5 Matchings in Bipartite Graphs and Their Applications

5.1 Matchings

Definition 5.1 A matching $M$ in a graph $G$ is a set of edges of $G$, none of which is a loop, such that no two edges in $M$ have a common endpoint. A matching $M$ saturates a vertex $u$ in $G$, and $u$ is called $M$-saturated, if $u$ is an endpoint of an edge in $M$; otherwise, $u$ is called $M$-unsaturated.

Vertices $u$ and $v$ are said to be matched under $M$ if $uv \in M$.

A matching in $G$ that saturates every vertex is called perfect. A matching $M$ of $G$ with the largest number of edges among all matchings of $G$ is called maximum.

Example 5.2 Show that every perfect matching is maximum, but not every maximum matching in perfect. Give an example of a maximal matching that is not maximum.

![Figure 23: A maximum matching that is not perfect, a maximal matching that is not maximum, and a perfect matching in a graph.](image)

Definition 5.3 Let $M$ be a matching and $P$ a path in a graph $G$. The path $P$ is called $M$-alternating if the edges of $P$ lie alternately in $M$ and in $E(G) - M$. A path $P$ with at least one edge is called $M$-augmenting if it is $M$-alternating and both endpoints of $P$ are $M$-unsaturated.

![Figure 24: An $M$-alternating path and an $M$-augmenting path (thick edges: matching $M$; dashed edges: the path).](image)

It is easy to see that if a graph $G$ contains an $M$-augmenting path, then $M$ is not maximum, and $P$ can be used to obtain a larger matching (explain!). The following theorem shows that the converse holds, too.
Theorem 5.4 [Berge] A matching \( M \) in a graph \( G \) is maximum if and only if \( G \) contains no \( M \)-augmenting path.

Proof. (⇒): Assume \( M \) is a maximum matching in \( G \). Suppose \( G \) contains an \( M \)-augmenting path \( P = v_0v_1 \ldots v_k \). Note that \( k \) (the number of edges of \( P \)) must be odd since every other edge lies in \( M \), and both \( v_0v_1 \) and \( v_{k-1}v_k \) do not.

Define a set of edges \( M' \subseteq E(G) \) by
\[
M' = (M - \{v_1v_2, v_3v_4, \ldots, v_{k-2}v_{k-1}\}) \cup \{v_0v_1, v_2v_3, \ldots, v_{k-1}v_k\}.
\]
Then \( M' \) is a matching in \( G \) with \( |M'| = |M| + 1 \), contradicting the maximality of \( M \). Hence \( G \) cannot have an \( M \)-augmenting path.

(⇐): Assume \( G \) has no \( M \)-augmenting path, and suppose \( M \) is not maximum. Then
\[ |M| < |M^*| \]
for any maximum matching \( M^* \) of \( G \). Consider the subgraph \( H = G[M \oplus M^*] \) of \( G \) induced by the symmetric difference of \( M \) and \( M^* \) (that is, \( H \) contains precisely those edges of \( G \) that lie in exactly one of the matchings \( M \) and \( M^* \)). Each edge in \( H \) is incident with at most one edge of \( M \) and at most one edge of \( M^* \); hence, it has degree either 1 or 2. Thus \( H \) consists of connected components that are either cycles or paths. Moreover, the edges in these paths and cycles alternate between \( M \) and \( M^* \), whence the cycles must be of even length. Since \( |M| < |M^*| \), at least one component of \( H \) must contain more edges of \( M^* \) than of \( M \); hence, it must be a path \( P \) that starts and ends with an edge of \( M^* \). But then the endpoints of \( P \) are both \( M \)-unsaturated and thus \( P \) is an \( M \)-augmenting path, a contradiction. Hence \( M \) must be a maximum matching.

Exercise 5.5 Show that a connected graph in which every vertex has degree one or two is a path or a cycle.

Exercise 5.6 Show that the \( k \)-dimensional cube \( Q_k \) has a perfect matching.

Exercise 5.7 Find the number of distinct perfect matchings in the (labelled) graphs \( K_n \) and \( K_{n,n} \).

Exercise 5.8 Show that any tree has most one perfect matching.
5.2 Matchings in Bipartite Graphs

In this section we shall prove Hall’s Theorem, which, together with Berge’s Theorem 5.4, forms the basis of the Hungarian Method, that is, the algorithm that can be used to solve the Personnel Assignment Problem.

But first, some new notation. For a set $S \subseteq V(G)$ of vertices in a graph $G$ we define the neighbourhood of $S$ as

$$N_G(S) = \{x \in V(G) : x \sim_G u \text{ for some } u \in S\}.$$  

Notice that

$$N_G(S) = \bigcup_{u \in S} N_G(u).$$

**Theorem 5.9 [Hall]** Let $G$ be a bipartite graph with bipartition $(X,Y)$. Then $G$ contains a matching that saturates every vertex in $X$ if and only if

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq X. \quad (*)$$

**Proof.** ($\Rightarrow$): Assume $G$ contains a matching $M$ that saturates every vertex in $X$. Let $S$ be any subset of $X$. Since every vertex in $S$ is matched to a vertex in $N(S)$ under $M$, and since two distinct vertices in $S$ are matched to two distinct vertices in $N(S)$, we have $|N(S)| \geq |S|$.

($\Leftarrow$): Assume $G$ satisfies (*), and suppose $G$ has no matching that saturates every vertex of $X$. Let $M^*$ be a maximum matching in $G$. By our supposition, there exists a vertex $u$ in $X$ that is $M^*$-unsaturated. Define the following sets of vertices in $G$:

$$Z = \{v \in V(G) : \text{there exists an } M^* - \text{alternating path from } u \text{ to } v\}$$

$$S = Z \cap X$$

$$T = Z \cap Y$$

Since $M^*$ is a maximum matching and $u$ is $M^*$-unsaturated, by Theorem 5.4, $u$ is the only $M^*$-unsaturated vertex in $Z$ (otherwise, we would have an $M^*$-augmenting path, contradicting the maximality of $M^*$). Hence every vertex in $S - \{u\}$ is matched under $M^*$ to a (unique) vertex in $T$ and vice-versa, and so $|T| = |S| - 1$.

By the definition of $S$ and $T$, clearly $N(S) = T$. But then $|N(S)| = |T| = |S| - 1 < |S|$, contradicting the assumption (*).

\[\Box\]
Corollary 5.10  If \( G \) is a \( k \)-regular bipartite graph with \( k > 0 \), then \( G \) has a perfect matching.

Proof. Let \( G \) be a \( k \)-regular bipartite graph with bipartition \((X, Y)\). Since \( G \) is \( k \)-regular, we have \(|E| = k|X|\) and, on the other hand, \(|E| = k|Y|\). Since \( k > 0 \), we conclude that \(|X| = |Y|\).

We shall now show that the sufficient condition (*) from Hall’s Theorem is satisfied. Let \( S \) be any subset of \( X \). Furthermore, let \( E_S \) and \( E_{N(S)} \) be the set of edges incident with the vertices in \( S \) and \( N(S) \), respectively. By the definition of \( N(S) \), we have \( E_S \subseteq E_{N(S)} \). Therefore,

\[
k|N(S)| = |E_{N(S)}| \geq |E_S| = k|S|.
\]

It follows that \(|N(S)| \geq |S|\) for all \( S \subseteq X \). By Hall’s Theorem 5.9, it follows that \( G \) has a matching \( M \) that saturates every vertex in \( X \). But since \(|X| = |Y|\), this matching must be perfect.

Corollary 5.10 is also known as the marriage theorem, since it can be stated as follows: if every girl in a village knows exactly \( k \) boys and every boy knows exactly \( k \) girls, then each girl can marry a boy she knows and each boy can marry a girl he knows.

Exercise 5.11  A 1-factorization of a graph \( G \) is a partition of \( E(G) \) into perfect matchings of \( G \) (that is, a collection \( \{M_1, \ldots, M_k\} \) of perfect matchings of \( G \) such that every edge of \( G \) lies in exactly one of the perfect matchings \( M_1, \ldots, M_k \)).

Use Corollary 5.10 to prove that every \( k \)-regular bipartite graph has a 1-factorization. (Hint: use induction on \( k \).)

Exercise 5.11 can be restated as follows. At a dance party, each woman knows exactly \( k \) men and each man knows exactly \( k \) women. Show that \( k \) dances can be scheduled so that each person dances with each person he or she knows exactly once.

Exercise 5.12  Show that \( K_{2m} \) has a 1-factorization.

Exercise 5.13  Show that it is impossible to exactly cover an \( 8 \times 8 \) chessboard from which opposite corners have been cut using \( 1 \times 2 \) dominoes. (Hint: Model the chessboard by an appropriate graph. Show that a covering of the chessboard with dominoes corresponds to a perfect matching of this graph.)

Exercise 5.14  The Art History Department wishes to offer six courses in a semester. There are seven professors in the department, each of which can teach only certain courses, as shown in the table. Is it possible to assign the six courses to the professors so that no professor teaches more than one course?

<table>
<thead>
<tr>
<th>Course</th>
<th>Professor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antique</td>
<td>Ant, Bat, Cat, Dodo</td>
</tr>
<tr>
<td>Renaissance</td>
<td>Ant, Frog, Gnat</td>
</tr>
<tr>
<td>Baroque</td>
<td>Ant, Frog</td>
</tr>
<tr>
<td>Impressionism</td>
<td>Frog, Gnat</td>
</tr>
<tr>
<td>Modern</td>
<td>Cat, Gnat, Hog</td>
</tr>
<tr>
<td>Contemporary</td>
<td>Ant, Gnat</td>
</tr>
</tbody>
</table>
Exercise 5.15  A manufacturing process consists of five operations that can be performed simultaneously on five machines. The table below gives for each operation the time in minutes it takes to perform on each machine. Is it possible to assign the five operations to the five machines so that the whole manufacturing process is completed within 4 minutes? (Hint: Use an appropriate bipartite graph to model the problem. Show that a required assignment of operations to the machines corresponds to a perfect matching in the graph.)

<table>
<thead>
<tr>
<th></th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$O_2$</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$O_3$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$O_4$</td>
<td>4</td>
<td>8</td>
<td>3</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$O_5$</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Hall’s Theorem 5.9 was originally stated and proved in the context of systems of distinct representatives; that is, as Theorem 5.19 below.

Definition 5.16 Let $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ be a family of subsets of a set $S$. A systems of distinct representatives (also called transversal) for the family $\mathcal{A}$ is a set $\{a_1, a_2, \ldots, a_k\}$ of $S$ such that $a_i \in A_i$ for all $i = 1, 2, \ldots, k$, and $a_i \neq a_j$ for $i \neq j$.

Example 5.17 Find a systems of distinct representatives for the family $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$ of the set $S = \{a, b, c, d, e\}$, where $A_1 = \{a, b\}$, $A_2 = \{b, c, d\}$, $A_3 = \{c, d, e\}$, $A_4 = \{d, e\}$, $A_5 = \{a, b, e\}$.

Example 5.18 In a class of five students, four committees are formed (see table below). Is it possible to choose a president for each committee so that no student is a president of more than one committee?

<table>
<thead>
<tr>
<th>Committee</th>
<th>Members</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>Ann, Ben, Dan</td>
</tr>
<tr>
<td>$C_2$</td>
<td>Ben, Dan</td>
</tr>
<tr>
<td>$C_3$</td>
<td>Ben, Cora</td>
</tr>
<tr>
<td>$C_4$</td>
<td>Ann, Ben, Cora</td>
</tr>
</tbody>
</table>

Theorem 5.19 [Hall] A family $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of subsets of a set $S$ has a system of distinct representatives if and only if

$$|\bigcup_{j \in J} A_j| \geq |J| \quad \text{for all subsets } J \subseteq \{1, 2, \ldots, k\}.$$ 

Exercise 5.20 Prove Theorem 5.19 using Theorem 5.9.
5.3 Vertex Coverings and König’s Theorem

Definition 5.21 A (vertex) covering in a graph \( G \) is a set of vertices \( K \) in \( G \) such that each edge of \( G \) has at least one endpoint in \( K \). A minimum covering of \( G \) is a covering of \( G \) with the smallest number of vertices.

Example 5.22 Consider a network of corridors in a museum. The management wishes to install cameras at some or the intersections so that each corridor is covered by at least one camera. Such a placement of cameras corresponds to a vertex covering of the corresponding graph. A minimum covering represents a placement with the smallest number of cameras.

Lemma 5.23 If \( M \) is any matching of a graph \( G \) and \( K \) is any covering, then \( |M| \leq |K| \).

Proof. The covering \( K \) has to contain at least one vertex from each of the edges in \( M \). Since no vertex in \( K \) can be an endpoint of more than one edge of \( M \), we have \( |M| \leq |K| \). \( \square \)

Corollary 5.24 If \( M^* \) is a maximum matching of a graph \( G \) and \( \tilde{K} \) is a minimum covering, then

\[ |M^*| \leq |\tilde{K}|. \quad (*) \]

Example 5.25 Give an example of a graph for which equality in (*) does not hold.

Notice that your graph in Example 5.25 is not bipartite. Below we shall prove König’s Theorem: In any bipartite graph the size of a maximum matching equals the size of a minimum covering.

Lemma 5.26 In a graph \( G \), let \( M \) and \( K \) be a matching and covering, respectively, such that \( |M| = |K| \). Then \( M \) is a maximum matching and \( K \) is a minimum covering.

Proof. Let \( M^* \) and \( \tilde{K} \) be a maximum matching and minimum covering in the graph \( G \). Then, by Corollary 5.24,

\[ |M| \leq |M^*| \leq |\tilde{K}| \leq |K|. \]

But since \( |M| = |K| \), all the inequalities above must be equalities; that is, \( |M| = |M^*| \) and \( |\tilde{K}| = |K| \). Hence \( M \) is a maximum matching and \( K \) is a minimum covering. \( \square \)

Theorem 5.27 [König] In a bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum covering.

Proof. Let \( G \) be a bipartite graph with bipartition \((X, Y)\), and let \( M^* \) be a maximum matching in \( G \). We shall prove the theorem by constructing a covering \( \tilde{K} \) such that \( |M^*| = |\tilde{K}| \). From Lemma 5.26 it will follow that \( \tilde{K} \) is a minimum covering, and the theorem will follow.

Let \( U \) denote the set of all \( M^* \)-unsaturated vertices in \( X \). Then, similarly to the proof of Hall’s Theorem 5.9, define the following sets of vertices in \( G \):

\[
\begin{align*}
Z &= \{ v \in V(G) : \text{there exists an } M^* - \text{alternating path from } u \text{ to } v \text{ for some } u \in U \} \\
S &= Z \cap X \\
T &= Z \cap Y
\end{align*}
\]
Then, as in the proof of Hall’s Theorem 5.9, we have that $N(S) = T$ and that every vertex in $S - U$ is matched under $M^*$ to a unique vertex in $T$ and vice-versa, whence $|S - U| = |T|$. Now, define

$$
\tilde{K} = (X - S) \cup T.
$$

We’ll show that $\tilde{K}$ is a covering, that is, every edge of $G$ has at least one endpoint in $\tilde{K}$.

Take an edge $e \in E(G)$. Suppose $e$ has no endpoint in $\tilde{K}$. Then $e$ must have one endpoint in $S$ and the other in $Y - T$. But since $T = N(S)$, all edges with one endpoint in $S$ have the other endpoint in $T$, a contradiction. Hence $e$ has at least one endpoint in $\tilde{K}$, and $\tilde{K}$ is a covering of $G$.

Now, by the definition of $U$, all vertices of $X - U$ are $M^*$-saturated and $|M^*| = |X - U|$. But since vertices in $S - U$ are matched to vertices in $T$, we have

$$
|M^*| = |X - U| = |S - U| + |X - S| = |T| + |X - S| = |\tilde{K}|.
$$

Since $|M^*| = |\tilde{K}|$, by Lemma 5.26, $\tilde{K}$ is a minimum covering, and the theorem follows. \hfill \Box

König’s Theorem 5.27 is sometimes stated in the language of 0-1 matrices (that is, matrices whose entries are 0’s and 1’s).

**Theorem 5.28** [König-Egerváry] Let $A$ be a 0-1 matrix. Then the maximum number of 1’s, no two of which lie in the same row or column, equals the minimum number of rows or columns containing all 1’s.

**Exercise 5.29** Prove Theorem 5.28 using Theorem 5.27.

### 5.4 The Personnel Assignment Problem and the Hungarian Method

The **Personnel Assignment Problem** asks the following. Given $n$ workers $w_1, w_2, \ldots, w_n$ and $n$ jobs $j_1, j_2, \ldots, j_n$, where each worker is qualified for one or more of these jobs, is it possible to assign one qualified person to each of the $n$ jobs, so that no person is assigned to more than one job? Such an assignment, of course, can be modelled by a perfect matching in the corresponding bipartite graph where vertices $w_i$ and $j_m$ are adjacent if and only if worker $w_i$ is qualified for the job $j_m$.

We shall consider a slightly more general type of a problem. Let $G$ be a bipartite graph with bipartition $(X, Y)$. In the algorithm, we shall either construct a matching $M$ that saturates every vertex in the set $X$, or else determine that such a matching does not exist. The algorithm starts with an arbitrary matching $M$. If $M$ saturates every vertex in $X$, then
$M$ is the required matching. Otherwise, we choose an $M$-unsaturated vertex $u$ in $X$ and systematically search for an $M$-augmenting path starting at $u$. If such a path exists, then it is used to construct a larger matching from $M$ (as in the proof of Berge’s Theorem 5.4). Otherwise, the set $Z$ of endpoints of all $M$-alternating paths starting at $u$ is found, and as in the proof of Hall’s Theorem 5.9, the set $S = Z \cap X$ satisfies $|N(S)| < |S|$ so no matching saturating all vertices in $X$ exists.

The search for an $M$-augmenting path starting at $u$ proceeds by growing an $M$-alternating tree $H$ rooted at $u$. That is, $H$ is a tree in the graph $G$ with the following properties:

- $u \in V(H)$ and
- for all $v \in V(H)$, the unique $(u,v)$-path in $H$ is an $M$-alternating path.

Let $S = V(H) \cap X$ and $T = V(H) \cap Y$. The $M$-alternating tree $H$ is grown so that at any stage, either

1. all vertices in $S - \{u\}$ are $M$-saturated and matched under $M$ to vertices in $T$, or
2. $T$ contains an $M$-unsaturated vertex $y$ different from $u$.

Whenever Case (ii) is reached, we have an $M$-augmenting $(u,y)$-path, which is then used to enlarge the matching $M$.

In Case (i) we have $T \subseteq N(S)$. If $T = N(S)$, then (as in the proof of Hall’s Theorem 5.9), $|S| = |T| + 1 > |N(S)|$, and $G$ has no matching saturating $X$; the algorithm terminates. If $T \neq N(S)$, however, then there exists a vertex $y \in N(S) - T$ adjacent to a vertex $x'$ in $S$. If $y$ is $M$-unsaturated, then the edge $x'y$ is added to the tree $H$, resulting in Case (ii). On the other hand, if $y$ is $M$-saturated, then there is an edge $xy \in M$ with $x \not\in S$ (since all vertices of $S - \{u\}$ are already matched under $M$ to vertices in $T$). We now add edges $x'y$ and $yx$ to the tree $H$, resulting in Case (ii).

**Example 5.30** Perform the Hungarian Method on the graph in Figure 27.
Algorithm 5.31 Hungarian Method

procedure Hungarian(G: bipartite graph with bipartition (X, Y); M: matching in G)
while X is not M-saturated
begin
  u := M-unsaturated vertex in X
  S := {u}
  T := ∅
  if N(S) = T then exit
  else
  begin
    y := vertex in N(S) − T
    while y is M-saturated and N(S) ≠ T
    begin
      x := vertex in X matched to y under M
      S := S ∪ {x}
      T := T ∪ {y}
      if N(S) ≠ T then y := vertex in N(S) − T
    end
    if N(S) = T then exit
    else {y is M-unsaturated}
    begin
      P := M-augmenting (u, y)-path
      M := matching M enlarged by using P
    end
  end
end

{if N(S) = T, then G has no matching that saturates every vertex in X; otherwise, M is such a matching}

Note that the Hungarian Method does not necessarily construct a maximum matching in a bipartite graph G. Namely, if the algorithm terminates with N(S) = T, then the resulting matching M is not necessarily maximum because there might be an M-augmenting path in another part of the graph. In such case, build M-alternating trees from each M-unsaturated vertex in X until a matching saturating X is constructed or else, all such vertices result in N(S) = T. In the latter case, the matching is maximum (but does not saturate X).

Using König’s Theorem 5.27, this extended version of the Hungarian Method can also be used to find a minimum vertex covering in a bipartite graph. From the proof of Theorem 5.27, recall the following: if M* is a maximum matching in a bipartite graph G with bipartition (X, Y), U is the set of M*-unsaturated vertices in X, Z the set of vertices connected to vertices in U by M*-alternating paths, S = Z ∩ X, and T = Z ∩ Y, then K = (X − S) ∪ T is a minimum covering in G.

Exercise 5.32 Use the Hungarian Method to find a maximum matching and minimum covering in the graphs in Figures 27 and 28.
5.5 The Optimal Assignment Problem
and the Kuhn-Munkres Algorithm

In this section we take the problem of personnel assignment one step further. Again, assume
that we have \( n \) persons \( p_1, p_2, \ldots, p_n \) and \( n \) jobs \( j_1, j_2, \ldots, j_n \), where each person is qualified
to perform some of these jobs. In addition, if person \( p_i \) is qualified to perform job \( j_m \), then
\( w(p_i, j_m) \) denotes a measure of effectiveness of person \( p_i \) on the job \( j_m \). We would like to assign
the \( n \) jobs to the \( n \) persons, one job per person, so that total effectiveness is maximized.
This is known as the Optimal Assignment Problem.

We shall assume that we have a weighted complete bipartite graph \( G \) with weight function
\( w \) and bipartition \( (X, Y) \), where \( |X| = |Y| \). We wish to develop an algorithm that computes
a perfect matching in \( G \) of largest weight — we shall call such a matching \textit{optimal}.

Instead of finding all perfect matchings in \( G \) and determining the one of largest weight
(highly inefficient, as \( K_{n,n} \) has \( n! \) perfect matchings!), we use the Hungarian Method en-
hanced by a labelling procedure due to Kuhn and Munkres.

Definition 5.33 A \textit{feasible vertex labelling} in a weighted graph \( G \) is a function \( \ell : V(G) \rightarrow \mathbb{R} \)
such that
\[
\ell(x) + \ell(y) \geq w(xy) \quad \text{for all } xy \in E(G).
\]

Example 5.34 Show that
\[
\ell(x) = \max\{w(xy) : y \in Y, x \sim y\} \quad \text{if } x \in X
\]
\[
\ell(y) = 0 \quad \text{if } y \in Y
\]
is a feasible vertex labelling on a bipartite weighted graph \( G \) with bipartition \((X, Y)\).
Definition 5.35  Let $G$ be a weighted graph with a feasible vertex labelling $\ell$. The spanning subgraph of $G$ with edge set

$$E_\ell = \{ xy \in E(G) : \ell(x) + \ell(y) = w(xy) \}$$

is called the equality subgraph corresponding to $\ell$, and denoted by $G_\ell$.

The following result explains the importance of equality subgraphs for optimal matchings. It will form the basis for the Kuhn-Munkres algorithm below.

**Theorem 5.36**  Let $G$ be a graph with weight function $w$ and feasible vertex labelling $\ell$. If the equality subgraph $G_\ell$ has a perfect matching $M^*$, then $M^*$ is an optimal matching for $G$.

**Proof.** Assume $G_\ell$ contains a perfect matching $M^*$. Since $G_\ell$ is a spanning subgraph of $G$, $M^*$ is also a perfect matching for $G$. Now

$$w(M^*) = \sum_{e \in M^*} w(e) = \sum_{v \in V(G)} \ell(v)$$

since each $e \in M^*$ belongs to $E(G_\ell)$ and $M^*$ saturates $V(G)$. On the other hand, if $M$ is any perfect matching of $G$, then by the definition of a feasible labelling,

$$w(M) = \sum_{e \in M} w(e) \leq \sum_{v \in V(G)} \ell(v).$$

Hence $w(M^*) \geq w(M)$ and $M^*$ is an optimal matching.

The Kuhn-Munkres algorithm starts with an arbitrary feasible vertex labelling $\ell$ (e.g. the one from Example 5.34). For this labelling, we determine the equality subgraph $G_\ell$ and choose an arbitrary matching $M$ in $G_\ell$. We then apply the Hungarian Method; if it terminates by finding a perfect matching $M'$ in $G_\ell$ (that is, a matching in $G_\ell$ that saturates $X$), then by Theorem 5.36, $M'$ is an optimal matching for $G$. Otherwise, the Hungarian Method terminates in a matching $M'$ that is not perfect, and the $M'$-alternating tree $H$ that can not be grown any further and contains no $M'$-augmenting path in $G_\ell$. At this stage, the feasible vertex labelling $\ell$ is replaced by a new labelling $\ell'$ defined as:

$$\ell'(v) = \begin{cases} 
\ell(v) - a & \text{if } v \in S \\
\ell(v) + a & \text{if } v \in T \\
\ell(v) & \text{otherwise}
\end{cases},$$

where

$$a = \min \{ \ell(x) + \ell(y) - w(xy) : x \in S, y \in Y - T \},$$

and, as usual, $S = V(H) \cap X$ and $T = V(H) \cap Y$. Note that at this stage in the algorithm, $N_{G_\ell}(S) = T$. The equality subgraph $G_\ell$ is now replaced by $G_\ell'$, which contains both the matching $M'$ and the $M'$-alternating tree $H$. Lemma 5.37 below shows that in the new equality subgraph $G_\ell'$, the $M'$-alternating tree $H$ can be grown further. This process is repeated until an $M'$-augmenting path is found, which is then used to extend the matching $M'$ in $G_\ell'$. The algorithm ends by finding a perfect matching $M^*$ is some equality subgraph $G_\ell^*$. By Theorem 5.36, $M^*$ is an optimal matching for $G$. 63
Lemma 5.37 The new labelling \( \ell' \) defined in (*) has the following properties:

(i) \( \ell' \) is a feasible vertex labelling for \( G \).

(ii) The matching \( M' \) and the \( M' \)-alternating tree \( H \) are contained in \( G_{\ell'} \).

(iii) \( T \subset N_{G_{\ell'}}(S) \) and hence the \( M' \)-alternating tree \( H \) can be extended in \( G_{\ell'} \).

PROOF. (i) First note that since \( \ell \) is a feasible vertex labelling, the number \( a \) defined in (**) is non-negative. Take any edge \( xy \) of \( G \), where \( x \in X \) and \( y \in Y \). If \( x \notin S \) and \( y \in T \), then

\[
\ell'(x) + \ell'(y) = \ell(x) + (\ell(y) + a) \geq \ell(x) + \ell(y) \geq w(xy).
\]

If \( x \notin S \) and \( y \notin T \), then

\[
\ell'(x) + \ell'(y) = \ell(x) + \ell(y) \geq w(xy). \tag{1}
\]

If \( x \in S \) and \( y \notin T \), then

\[
\ell'(x) + \ell'(y) = (\ell(x) - a) + \ell(y) = (\ell(x) + \ell(y) - w(xy)) - a + w(xy) \geq a - a + w(xy) = w(xy).
\]

Finally, if \( x \in S \) and \( y \in T \), then

\[
\ell'(x) + \ell'(y) = (\ell(x) - a) + (\ell(y) + a) = \ell(x) + \ell(y) \geq w(xy). \tag{2}
\]

Hence \( \ell' \) is a feasible vertex labelling.

(ii) If \( xy \in E(G_\ell) \) such that either \( x \in S \) and \( y \in T \), or \( x \in X - S \) and \( y \in Y - T \), then (1) and (2) show that \( \ell'(x) + \ell'(y) = \ell(x) + \ell(y) \). Since \( xy \in E(G_\ell) \), we have \( \ell(x) + \ell(y) = w(xy) \), and hence \( \ell'(x) + \ell'(y) = w(xy) \) and \( xy \in E(G_{\ell'}) \).

Now, if \( xy \in M' \), then, indeed, either \( x \in S \) and \( y \in T \), or \( x \in X - S \) and \( y \in Y - T \), and hence \( xy \in E(G_{\ell'}) \). Furthermore, if \( xy \in E(H) \), then \( x \in S \) and \( y \in T \), and hence \( xy \in E(G_{\ell'}) \). Hence the matching \( M' \) and the \( M' \)-alternating tree \( M' \) are contained in \( G_{\ell'} \).

(iii) By (**), there exist \( x \in S \) and \( y \in Y - T \) such that \( a = \ell(x) + \ell(y) - w(xy) \). Hence

\[
\ell'(x) + \ell'(y) = (\ell(x) - a) + \ell(y) = (\ell(x) + \ell(y) - w(xy)) - a + w(xy) = a - a + w(xy) = w(xy).
\]

It follows that \( xy \in E(G_{\ell'}) \) and \( y \in N_{G_{\ell'}}(S) \). Thus \( T \subset N_{G_{\ell'}}(S) \) and the \( M' \)-alternating tree \( H \) can be extended in \( G_{\ell'} \). \hfill \Box

Example 5.38 It is convenient to represent a weighted complete bipartite graph \( G \) with weight function \( w \) and bipartition \( (X, Y) \), where \( X = \{x_1, x_2, \ldots, x_n\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \), by an \( n \times n \) matrix \( W = [w_{ij}] \), where \( w_{ij} = w(x_iy_j) \).

Use the Kuhn-Munkres Algorithm to find an optimal matching in the complete bipartite graph \( G \) with the matrix

\[
W = \begin{bmatrix}
3 & 5 & 5 & 4 & 1 \\
2 & 2 & 0 & 2 & 2 \\
2 & 4 & 4 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 3 & 3
\end{bmatrix}.
\]
\[
\begin{bmatrix}
3 & 5 & 5 & 4 & 1 \\
2 & 2 & 0 & 2 & 2 \\
2 & 4 & 4 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 3 & 3
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]
Algorithm 5.39 The Kuhn-Munkres Algorithm

procedure \textit{Kuhn − Munkres}($G$: weighted complete bipartite graph with bipartition $(X, Y)$; $\ell$: feasible vertex labelling)
{ assuming $|X| = |Y|$}
\begin{align*}
E_\ell & := \{xy \in E(G) : w(xy) = \ell(x) + \ell(y)\} \\
G_\ell & := (V(G), E_\ell) \\
M & := \text{arbitrary matching in } G_\ell
\end{align*}

\textbf{while} $X$ is not $M$-saturated in $G_\ell$
\begin{align*}
u & := M\text{-unsaturated vertex in } X \\
S & := \{u\} \\
T & := \emptyset
\end{align*}
repeat
\begin{itemize}
\item [if] $N_{G_\ell}(S) = T$ then \{update $\ell$\}
\begin{align*}
a & := \min\{\ell(x) + \ell(y) - w(xy) : x \in S, y \in Y - T\} \\
\text{for all } x \in S \text{ do } \ell(x) := \ell(x) - a \\
\text{for all } y \in T \text{ do } \ell(y) := \ell(y) + a \\
\text{update } G_\ell \{\text{now } T \subset N_{G_\ell}(S)\}
\end{align*}
\end{itemize}
y := vertex in $N_{G_\ell}(S) - T$
\textbf{while} $y$ is $M$-saturated and $N_{G_\ell}(S) \neq T$
\begin{align*}
x & := \text{vertex in } X \text{ matched to } y \text{ under } M \\
S & := S \cup \{x\} \\
T & := T \cup \{y\} \\
\text{if } N_{G_\ell}(S) \neq T \text{ then } y & := \text{vertex in } N_{G_\ell}(S) - T
\end{align*}
\textbf{end}
\textbf{until} $y$ is $M$-unsaturated
\begin{align*}
P & := \text{an } M\text{-augmenting } (u, y)\text{-path} \\
M & := \text{matching } M \text{ enlarged by using } P
\end{align*}
\textbf{end}
{ $M$ is an optimal matching}

Exercise 5.40 A diagonal of an $n \times n$ matrix is a set of $n$ entries such that no two belong to the same row or the same column. The weight of a diagonal is the sum of its entries.

Find a minimum-weight diagonal in the following matrix:
\[
\begin{bmatrix}
4 & 5 & 8 & 10 & 11 \\
7 & 6 & 5 & 7 & 4 \\
8 & 5 & 12 & 9 & 6 \\
6 & 6 & 13 & 10 & 7 \\
4 & 5 & 7 & 9 & 8
\end{bmatrix}
\]
Exercise 5.41 Estimate the order of time complexity of the Hungarian Method and the Kuhn-Munkres Algorithm.

References for Chapter 5: Bondy, Gross