [5].

**Lemma 7.6.** Let $\theta \in \mathbb{R}$. Suppose that there exist $k_0, l_0 > 0$ and $E, Q > 1$ such that for all $r \in \mathbb{N}$, there are rational integers $p_r$ and $q_r$ with $|q_r| < k_0 Q^r$ and $|q_r \theta - p_r| \leq l_0 E^{-r}$ satisfying $p_r q_{r+1} \neq p_{r+1} q_r$. Then for any rational integers $p$ and $q$ with $|q| \geq 1/(2l_0)$, we have

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{c |q|^\kappa + 1},$$

where $c = 2k_0 Q(2l_0 E)\kappa$ and $\kappa = \frac{\log Q}{\log E}$.

**proof** This is Lemma 2.8 from [5].

For the remainder of this section, we shall assume that $t$ is a fixed integer greater than 37. We shall also simplify our notation here to reflect the fact that we have $n = 4$. We shall use $R_r$ and $X_r$ instead of $R_{4,r}$ and $X_{4,r}$.

We now determine the quantities defined in the Lemma 7.1. Put

$$\alpha_1 = \sqrt{-2t}, \quad \alpha_2 = -\sqrt{-2t}, \quad c_1 = \frac{1 + \sqrt{-t/2}}{2}, \quad c_2 = \frac{1 - \sqrt{-t/2}}{2},$$

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then

\[ p_t(X) = X^4 + 4tX^3 - 12tX^2 - 8t^2X + 4t^2. \]

We define

\[ \tau = \frac{\sqrt{t} + \sqrt{t + 2}}{\sqrt{2}} \quad \text{and} \quad \rho = \sqrt{\tau^2 + 1} = \sqrt{t + 2 + \sqrt{t^2 + 2t}}. \]

for any positive integer \( t \).

The preliminary results above will now be used in order to obtain an effective measure of approximation to \( \beta^{(3)} \). By Lemma 7.2, we want to choose \( x \) so that

\[ \beta^{(3)} = \frac{c(x)w(x)^{1/4} + d(x)}{a(x)w(x)^{1/4} + b(x)}. \]

and for this purpose we will select \( x = 0 \). We have

\[ w = w(0) = \frac{2 + \sqrt{-2t}}{-2 + \sqrt{-2t}}, \]

\[ \left( \frac{\tau - i}{\tau + i} \right)^2 = w, \quad \left( \frac{\tau - i}{\rho} \right)^2 = \frac{\tau - i}{\tau + i}, \]

and so

\[ w^{1/4} = \frac{\tau - i}{\rho}. \]

Using the fact that \( \rho^2 = \tau^2 + 1 \), one can check that

\[ \frac{-i\tau - 1 + 4\rho}{-\tau + i - \rho} = -\tau + \rho, \]

and since

\[ a(0) = -10t, \quad b(0) = -10t, \quad c(0) = -10t\sqrt{-2t} \quad \text{and} \quad d(0) = 10t\sqrt{-2t}, \]

it follows that

\[ \beta^{(3)} = \frac{c(0)w^{1/4} + d(0)}{a(0)w^{1/4} + b(0)}. \]

Therefore, the first term in the expression for \( (-2t)^{r/2}C_r(0) \) in Lemma 7.2 disappears.

We now construct our sequence of rational approximations to \( \beta^{(3)} \). By Lemma 7.1, Lemma 7.2, we have that \( \lambda = -2t \), and moreover

\[ (-2t)^{r/2}A_r(0) = a(0)X_r^*(z(0), u(0)) + b(0)X_r^*(u(0), z(0)), \]
\[ (-2t)^{r/2}B_r(0) = c(0)X_r^*(z(0), u(0)) + d(0)X_r^*(u(0), z(0)), \]
\[ (-2t)^{r/2}C_r(0) = - (\beta^{(3)}a(0) - c(0)) u(0)^r R_r(w). \]

Therefore, we have

\[ (-2t)^{r/2}A_r(0) = -10t \left[ X_r^*(z(0), u(0)) + X_r^*(u(0), z(0)) \right], \]
\[ (-2t)^{r/2}B_r(0) = 5(-2t)^{3/2} \left[ X_r^*(z(0), u(0)) - X_r^*(u(0), z(0)) \right], \]
\[ (-2t)^{r/2}C_r(0) = 10t^{2r+1} \left[ \beta^{(3)} - \sqrt{-2t} \right] (-2 + \sqrt{-2t})^r R_r(w). \]
These quantities will form the basis for our approximations. We first eliminate some common factors. We can write $u(0) = t^2 (2 + \sqrt{-2t})$ and $z(0) = t^2 (2 + \sqrt{-2t})$. Using Lemma 7.2, Lemma 7.3, and triangular inequality, we obtain

\[
|A_r(0)| \leq 80t^{(3r+2)/2} (2 + t)^{r/2} \frac{\Gamma(3/4)r!}{\Gamma(r + 3/4)} |1 + \sqrt{w}|^{2r-2};
\]

\[
|B_r(0)| \leq 80\sqrt{r} t^{(3r+3)/2} (2 + t)^{r/2} \frac{\Gamma(3/4)r!}{\Gamma(r + 3/4)} |1 + \sqrt{w}|^{2r-2};
\]

and

\[
|C_r(0)| \leq 40 t^{(3r+2)/2} \varphi \left|\beta^{(3)} - \sqrt{-2t}\right| (2 + t)^{r/2} \frac{\Gamma(r + 5/4)}{r!(1/4)} |1 - \sqrt{w}|^{2r}.
\]

In the other hand, after some routine manipulations, we find that

\[
(-2t)^{r/2} A_r(0) = -\frac{10t^{2r+1}}{D_{4,r}} N_{4,r} \left\{ D_{4,r} \left[ (2 + \sqrt{-2t})^r X_r (1 - 2\eta) + (-2 + \sqrt{-2t})^r X_r (1 - 2\eta) \right] \right\}
\]

and

\[
(-2t)^{r/2} B_r(0) = \frac{5(-2)^{3/2} t^{2r+3/2} N_{4,r}}{D_{4,r}} \left\{ D_{4,r} \left[ (-2 + \sqrt{-2t})^r X_r (1 - 2\eta) - (2 + \sqrt{-2t})^r X_r (1 - 2\eta) \right] \right\};
\]

where $\eta = \frac{2}{2 + \sqrt{-2t}}$.

By Lemma 7.4, the quantities inside the braces can be expressed as $(-1)^r(e - f \sqrt{-2t}) \pm (e - f \sqrt{-2t})$, where $e$ and $f$ are rational integers, and recalling from Lemma 2.4 that $N_{4,r} = 2^r$, considering the cases of $r$ being even or odd separately, we find that

\[
P_r = \frac{D_{4,r} B_r(0)}{20 \cdot t^{(3r+3)/2}} \quad \text{and} \quad Q_r = \frac{D_{4,r} A_r(0)}{20 \cdot t^{(3r+3)/2}}
\]

are rational integers. We note for future reference that if $r$ is even, then $P_r$ will be divisible by $t$. The numbers $P_r/Q_r$ are those that will be used as the rational approximations to $\beta^{(3)}$. We have

\[
Q_r \beta^{(3)} - P_r = S_r,
\]

where

\[
S_r = \frac{D_{4,r} C_r(0)}{20 \cdot t^{(3r+3)/2}}.
\]

We wish to show that these are good approximations. This will be done by estimating $|P_r|$, $|Q_r|$ and $|S_r|$ from above. It is readily verified that

\[
|1 + \sqrt{w(0)}| = \frac{2t}{\sqrt{t^2 + 1}} = 2 - \frac{1}{2t} + O \left( \frac{11}{16t^2} \right);
\]

and in particular we have that

\[
|1 + \sqrt{w(0)}| < 2.
\]

Therefore, we have, for $t \geq 37$, that

\[
|Q_r| \leq 3.36 (22 \sqrt{t})^r.
\]

Similarly, one obtains for $t \geq 37$ that

\[
|P_r| \leq 4.75 \sqrt{t} (22 \sqrt{t})^r.
\]
Also we have
\[ \left| 1 - \sqrt{w(0)} \right|^2 = \frac{4}{\tau^2 + 1} \leq \frac{2}{t} . \]
With \( \phi \) as in Lemma 7.2, it can be shown that \( 2\phi/\pi \leq \sin \phi \) and
\[ \sin \phi = \text{Im } w(0) = -2\sqrt{2t/(t + 2)} \geq -2\sqrt{2/t} . \]
From the estimates for the roots of \( p_t(X) \) given in section 3, we know that \( 0 < \beta^{(3)} < 0.5 \), and so
\[ \varphi \left| \beta^{(3)} - \sqrt{-2t} \right| \leq \pi \left( 2 + \frac{1}{\sqrt{2t}} \right) \leq \pi \left( 2 + \frac{1}{\sqrt{2}} \right) . \]
Combining these inequalities with Lemma 7.4, we obtain
\[ |S_r| < 3.3 \left( \frac{11}{\sqrt{t}} \right)^r ; \]
for \( t \geq 37 \). Note also that since \( \beta^{(3)} \beta^{(4)} = -2t \), we have
\[ 2tQ_r + \beta^{(4)} P_r = -\beta^{(4)} S_r . \]
With these estimates, Lemma 7.6 gives the following.

**Lemma 7.7.** Suppose that \( t \geq 37 \). Define
\[ \kappa = \frac{\log (22\sqrt{t})}{\log (\sqrt{t}/11)} . \]
For \( j = 3, 4 \) and any rational integers \( p \) and \( q \), we have
\[ \left| p - \beta^{(j)} q \right| > \frac{1}{c_j |q|^\kappa} \]
for \( |q| \geq 1 \), where
\[ c_3 = 147.84\sqrt{t} \left( .6\sqrt{t} \right)^\kappa \quad \text{and} \quad c_4 = 4598\sqrt{t} \left( .44t \right)^\kappa . \]

**proof** In each case we will apply Lemma 7.5 and Lemma 7.6. First notice that \( P_r Q_{r+1} - P_{r+1} Q_r \) is a non-zero multiple of \( A_{r+1}(0)B_r(0) - A_r(0)B_{r+1}(0) \). Applying Lemma 7.5, with \( a = d = 1, b = c = 0 \) and \( x = 0 \), we see that \( P_r Q_{r+1} \neq P_{r+1} Q_r \). For \( \beta^{(3)} \), we put \( p_r = P_r \) and \( q_r = Q_r \), and apply Lemma 7.6 with \( k_0 = 3.36, l_0 = 3.3, E = \sqrt{t}/11 \) and \( Q = 22\sqrt{t} \). For \( \beta^{(4)} \), we take advantage of the fact that \( P_{2r} \) is divisible by \( t \). In this case, we set \( p_r = -2Q_{2r}, q_r = P_{2r}/t, s_r = S_{2r}/\beta^{(4)}/t \), and apply Lemma 7.6 accordingly. Since \( -4t - 2 < \beta^{(4)} < -4t \), we can put \( k_0 = 4.75/\sqrt{t}, l_0 = 13.2, E = t/11^2 = (\sqrt{t}/11)^2 \), and \( Q = 484t = (22\sqrt{t})^2 \). We see therefore that the value of \( \kappa \) in this case is the same as in the case of \( \beta^{(3)} \).

8 **Completion of the proof of Theorem 1.1**

The goal now is to solve equation (2.1) for all \( t > 40,000 \). We remark that the analysis here will use the fact that, by not restricting that a solution \((x, y)\) have the property that \( x/y \) is close to \( \beta^{(4)} \), as asserted in Lemma 3.1, we can restrict to the case that \( s \) in equation (3.3) satisfies the inequality \( s < \sqrt{2t} \). The can be seen by considering once again the equation
\[ s^2B^4 - 4tsAB^3 - 12tA^2B^2 + 4rtBA^3 + r^2A^4 = 1 \]
We will assume, in the construction of \( \mathcal{p} \), we define \( s_0 = \min(r, s) \), and multiply the above equation through by \( s_0^2 \). By defining \( (x, y) = (-sB, A) \) if \( s_0 = s \) and \( (x, y) = (rA, B) \) if \( s_0 = r \), one obtains the Thue equation
\[
x^4 + 4tx^3y - 12tx^2y^2 - 8txy^3 + 4t^2y^4 = s_0^2,
\]
where, as discussed above, \( s_0 \) divides \( 2t \), and also, \( s_0 \approx \sqrt{2t} \). It is not difficult to verify that if \( s_0 = s \), then the closest root of \( p_t(X) \) to \( x/y \) is \( \beta(4) \), while if \( s_0 = r \), then the closest root of \( p_t(X) \) to \( x/y \) is \( \beta(3) \).

The consequence of this remark is that one then needs only to solve the Thue inequality
\[
|p_t(x, y)| \leq 2t,
\]
as opposed to having \( 4t^2 \) on the right hand side.

We first obtain a lower bound for \( |y| \) for any solution to the Thue inequality.

We will assume, in the construction of \( \mathcal{p} \), given in section 3, that \( r < \sqrt{2t} \), as the case \( s < \sqrt{2t} \) actually gives a larger lower bound for \( |y| \). Recall that \( y = A \), where
\[
X - V_{2k+1} = \sqrt{V_{4k+1} - V_{2k+1}} = 2rA^2.
\]
Recall also that \( r < 2t \). We make use of the inequality \( U_{2k} > (2t)^{k-1} \), which is easily proved by induction. We note that because of the relation \( V_{4k+1} = V_{2k+1} + 2tU_{2k}^2 \), we can deduce the following expression
\[
\sqrt{V_{4k+1} - V_{2k+1}} = \frac{\sqrt{2tU_{2k}}}{\sqrt{V_{2k+1}/\sqrt{2tU_{2k}} + 1}}.
\]
Therefore,
\[
y^2 = A^2 > \frac{\sqrt{2tU_{2k}}}{4t} > (2t)^{2k-1}.
\]
By the fact that \( k > 24 \), we deduce that
\[
|y| > (2t)^{47/4}.
\]
As remarked earlier, the assumption \( s < \sqrt{2t} \) implies that \( x/y \) is closest to \( \beta(3) \). In other words, \( |x - \beta(3)y| = \min_{i=1,2,3,4} |x - \beta(i)y| \), and since \( |p_t(x, y)| \leq 2t \), it follows that \( |x - \beta(3)y| < (2t)^{1/4} \). Therefore, as \( y > 4 \), \( x/y > \beta(3) - (2t)^{1/4} / 4 \), and so
\[
|x/y - \beta(4)| > \beta(3) - (2t)^{1/4} / 4 - \beta(4) > 4t - (2t)^{1/4} / 4 + 3 - 21/(16t) \ldots.
\]
Similarly for \( i = 1, 2, 3 \),
\[
|x/y - \beta(i)| > \sqrt{2t} - (2t)^{1/4} / 4 - 1/2 \ldots.
\]
Therefore, because \( t > 40,000 \), it is readily deduced that
\[
|x/y - \beta(3)| < \frac{1}{15.9ty^4}.
\]
A similar argument for \( \beta(4) \) gives
\[
|x/y - \beta(4)| < \frac{1}{31.9ty^4}.
\]
Combining the above upper bound for \( |x/y - \beta(3)| \) with the lower bound proved in Lemma 7.7, it follows that
\[
|y|^{3 - \kappa} < \frac{c_3(t)}{15.9t}.
\]
For \( t > 40,000, \kappa < 2.9, \) and we conclude that

\[
3t^2 > \frac{147.84 \sqrt{t} \cdot (6 \sqrt{t})^3}{15.9t} > \frac{147.84 \sqrt{t} (6 \sqrt{t})^\kappa}{15.9t} > |y|^{-1} > (2^{21} t^{23})^{-1} > 4t^2,
\]

which is not possible.

Similarly for \( \beta^{(4)} \), combining the upper and lower bounds for \( |x/y - \beta^{(4)}| \) gives

\[
|y|^{3-\kappa} < \frac{c_3(t)}{31.9t^2}.
\]

Again since \( t > 40000 \), we have that \( \kappa < 2.9, \) and we therefore conclude that

\[
13.15 > \frac{4598 \sqrt{t} (44t)^3}{31.9t^2} > \frac{4598 \sqrt{t} (44t)^\kappa}{31.9t^2} > |y|^{-1} > (2^{21} t^{23})^{-1} > 4t^2,
\]

which is not possible for \( t > 10 \).

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