Positive Dependence of Exchangeable Sequences.

R. M. Burton and André Dabrowski

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Abstract

We show that infinite sequences of binary exchangeable random variables are strong FKG, a strong positive dependence property. This does not hold in general for multi-valued exchangeable sequences, and we obtain necessary and sufficient conditions for exchangeable sequences to be associated, a weaker property than FKG. For finite sequences, we obtain analogous results for weak association. As examples we look at Polyà urn processes and the match set distribution.

Positive Dependence of Exchangeable Sequences.

1. Definitions

2. Theorems.

3. Examples (Urn processes and match set distribution).

4. Comments on proofs.
1. Definitions.

- \( \{X_i : i \in I\} \) is exchangeable if its distribution is invariant under permutations.

  (a) Polyà urn process;

  (b) match set distribution.
1. Definitions.

- \( \{X_i : i \in I\} \) is strong FKG\(^*\) if the random variables have discrete distributions and for all \( n \) and \( \alpha, \beta \in \mathbb{R}^n \) we have
  \[
  P[\vec{X}_n > \alpha \land \beta]P[\vec{X}_n > \alpha \lor \beta] \geq P[\vec{X}_n \geq \alpha]P[\vec{X}_n \geq \beta].
  \]
  Operations are taken coordinate-wise.

- \( \{X_i : i \in I\} \) is associated\(^†\) if for all \( n \) and coordinate-wise non-decreasing functions \( f_1, f_2 : \mathbb{R}^n \mapsto \mathbb{R} \) we have
  \[
  \text{Cov}(f_1(X_1, \ldots, X_n), f_2(X_1, \ldots, X_n)) \geq 0,
  \]
  whenever this covariance exists.


1. Definitions.

• \( \{X_i : i \in I\} \) is \textbf{weakly associated}\(^*\) if for all \( k \) and \( \ell \) and coordinate-wise non-decreasing functions \( f_1 : \mathbb{R}^k \mapsto \mathbb{R} \) and \( f_2 : \mathbb{R}^\ell \mapsto \mathbb{R} \) we have

\[
\text{Cov}(f_1(X_i(1), \ldots, X_i(k)), f_2(X_i(k+1), \ldots, X_i(k+\ell)) \geq 0,
\]

whenever we have two disjoint sets of indices.

• \( \{X_i : i \in I\} \) is \textbf{PQD} if the \( f_1 \) and \( f_2 \) of weak association are indicators of events of the form \( [X_i(1) > u_1, \ldots, X_i(k) > u_k] \).

2. Theorems. Do exchangeable sequences have positive dependence properties?

\[ Y \sim Bern(0.5) \]

\[ X_i | Y \sim iid \begin{cases} 2 \times Bern(0.5) & \text{if } Y = 0 \\ 1 & \text{if } Y = 1 \end{cases} \]

\[ P[X_1 \in \{1, 2\}, X_2 \in \{2\}] - P[X_1 \in \{1, 2\}]P[X_2 \in \{2\}] = -1/16 < 0 \]

and so \( \{X_i\} \) is not even PQD. On the other hand,

**Theorem 1** A two-valued infinite exchangeable sequence is strong FKG.
2. Theorems.

**Theorem 2** Represent an infinite exchangeable real sequence as conditionally independent sequences with common distribution $M$, where $M$ is a random measure with values in $\mathcal{M}$. If $M$ takes values (with probability one) in a totally stochastically ordered subset of $\mathcal{M}$, then the sequence is associated.

A two-valued sequence is associated since any set of probability measures on $\{0, 1\}$ is totally ordered. However the condition is not necessary as there are $\{0, 1, 2\}$-valued associated exchangeable sequences where $M$ is not concentrated on a totally ordered subset of $\mathcal{M}$.
2. Theorems.

**Theorem 3** Let \( \{X_i : i \geq 1\} \) be an infinite (finite) sequence of exchangeable real-valued random variables. Then \( \{X_i : i \geq 1\} \) is (weakly) associated iff

\[
f(k + m, l + n) \geq f(k, l) \times f(m, n)
\]

for any positive integers \( k, l, m \) and \( n \), and where for every choice of non-decreasing function \( g : \mathbb{R} \mapsto \{0, 1, 2\} \)

\[
f(i, j) = P[g(X_1) \geq 1, \ldots, g(X_i) \geq 1, g(X_{i+1}) \geq 2, \ldots, g(x_{i+j}) \geq 2].
\]

In other words, (weak) association follows from PQD of all non-decreasing \( \{0, 1, 2\} \)-valued images of \( \{X_i : i \geq 1\} \) (\( \{X_i : i = 1 \ldots n\} \)).
Examples.

Association of infinite exchangeable sequences comes down to association of \(\{0, 1, 2\}\)-valued exchangeable sequences, and this follows if the set of measures on which the de Finetti \(M\) is concentrated is totally ordered.

Let \(m(q, p)\) put mass \(p\) on 0, \(q\) on 1 and \(1 - p - q\) on 2, and let \(A\) be the set of de Finetti measures \(M\) for which \(\{X_i : i \geq 1\}\) is associated.

Example: \(M\) is uniform on \(\{m(0.5, 0.5), m(0.5, 0), m(0, 0.5), m(1/3, 1/3)\}\). This an associated sequence but \(M\) is not concentrated on a totally ordered set.
3. Examples.

Fact: \( \mathcal{A} \) is a closed set.

Fact: \( \mathcal{A} \) contains every ‘pure’ measure, i.e. \( M \equiv m(p, q) \), since iid sequences are associated.

Fact: If \( \{X_i : i \geq 1\} \) is associated, so is \( \{2 - X_i : i \geq 1\} \). Thus \( \mathcal{A} \) is symmetric wrt this transform.

Fact: No combination of \( m(0, 1) \) and \( m(.5, 0) \) is associated. Thus \( \mathcal{A} \) is not convex.

Question: Can we characterize the geometry of \( \mathcal{A} \)? Is there a notion of positive dependence which will yield a more regular \( \mathcal{A} \)?
3. Examples.

Consider an urn containing a finite number of coloured balls, each colour assigned a distinct real value. Mix the balls, choose one and note the colour, and replace it in the urn with $h$ more of this colour. The sequence $\{X_i : i \geq 1\}$ of the numeric values observed is an exchangeable Polyà urn process*. 

3. Examples. Example: 3 colours labelled \( \{0, 1, 2\} \), \( h = 1 \). Then

\[
P[X_1 = i, \ X_2 = j] = \begin{cases} 
\frac{1}{12} & i \neq j \\
\frac{1}{6} & i = j
\end{cases}
\]

Thus \( P[X_1 = 2, X_2 = 1]P[X_1 = 1, X_2 = 0] < P[X_1 = X_2 = 1]P[X_1 = 2, X_2 = 0] \)
and so this exchangeable Polyà process is not strong FKG.

However,

**Theorem 4** *Every exchangeable Polyà urn process is associated.*
3. Examples.

The CLT* for exchangeable \( \{X_i : i \geq 1\} \) holds iff \( EX_1 = 0, EX_2 = 0, EX_1 X_j = 0 \) and \( E(X_1^2 - 1)(X_j^2 - 1) = 1 \). The first and third conditions imply \( Cov(X_1, X_j) = 0 \) for all \( j > 1 \), which for associated sequences implies that \( \{X_i : i \geq 1\} \) is in fact iid.

Associated exchangeable infinite sequences are either iid or do not satisfy the CLT. In particular exchangeable Polyà sequences do not satisfy the CLT.

Question: What possible limits are there for sums of Polyà urn variables?

3. Examples.

Let \( N > 1 \) and choose a permutation

\[
\sigma : \{1, 2, \ldots, N\} \mapsto \{1, 2, \ldots, N\}
\]

with uniform probability. Consider \( \{X_1, \ldots, X_N\} \) where \( X_i = I\{\sigma(i) = i\} \). This match set distribution is associated* by a lengthy examination of cases and coupling arguments. It is immediate from Theorem 3 that

**Theorem 5** The match set random variables \( \{X_1, \ldots, X_N\} \) are weakly associated.

4. Comments on proofs.

**Theorem 6**  TFAE for \( \{X_i : i \geq 1 \} \).

- \( Cov(I_A(X_1), I_B(X_2)) \geq 0 \) for increasing permutation-invariant sets \( A \) and \( B \).
- \( Cov(g_1(X_1), g_2(X_2)) \geq 0 \), for \( g_i \) non-negative combinations of indicators
- \( Cov(g_1(X_1), g_2(X_2)) \geq 0 \) for real-valued \( g_i \) applied to disjoint sets of indices.
- \( Cov(g_1(X_1), g_2(X_2)) \geq 0 \) for non-decreasing permutation symmetric real-valued \( g_i \).
- weak association
- association.
4. Comments on proofs. Weak association $\Rightarrow$ association.

Consider $f(X_1, \ldots, X_n) = I[X_1 > f_1, \ldots, X_n > f_n] = \prod I[X_i > f_i]$ and $g$ likewise. Then

$$Ef(X_1, \ldots, X_n)g(X_1, \ldots, X_n)$$

$$= \int Ef[\prod I[X_i > f_i]I[X_i > g_i]] \ dM(F) \text{ exch.}$$

$$= \int [\prod Ef(I[X_i > f_i]I[X_i > g_i])] \ dM(F) \text{ cond. iid}$$

$$\geq \int [\prod Ef(I[X_i > f_i])Ef(I[X_i > g_i])] \ dM(F) \text{ single } \{X_i\} \text{ is assoc.}$$

$$= \int [\prod Ef(I[X_i > f_i])Ef(I[X_{i+n} > g_{i+n}])] \ dM(F) \text{ Inf seq used here.}$$

$$= Ef(X_1, \ldots, X_n)g(X_{n+1}, \ldots, X_{2n})$$

$$\geq Ef(X_1, \ldots, X_n) \ Eg(X_1, \ldots, X_n). \text{ by assumed weak assoc.}$$
4. Comments on proofs.

Question: Can we obtain versions of these results for contractable sequences? Can we obtain extensions to exchangeable (contractable) arrays?