Transcendence measures for algebraic points of Siegel modular functions

Villani Eric

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Abstract

We give an effective version of a result of Cohen, Shiga and Wolfart, which is a generalization to the case of Siegel spaces of arbitrary degree, of the classical theorem of Schneider on the modular invariant \( j(\tau) \). Given a point \( \tau \) of the Siegel space parametrizing a principally polarized abelian variety \( \mathcal{A} \) defined over \( \overline{\mathbb{Q}} \), we obtain a lower bound for the distance between \( \tau \) and algebraic points \( \beta \) of the Siegel space, in terms of the geometrical data of the problem. To achieve this, we establish a simultaneous measure of linear independence for periods of abelian integrals, using Baker’s method.

1 Introduction

1.1 Objective

A well known result from Schneider states that an elliptic curve \( \mathcal{E} \) defined over \( \overline{\mathbb{Q}} \) has complex multiplication if and only if the point \( \tau \) of the Poincaré half-plane \( \mathbb{H} \) parametrizing \( \mathcal{E} \) is algebraic. Faisant and Philipbert gave an effective demonstration of this theorem in 1987 (see [7]).

The theorem of Schneider was generalized in higher dimension by Cohen, Shiga and Wolfart (see [4, 18]) : an abelian variety \( \mathcal{A} \) of dimension \( g \) defined over \( \overline{\mathbb{Q}} \) has complex multiplication if and only if the point \( \tau \) of the Siegel half-space \( \mathbb{H}_g \) parametrizing \( \mathcal{A} \) is algebraic. The goal of this work is to give an effective version of this generalization.

Notations

- Let \( \overline{\mathbb{Q}} \) be the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \).
- For any integer \( g > 0 \), \( \mathbb{H}_g \) is the Siegel half-space of dimension \( g \) : the set of \( g \times g \) matrices with complex coefficients, with definite and positive imaginary part.

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• If $\tau \in \mathcal{H}_g$, we denote $A_\tau = C^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$ the principally polarized abelian variety of dimension $g$ attached to $\tau$; when it is defined over $\overline{\mathbb{Q}}$, we denote $h_F(A_\tau)$ its Faltings height (see [2]) and $K_{A_\tau}$ the definition field of $A_\tau$ and its 4-torsion points.

• For any element $\beta = (\beta_{i,j})_{1 \leq i,j \leq g}$ of $\mathcal{H}_g(\mathbb{Q})$, we denote $h(\beta)$ the logarithmic absolute height of the point $(1 : \beta_{1,1} : \ldots : \beta_{1,g} : \ldots : \beta_{g,1} : \ldots : \beta_{g,g})$ of $\mathbb{P}_g^2$.

• If $M = (m_{i,j})_{1 \leq i,j \leq g}$ is a complex matrix, we denote $||| M ||| = \sup_{1 \leq i,j \leq g} |m_{i,j}|$.

• For any real number $x > 0$, we denote finally $\log^+ (x) = \max \{1, \log(x)\}$, where $\log$ is the neperian logarithm.

We want to find a function $f$, such that for any $g$, $A_\tau$ and $\beta$ defined over a number field of degree $D$ over $\mathbb{Q}$, we have a lower bound, if $A_\tau$ is not CM:

$$||| \tau - \beta ||| \geq \exp \left( -C_1(g) \log B \left( Dh^+(A_\tau) \right)^{4g^2+1} \right)$$

If moreover $\text{End}(A_\tau) = \mathbb{Z}$, we have the lower bound:

$$||| \tau - \beta ||| \geq \exp \left( -C_2(g) \log B \left( Dh^+(A_\tau) \right)^5 \right)$$

The constants $C_1(g)$ and $C_2(g)$ depend only of $g$ and are (theoretically) effectively calculable. This theorem is a consequence of the following theorem 1.2.
1.2 Result in general situation

Let $\tau$ be an element of the Siegel space, parametrizing a principally polarized abelian variety $\mathcal{A} = \mathcal{A}_\tau$ over $\mathbb{C}$. The relation $\mathcal{A}/\mathbb{C} = \mathbb{C}^g/(\mathbb{Z}^g \oplus \tau\mathbb{Z}^g)$ allows to identify the tangent space at the origin $T_0\mathcal{A}$ to $\mathbb{C}^g$, and to define an embedding $\Psi$ of $\mathcal{A}$ into a projective space $\mathbb{P}_\nu$ with

$$\tilde{\Psi} : \mathbb{C}^g \to \mathbb{C}^{g+1} ; z \mapsto (\theta_{a,b}(\tau, 2z))_{(a,b) \in (\mathbb{Z}^g/2\mathbb{Z}^g)^2}$$

where $\theta_{a,b}$ are the usual theta functions with half integer characteristic and $\nu = 2^{2g}$. We will call "Siegel basis" the canonical basis $\left( \frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_n} \right)$ of $T_0\mathcal{A}$ (as seen as $\mathbb{C}^g$).

Since the jacobian matrix of $\tilde{\Psi}$ in $0$ is of rank $g$, we can extract from it a $g \times g$ invertible matrix $M$, and we define $\Omega_2 = \Omega_1^{-1} \tau$ where $\Omega_1 = \theta_{0,0}(\tau, 0)^{-1} M$. The column vectors of $\Omega_1^{-1}$ form a basis $B = \left( \frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_n} \right)$ of $T_0\mathcal{A}$ that we will call "Shimura basis" of the tangent space, in which the lattice of periods is written $\Omega_1\mathbb{Z}^g \oplus \Omega_2\mathbb{Z}^g$ (cf. [19] th.30.3). In addition, if $\mathcal{A}$ is defined over $\mathbb{Q}$, the $K_\mathcal{A}$-algebra generated by the functions $\frac{\theta_{a,b}}{\theta_{0,0}}$ is stable under the Shimura derivatives.

For any element $z$ of $T_0\mathcal{A}$, we denote $\|z\|_R$ the Riemann norm of $z$ attached to the polarization of $\mathcal{A}$. If we represent $z$ with the column vector $z$ of its coordinates in the Siegel basis of $T_0\mathcal{A}$, we have: $\|z\|_R = (z^* (3m \tau)^{-1} z)^{1/2}$ (where $z^*$ is the conjugate transpose of $z$).

**Theorem 1.2** Let $t, n, g_1, \ldots, g_n$ be integers $> 0$. We suppose $g = \sum g_i > t$. There exist real numbers $C_3(g)$ and $C_4(g)$ verifying the following property.

- let $K$ be a number field and $D$ its degree over $\mathbb{Q}$;
- for $1 \leq i \leq n$, let $\mathcal{A}_i$ be a principally polarized abelian variety of dimension $g_i$, defined over $K$; $\mathcal{Z}_i$ an element of the lattice of periods of $\mathcal{A}_i/\mathbb{C}$, of coordinates $(\omega_{i,k})_{1 \leq k \leq g_i}$ in a Shimura basis $\mathcal{B}_i$ of $T_0\mathcal{A}_i$;
- let $\beta_{k,j}$ ($k \in \{1, \ldots, t\}$, $j \in \{1, \ldots, g\}$) be elements of $K$, such that the matrix $[\beta_{k,j}]$ is of rank $t$;
- let $B \geq 1$ be a real number such that $\log B \geq \max_{1 \leq k \leq t} \{ h(\beta_{k,1}; \ldots; \beta_{k,g}) \}$

Under these conditions, we consider the abelian variety $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, and $\mathcal{W}$ the vector subspace of $T_0\mathcal{A}$ defined in the basis $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ by the equations $\sum_{j=1}^g \beta_{k,j} z_j = 0$, $k = 1, \ldots, t$ and we denote $\omega = (\omega_j)_{1 \leq j \leq g} = ((\omega_{i,k})_{1 \leq k \leq g_i})_{1 \leq i \leq n}$. Then one of the following is true:

- Because of the extraction process of $M$, the Shimura basis is not unique, but its arithmetical properties are the same whatever choice has been made.
1. the \( t \) numbers \( \Delta_k = \sum_{j=1}^{g} \beta_{k,j} \omega_j \) (\( 1 \leq k \leq t \)) are not all null and we have a lower bound:

\[
\log \max_{1 \leq k \leq t} |A_k| \geq -C_3(g)(D \log B + D \max_{1 \leq i \leq n} \{ h^+(A_i) \}) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^\tau\|_R \times \prod_{i=1}^{n} \left( D \max_{1 \leq i \leq n} \{ h^+(A_i) \}) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^\tau\|_R \right)^{g_i/t}
\]

2. there is an abelian subvariety \( \tilde{A} \) of \( A \), with a polarization of degree at most

\[
C_4(g) \deg \Psi A \times \max_{\sum a_i \omega_i \in \Omega A} \prod_{i=1}^{n} \left( D \max_{1 \leq i \leq n} \{ h^+(A_i) \}) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^\tau\|_R \right)^{d_i}
\]

such that the tangent space of \( \tilde{A} \) at the origin contains \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \), and verifies \( T_0 \tilde{A} + \mathcal{W} \neq T_0 A \).

This theorem will be proven with the use of Baker’s method, following the scheme of proof developed in the series of effective versions of Wüstholz theorem [21], by Philippon and Waldschmidt [16, 17], Hirata-Kohno and David [11, 6], and Gaudron [8].

### 1.3 Proof of theorem 1.1

We define \( n = 2g \), take \( A_1 = \cdots = A_n = A_{\tau} \), and for \( (\omega_1, \ldots, \omega_n) \) the usual basis of the lattice of periods of \( A_{\tau}/\mathbb{C} \). With this definition, the coefficients of the matrix \( \Omega_2 - \Omega_1 \beta \) can be seen as \( g^2 \) linear forms \( l_1, \ldots, l_{2g} \) in \( (\omega_1, \ldots, \omega_n) \).

We can write \( \|\omega_i^\tau\|_R^2 \) in terms of \( h(A_{\tau}) \) and \( D \). The column vector representing \( \omega_i^\tau \) in the Siegel basis of \( T_0 A_{\tau} \) can be

- \( \mathbf{1}_j \): a single 1 in \( j \)-th position and zeros, hence

\[
\|\omega_i^\tau\|_R^2 = \mathbf{1}_j^\tau (3m\tau)^{-1} (\mathbf{1}_j) = (3m\tau)^{-1}_{j,j}
\]

\[
\leq |||3m\tau^{-1}|||
\]

- a column \( \tau_j \) of the matrix \( \tau \) (we write the real and imaginary part of \( \tau_j = a + ib \). Since \( \tau \) is Siegel-reduced, the coefficients of \( a \) are in \([-1/2; 1/2]\]):

\[
\|\omega_i^\tau\|_R^2 = (a + ib)^\tau (3m\tau)^{-1} (a + ib) = a^\tau (3m\tau)^{-1} a + b^\tau (3m\tau)^{-1} b = a^\tau (3m\tau)^{-1} a + b^\tau \mathbf{1}_j = \sum_{1 \leq j,k \leq g} a_j a_k (3m\tau)^{-1}_{j,k} + (3m\tau)_{j,j}
\]

\[
\leq \frac{g^2}{4} |||3m\tau^{-1}||| + |||3m\tau|||
\]
From [12] V.4, we can deduce that there is a constant $c(g)$ such that in any case

$$\|\omega_i\|_R^2 \leq c(g) Dh(A_\tau).$$

We will use lemma 2.2 and the fact that

$$\Omega_2 - \Omega_1 \beta = \Omega_1 (\tau - \beta)$$

to link estimates from theorems 1.1 and 1.2.

1.3.1 if $A_\tau$ has not CM

To prove the first result of theorem 1.1, we suppose that the conclusion does not stand, so that we have for an abelian variety $A_\tau$ which is defined on $\overline{\mathbb{Q}}$ and has not CM:

$$||| \Omega_2 - \Omega_1 \beta ||| \leq g ||| \Omega_1 ||| ||| \tau - \beta |||$$

$$\leq g \exp(c_2(g)Dh(A_\tau)) \exp \left( -C_1(g) \log B (Dh(A_\tau))^{4g^2+1} \right) \exp \left( -C_2(g) \log B (Dh(A_\tau))^{4g^2+1} \right)$$

We now apply $g^2$ times the theorem 1.2 (once for each linear form $l_i$ with $t = 1$). If the first conclusion was true, we would have, taking (1) into account:

$$||| \Omega_2 - \Omega_1 \beta ||| \geq ||| \Omega_1 ||| ||| \tau - \beta |||$$

$$\geq \exp \left[ -C_3(g) (D \log B + Dh(A_\tau) + \frac{1}{2} \log^+ (c(g)Dh(A_\tau))) \right] \left( (Dh(A_\tau) + \frac{1}{2} \log^+ (c(g)Dh(A_\tau))) (c(g)Dh(A_\tau)) \right)^{2g^2+1} \exp \left( -C_2(g) \log B (Dh(A_\tau))^{4g^2+1} \right)$$

The values of $c_2(g)$ and $c'(g)$ are fixed (from lemma 2.2 and theorem 1.2), so if $C_1(g)$ is chosen big enough, this is not possible and the second conclusion of theorem 1.2 must be true. There is an abelian subvariety $\tilde{A}$ such that $\omega \in T_{0\tilde{A}}$ and $T_{0\tilde{A}} \cap \ker l_i \neq T_{0\tilde{A}}$. Since $\ker l_i$ has codimension 1, we necessarily have $T_{0\tilde{A}} \subset \ker l_i$, and $l_i(\omega) = 0$.

This is true for all $i$, so we have that $\tau = \beta$, and $\tau$ has its coefficients in $\overline{\mathbb{Q}}$. Hence by [18], $A_\tau$ has CM, and we have a contradiction.

1.3.2 if $\text{End}(A_\tau) = \mathbb{Z}$

To prove the second result of theorem 1.1, we suppose that we have for an abelian variety $A_\tau$ defined on $\overline{\mathbb{Q}}$ such that $\text{End}(A_\tau) = \mathbb{Z}$ and:

$$||| \Omega_2 - \Omega_1 \beta ||| \leq g ||| \Omega_1 ||| ||| \tau - \beta |||$$

$$\leq g \exp(c_2(g)Dh(A_\tau)) \exp \left( -C_2(g) \log B (Dh(A_\tau))^{4g^2+1} \right)$$

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We then apply theorem 1.2 (simultaneously with all linear forms $l_i$, with $t = g^2$). If the first conclusion was true, we would have

$$|||\Omega_2 - \Omega_1\beta||| \geq \max_{1 \leq i \leq g^2} |A_i|$$

$$\geq \exp\left[ -C_3(g)(D \log B + Dh^+(A_\tau) + \frac{1}{2} \log^+ (c(g)Dh^+(A_\tau)) \right]$$

$$\left( \left( Dh^+(A_\tau) + \frac{1}{2} \log^+ (c(g)Dh^+(A_\tau)) \right) (c(g)Dh^+(A_\tau)) \right)^{g^2/2}$$

$$\geq \exp\left( -c''(g) \log B \left( Dh^+(A_\tau) \right)^5 \right)$$

If $C_2(g)$ is chosen big enough, this is not possible and, from the second conclusion of theorem 1.2, there is a proper abelian subvariety $\tilde{A}$ of $A_2^{2g}$ such that $\omega \in T_0\tilde{A}$.

From [13], theorem 5.1, using duality and a Siegel lemma (prop. 8.1 and 9.1), there are $(\gamma_i)_{1 \leq i \leq 2g} \in \text{End}(A_\tau)^{2g}$ not all null such that

$$\sum_{i=1}^{2g} \gamma_i \omega_i = 0$$

Since the periods $(\omega_i)$ are linearly independent over $\mathbb{Z} = \text{End}(A_\tau)$, this is not possible.

2 Generalities on abelian varieties and their vector fields

In this section, we collect various results that may be of interest independently of the theorems. Proposition 2.4 gives explicit estimates in terms of the height of the abelian variety for the formal logarithms.

We consider here a point $\tau$ of Siegel half-space $H_g$, and the principally polarized abelian variety $A_\tau$ of dimension $g$ attached to $\tau$.

2.1 Geometry of embeddings

2.1.1 Classical theta embeddings

We say that $\tau$ is Siegel-reduced if:

1. for all $k$, $1 \leq k \leq g$, and all $\xi \in \mathbb{Z}^g$ with relatively prime coordinates $\xi_k, \ldots, \xi_g$, we have $^t \xi (3m(\tau)) \xi \geq (3m(\tau))_{k,k}$.

2. for all $k$, $1 \leq k \leq g - 1$, $(3m(\tau))_{k,k+1} \geq 0$.  

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3. the coefficients of $\Re(\tau)$ are in $[-1/2; 1/2]$.

4. for any $\sigma \in Sp(2n, \mathbb{C})$, $|\det(\sigma \tau)| \leq |\det \tau|$.

We identify the complex abelian variety $A_{\tau}/\mathbb{C}$ with the analytic manifold $\mathbb{C}^g/(\mathbb{Z}^g \oplus \tau \mathbb{Z}^g)$.

We define the theta function with half-integer characteristic in variable $z \in \mathbb{C}^g$:

$$\theta_{a,b}(\tau, 2z) = \sum_{n \in \mathbb{Z}^g} \exp \left( 2i\pi \left( \frac{1}{2} t(n + a)\tau(n + a) + t(n + a)(2z + b) \right) \right)$$

where $a$ and $b$ are elements of $\mathbb{Z}_g = (\frac{1}{2} \mathbb{Z}/\mathbb{Z})^g$.

To simplify the notation, we will denote them by $\theta_0(\tau, 2z)$, $\theta_1(\tau, 2z)$, ..., $\theta_\nu(\tau, 2z)$ (where $\nu = 4g - 1$), with $\theta_0$ such that $|\theta_0(\tau, 0)| = \max_j |\theta_j(\tau, 0)|$, so that in the neighborhood $0, z \mapsto \theta_0(\tau, 2z)$ doesn’t vanish.

Identifying $T_0A_{\tau}$ and $\mathbb{C}^g$ with the Siegel basis, we rewrite them as

$$\vartheta_j : T_0A_{\tau} \rightarrow \mathbb{C}$$

$$\mathbb{C}^g \mapsto \theta_j(\tau, 2z)$$

We have now an analytic map :

$$\varphi : T_0A_{\tau} \rightarrow \mathbb{P}_\nu$$

$$\mathbb{C}^g \mapsto (\vartheta_0(\tau, 2z) : \vartheta_1(\tau, 2z) : \ldots : \vartheta_\nu(\tau, 2z))$$

which by taking quotient, defines a projective embedding $A_{\tau} \rightarrow \mathbb{P}_\nu$. In this embedding in $\mathbb{P}_\nu$, the principally polarized abelian variety $A_{\tau}$ has degree (cf. [12]) $\deg A_{\tau} = 4^g$.

When $A_{\tau}$ is defined over $\overline{\mathbb{Q}}$, so is $\varphi(0)$, and we set

$$h(A_{\tau}) = h(\vartheta_0(0) : \vartheta_1(0) : \ldots : \vartheta_\nu(0))$$

We also note, in a neighborhood $U$ of the origin :

$$\psi : U \rightarrow \mathbb{C}^{\nu+1}$$

$$z \mapsto \left( \vartheta_0(\tau, 2z) : \vartheta_1(\tau, 2z) : \ldots : \vartheta_\nu(\tau, 2z) \right)$$

We will need analytic estimates on the thetas, which we collect in the following lemma.

**Lemma 2.1** There is a constant $c_1$ (depending only of $g$) such that for any $z$ in $\mathbb{C}^g$, we have :

$$\left| \log \max_{0 \leq j \leq \nu} |\vartheta_j(\tau, 2z)| - 4\pi \|3m\tau\|_R^2 \right| \leq c_1 Dh(A_{\tau})$$

**Proof:** This is theorem 3.1 of [5]. (Slightly sharper estimates exists, see for example lemma A.17 of [9], but their use in our proof would not improve the result)
2.1.2 Embedding of a product

Using the notations of theorem 1.2, we now consider for $1 \leq i \leq n$, points $\tau_i$ of Siegel spaces $H_{g_i}$, and we let $\mathcal{A}_i = \mathcal{A}_{g_i}$ be the corresponding abelian varieties, which we suppose to be defined over $\overline{\mathbb{Q}}$.

We denote $\varphi_i : T_0 \mathcal{A}_i \to \mathbb{P}_{\nu_i}$ and $\psi_i : T_0 \mathcal{A}_i \to \mathbb{C}^{\nu_i+1}$ the functions attached to $\mathcal{A}_i$ defined earlier (in 2.1.1).

We denote $\nu_a = t$ and define the standard embedding of $G^t_a = G^{\nu_a}_a$ in $\mathbb{P}_{\nu_a}$ by

$$\varphi_a : G^t_a \to \mathbb{P}_{\nu_a} \quad z \mapsto (1 : z_1 : \ldots : z_{\nu_a}).$$

We denote by $G := \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \times G^t_a$, and consider the multiprojective embedding in $\mathbb{P} = \mathbb{P}_{\nu_1} \times \ldots \times \mathbb{P}_{\nu_n} \times \mathbb{P}_{\nu_a}$:

$$\Phi : T_0 G \to \mathbb{P} \quad z \mapsto (\varphi_1(z_1), \ldots, \varphi_n(z_n), \varphi_a(z_a))$$

and in a neighborhood $\mathcal{U}$ of the origin:

$$\Psi : \mathcal{U} \to \mathbb{C}^{\nu_1+1} \times \ldots \times \mathbb{C}^{\nu_n+1} \times \mathbb{C}^{t+1} \quad z \mapsto (\psi_1(z_1), \ldots, \psi_n(z_n), 1, z_a)$$

2.2 Derivatives

Shimura basis and derivatives

Supposing now that $\mathcal{A}_a$ is defined over a number field $K$ of degree $D$ over $\mathbb{Q}$, we have explained in the introduction a way of defining the Shimura basis.

For a vector $\overline{z} \in T_0 \mathcal{A}_a$, if we denote its coordinates in the Siegel basis by $(z_i)_{1 \leq i \leq g} = z$ and the ones in the Shimura basis by $(\zeta_i)_{1 \leq i \leq g} = \zeta$, then we have:

$$\zeta = \Omega_1 z, \quad z = \Omega_1^{-1} \zeta, \quad \vartheta_j(\overline{z}) = \theta_j(\tau, 2z) = \theta_j(\tau, 2\Omega_1^{-1} \zeta).$$

**Lemma 2.2** There is a constant $c_2$ (depending only of $g$) such that the coefficients of the matrices $\Omega_1$ and $\Omega_1^{-1}$ are upper bounded by $\exp(c_2 Dh(\mathcal{A}_a))$.

**Proof:** This is Lemma 4.14 of [5] $\square$

**Lemma 2.3** There is a constant $c_3$ (depending only of $g$) such that for any $(m, j) \in \{0, \ldots, \nu\} \times \{1, \ldots, g\}$, there exist elements $(a_{k,l})_{0 \leq k,l \leq \nu}$ of $K$ of height at most $c_3 h(\mathcal{A}_a)$ such that in a neighborhood of the origin:

$$\frac{\partial}{\partial \zeta_j} \left( \frac{\partial m}{\partial \nu_0} \right) = \sum_{0 \leq k,l \leq \nu} a_{k,l} \left( \frac{\partial_k}{\partial \nu_0} \right) \left( \frac{\partial_l}{\partial \nu_0} \right)$$
proof This is theorem 4.2 of [5].

Formal logarithm on abelian variety

We will also need estimates of derivatives for non-archimedian places. They are given by the

Proposition 2.4 Let g be an integer \( \geq 1 \). There is a constant \( c_4 \) depending only of \( g \) satisfying the following property. Let \( \tau \) be an element of \( S_g \) such that the abelian variety \( A = A_\tau \) is defined over \( \mathbb{Q} \). Denote by \( D \) the degree over \( \mathbb{Q} \) of the field \( K = K_A \), by \( \mathcal{O}_K \) its integers ring and by \( (d\zeta_1, \ldots, d\zeta_g) \) the dual basis of a Shimura basis of \( T_0A \).

There is a non-zero integer \( r \leq \exp (c_4 Dh_F(A)) \), and \( K \)-rational functions \( X_1, \ldots, X_g \) on \( A \) forming a system of parameters for \( A \) in \( 0 \), and such that if we write \( T_i = X_i \circ \exp_A \),

\[ i) \ rd\zeta_j \in \bigoplus_{k=1}^g \mathcal{O}_K[[T_1, \ldots, T_g]]dT_k \text{ for all } j = 1, \ldots, g. \]

\[ ii) \ r\psi \circ [r] \in (\mathcal{O}_K[[T_1, \ldots, T_g]])^r + 1, \text{ where } [r] \text{ denote the homothety of ratio } r \text{ on } T_0A, \text{ and } \psi = \left( \frac{\theta_0}{\varepsilon_0}, \ldots, \frac{\theta_g}{\varepsilon_0} \right). \]

Remark : The integration of the formulas \( rd\zeta_j \in \bigoplus_{k=1}^g \mathcal{O}_K[[T_1, \ldots, T_g]]dT_k \) gives an expression of the formal logarithm of \( A \) under the form \( \zeta_j = Z_j(T_1, \ldots, T_g) \) where the \( Z_j \) are formal series. In the Formula (ii), one must see \( \psi(r^{\varepsilon}) \) as the formal serie obtained by composing \( (T_1, \ldots, T_g) \mapsto \zeta_j = Z_j(T_1, \ldots, T_g), \varepsilon = \sum_{j=1}^g \zeta_j e_j \) and \( r^{\varepsilon} \mapsto \psi(r^{\varepsilon}) \).

This proposition is a generalization of Chudnovsky trick [3] (1.12 p359) in the elliptic case by David-Hirata [6]. To prove it, we first need a multidimensional effective version of Eisenstein theorem, inspired from Grinspan [10] prop. 4.1.

Lemma 2.5 (Eisenstein) Let \((m, n)\) be two strictly positive integers; \( L \) a number field; \( P_1, \ldots, P_g \) polynomials in \((m+n)\) variables, with coefficients in \( \mathcal{O}_L \), of heights at most \( h \); and \( n \) formal series \((Y_1, \ldots, Y_n) \in (L[[X_1, \ldots, X_m]])^n \) without constant terms and verifying the system:

\[ \forall 1 \leq i \leq n; \ P_i(X_1, \ldots, X_m, Y_1, \ldots Y_n) = 0. \]

Suppose that

- for all \( 1 \leq i \leq n \), \( P_i(0, \ldots, 0) = 0 \)

- The matrix \( M = \left[ \frac{\partial P_i}{\partial X_j}(0) \right]_{1 \leq i,j \leq n} \) is invertible, and we denote \( \Delta \in \mathcal{O}_L \) its determinant.

Then, \( h(\Delta) \leq n(h + \log n) \) and

\[ \forall 1 \leq j \leq n; \ Y_j(\Delta^2 X_1, \ldots, \Delta^2 X_m) \in \mathcal{O}_L[[X_1, \ldots, X_m]]. \]
Proof: Denote $\mathbf{X} = (X_1, \ldots, X_m)$ et $\mathbf{Y} = (Y_1, \ldots, Y_n)$. Rewriting each $P_i(\mathbf{X}, \mathbf{Y})$ under the form

$$P_i(\mathbf{X}, \mathbf{Y}) = Q_i(\mathbf{X}) + \sum_{j=1}^{n} R_{i,j}(\mathbf{X})Y_j + S_i(\mathbf{X}, \mathbf{Y})$$

where the polynomials $S_i$ are of degree at least 2 in $\mathbf{Y}$. We write now the system in vectorial notation, with $R(\mathbf{X}) = [R_{i,j}(\mathbf{X})]_{1 \leq i,j \leq n}$:

$$[P_i(\mathbf{X}, \mathbf{Y})]_{1 \leq i \leq n} = 0 \iff R(\mathbf{X}).\mathbf{Y} = -Q(\mathbf{X}) - S(\mathbf{X}, \mathbf{Y})$$

$M = R(\overline{0})$ is invertible, so $R(\mathbf{X}) \in GL_n(L[[\mathbf{X}]])$:

$$\det R(\mathbf{X}) = \Delta + \rho(\mathbf{X})$$

where $\rho(\mathbf{X}) \in O_L[\mathbf{X}]$, without constant term, and we can write:

$$R(\mathbf{X})^{-1} = t(\text{com} R(\mathbf{X})) \frac{1}{\Delta} \left( \sum_{k \geq 0} \left( \frac{-\rho(\mathbf{X})}{\Delta} \right)^k \right)$$

So, in $(L[[\mathbf{X}]])^n$, we have:

$$\mathbf{Y} = -R(\mathbf{X})^{-1}. (Q(\mathbf{X}) + S(\mathbf{X}, \mathbf{Y}))$$

By identifying the coefficients before each monomial $\mathbf{X}^e = X_1^{k_1} \ldots X_n^{k_n}$, we get induction relations on $|e|$, by observing that there is no constant term, and that we can work on monomials of given length in arbitrary order. We then see that the $Y_j$ can be written as announced. $\square$

Proof of proposition 2.4:

We denote $T_i = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial y_0}(0)$. Denote $K$ the field of thetanullwertes and $D$ its degree.

By construction, $T_1, \ldots, T_g$ form a parameters system (composed of rational functions on $\mathcal{A}$ and regular in 0) of the local ring of $\mathcal{A}$ in 0. Hence we can write for all $i > g$, $T_i \in K[[T_1, \ldots, T_g]]$.

Moreover $(T_1, \ldots, T_g)$ verify a system of quadratic polynomial equations with coefficients in $K$, of heights at most $\exp(Dh(\mathcal{A}))$ (cf. [14] Chapter II, paragraph 6). Since $\mathcal{A}$ is smooth at the origin, the differentials of these polynomials in 0 generate in $T_{0,\mathcal{A}}^{\mathcal{A} \leftrightarrow \nu}$ a subspace of codimension $g$, and we can extract a system of $\nu - g$ polynomials verifying the hypothesis of lemma 2.5, with $n = \nu - g$, $m = g$, $L = K$, $h = Dh(\mathcal{A})$, $Y_1 = T_{g+1}, \ldots, Y_n = T_{\nu}$ and $X_1 = T_1, \ldots, X_g = T_g$.

We conclude then

$$\forall i \in [g + 1; \nu]; T_i \in O_K[[\Delta^2 T_1, \ldots, \Delta^2 T_g]].$$
and since \( \psi = (1, T_1, \ldots, T_\nu) \), this establish the second conclusion.

From lemma 2.3, the matrix \( (\partial T_i / \partial \zeta_j) \) is an element of \( \mathcal{M}_g \left( \frac{1}{\delta} \mathcal{O}_K[[\Delta^2 T_1, \ldots, \Delta^2 T_\nu]] \right) \)

where \( \delta \) is an integer of height at most \( c_3(g) h(A_\tau) \).

Moreover, when \( (T_i)_{i=1}^\nu = 0 \), \( (\partial T_i / \partial \zeta_j) = I_g \). So \( (\partial T_i / \partial \zeta_j) \) is invertible in \( \mathcal{M}_g(K[[T_1, \ldots, T_\nu]]) \), and

\[ (\partial T_i / \partial \zeta_j)^{-1} \in \mathcal{M}_g \left( \frac{1}{\delta} \mathcal{O}_K[[\Delta^2 T_1, \ldots, \Delta^2 T_\nu]] \right). \]

By taking \( r = \delta \Delta^2 \) and inverting \( (d T_i)_{1 \leq i \leq g} = \left( \frac{\partial T_i}{\partial \zeta_j} \right)_{i,j} \) \( (d \zeta)_{1 \leq j \leq g} \), we finally get the proposition. \( \square \)

3 Differential operators

We take in this section the notation introduced in paragraph 2.1.2: we are working with a group \( G := A_1 \times \cdots \times A_n \times G'_{\alpha} \).

3.1 Upper bound of derivatives

**Lemma 3.1** Let \( P \) be a polynomial in \( \mathbb{C}[X_1, \ldots, X_n, X_a] \), such that the sum of the modulus of its coefficients is at most \( H \), and for all \( X_i = (X_{i,0}, X_{i,1}, \ldots, X_{i,\nu_i}) \) (resp. \( X_a = (X_{a,0}, X_{a,1}, \ldots, X_{a,t_a}) \) \( P \) is homogenous of degree \( N_i \) (resp. \( N_a \)).

Let \( t = (t_1, \ldots, t_m) \in \mathbb{N}^m \), \( (x_1, \ldots, x_m, u) \) be vectors of \( T_0 G \), and for \( 1 \leq i \leq m \), we denote \( (x_{i,j})_{1 \leq j \leq g + t} \) the coordinates of \( x_i \) in the Shimura basis, and \( X = \{ x_1, \ldots, x_m \} \). Then:

\[
\log \left| \frac{1}{t!} D_X^t P \circ \Phi(u) \right| \leq \\
\min(N_a, |t|) \log \xi_a + |t| \log \xi + |t| \log m + \log H \\
+ t N_a \log \max_{i \in \{g+1, \ldots, g+t\}} (1 + |u_i|) + c_1 \sum_{i=1}^n N_i (1 + \sqrt{|u_i| + |u_i|})^2
\]

where \( \xi_a = \max_{j \in \{g+1, \ldots, g+t\}} (1, |x_{i,j}|) \), \( \xi = \max_{j \in \{1, \ldots, g\}} (1, |x_{i,j}|) \)

**Proof:** This is lemma 6.22 of [8]. (The idea is to write each \( x_i \) as a sum of elements of the Siegel basis of \( G \), and develop the derivative \( D_X^t P \circ \Phi(u) \). Then a direct use of Cauchy formula and theta-estimates from 2.1 gives the result) \( \square \)
3.2 Lower bounds of derivatives

In the following lemma, we consider a period $\omega$ of $G = A_1 \times \ldots A_n \times \mathbb{C}^t$ and a family of $g$ vectors $f = \{f_1, \ldots, f_g\}$ of $T_0 G$:

$$f_j = e_j - \sum_{k=1}^{t} \beta_{j,k} e_{g+k}$$

where the $\beta_{j,k}$ are algebraic of height at most $\log B$ and the $(e_j)$ form the Shimura basis $E$ of $T_0 G$.

**Lemma 3.2** Let $P \in \mathbb{C}[X_1, \ldots, X_n, X_a]$, with algebraic coefficients of height at most $\log H$, homogenous of degree $N_i$ in $X_i = (X_{1,1}, \ldots, X_{1,\nu_i})$, and of degree $N_a$ in $X_a$, and denote $F := P \circ \Phi$.

Suppose that $t = (t_1, \ldots, t_k)$ is the smallest element of $\mathbb{N}^k$ (for the lexicographic order) such that $D_t f P \circ \Phi(\omega) \neq 0$. There is a constant $c_5$ depending only of $g$, such that:

$$\log \left| \frac{1}{t!} D_t F(\omega) \right| \geq -c_5 D \left( \log H + N_a \log B + T \log N_a + \sum_{i=1}^{n} N_i (1+\left\|\omega_i\right\|^2) + h(A_i) (T + N_i) \right)$$

**Proof:**

Using homogeneity of $P$, we write:

$$F(\omega) = P \circ \Psi(\omega). \prod_{i=1}^{n} \left( \varphi_{0}^{(i)}(\omega_i) \right)^{N_i}$$

Since $\varphi_{0}^{(i)}(\omega_i)$ doesn’t vanish, the hypothesis $\forall \pi < t$, $D_{\pi} F(\omega) = 0$ and Newton formulas allow us to write

$$\forall \pi < t, D_{\pi} F \circ \Psi(\omega) = 0 \quad \text{and} \quad D_{\pi} F(\omega) = \prod_{i=1}^{n} \left( \varphi_{0}^{(i)}(\omega_i) \right)^{N_i} . D_{\pi} F \circ \Psi(\omega) \quad (2)$$

By lemma 2.1,

$$\left| \log \left( \prod_{i=1}^{n} \varphi_{0}^{(i)}(\omega_i) \right)^{N_i} \right| - \sum_{i=1}^{n} 4\pi N_i \left\|\omega_i\right\|^2_R \leq \sum_{i=1}^{n} N_i c_1 D h(A_i)$$

We now focus on function

$$f : \mathbb{C}^g \rightarrow \mathbb{C}$$

$$z \mapsto P \circ \Psi(\omega + \sum_{i=1}^{g} z_i f_i)$$

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Since $\omega$ is a period and $\Psi$ is expressed by mean of quotients of $\vartheta$ and $\vartheta(i_0)$, we can write:

$$f(z) = P \circ \Psi(\sum_{j=1}^{g} z_j f_j)$$

From (2), all the derivatives of $f$ at order strictly less than $|\tau|$ vanish, and $\frac{1}{\tau} D_P^\tau P \circ \Psi(\omega)$ is the first non-zero Taylor coefficient of $f$ in 0.

We now use the formal logarithm from lemma 2.4: for any $j \leq g$, we get by integration:

$$\zeta_j = l_j(T_1, \ldots, T_g) = \sum_{\pi \in \mathbb{N}^g} \lambda_{\pi, j} T_1^{\pi_1} \cdots T_g^{\pi_g},$$

where $\lambda_{\pi, j}$ can be written as $\frac{\alpha_{\pi, j}}{r \cdot m_{\pi, j}}$, with $\alpha_{\pi, j} \in \mathcal{O}_K$.

Hence we can write

$$\lambda_{\pi, j} = \frac{\alpha_{\pi, j}}{r \cdot m_{\pi, j}},$$

with $\alpha_{\pi, j} \in \mathcal{O}_K$, $m_{\pi, j} \in \mathbb{N}^*$. 

Let $(\zeta_j)$ be the coordinates of an element $z_0$ of $T_0G$ in Shimura basis $E$.

With this change of variable, we get $\psi_r(rz_0) = (s_{i,0}, \ldots, s_{i,\nu_i})(T_1, \ldots, T_g)$ where the $s_{i,j}$ are formal series with coefficient in $\frac{1}{r} \mathcal{O}_K$.

Hence working on $rz_0$ instead of $z_0$,

$$P \circ \Psi(rz_0) = P \circ \Psi \left( \sum_{j=1}^{g+t} r_\zeta_j e_j \right) = P((s_{0,0}, \ldots, s_{n,\nu_n})(T_1, \ldots, T_g), 1, r\zeta_{g+1}, \ldots, r\zeta_{g+t})$$

We develop:

$$P \circ \Psi \left( \sum_{j=1}^{g+t} r_\zeta_j e_j \right) = \sum_{\pi \in \mathbb{N}^g} \mu_{\pi} T_1^{\pi_1} \cdots T_g^{\pi_g} \zeta_{g+1}^{m_{g+1}} \cdots \zeta_{g+t}^{m_{g+t}}$$

where $\mu_{\pi}$ are written as $\sum b_{\lambda, \pi} p_{\lambda}$, with $p_{\lambda}$ denoting coefficients of $P$, $b_{\lambda, \pi}$ elements of $\frac{1}{r \cdot \tau} \mathcal{O}_K$, $0 \leq M \leq \sum_{i=1}^{n_i} N_i$, and $m_{g+1} + \ldots + m_{g+t} \leq N_a$, since $P$ is homogenous of degree $N_a$ in $\mathcal{X}_a$.

We now apply this formula to the particular case where $z_0 = \sum_{j=1}^{g} z_j f_j$.

$$\zeta_{g+k} = - \sum_{j=1}^{g} \beta_{k,j} \zeta_j = - \sum_{j=1}^{g} \beta_{k,j} l_j(T_1, \ldots, T_g)$$

$$= \sum \eta_{\pi, k} T_1^{\pi_1} \cdots T_g^{\pi_g}$$

where $\eta_{\pi, k} = - \sum_{j=1}^{g} \beta_{k,j} \lambda_{\pi, j}$

and we substitute the $\zeta_{g+k}$ in (3) to get
\( P \circ \Psi \left( \sum_{i=1}^{g} r_i z_i f_i \right) = \sum_{\pi} \left[ \mu_{\pi} T_1^{m_1} \cdots T_g^{m_g} \left( \sum_{\pi} \eta_{\pi i} T_1^{n_1} \cdots T_g^{n_g} \right)^{m_{g+1}} \cdots \left( \sum_{\pi} \eta_{\pi i} T_1^{n_1} \cdots T_g^{n_g} \right)^{m_{g+1}} \right]. \)

The coefficient of \( T_1^{t_1} \cdots T_g^{t_g} \), is equal to \( r_{|t|} D_{tf} P \circ \Psi(\omega) \), and it is a finite linear combination, with integer coefficients of elements of the form

\[ \mu_{\pi} \prod_{k=1}^{N} \eta_{\pi k} \text{ with } N \leq N_a, \sum |\pi_k| \leq |\pi| \]

By definition of \( \mu_{\pi} \) and \( \eta_{\pi i} \), it is a finite linear combination, with integer coefficients of elements of the form

\[ b_{,p} \lambda_{i,j} \left( \prod_{k=1}^{N} n_k \right)^{-1} r^{N-M} \]

with \( 0 \leq M \leq \sum_{i=1}^{n} N_i, N \leq N_a, \sum n_k \leq |\pi|, b \in \mathcal{O}_K \).

The upper-bound we deduce for the denominator of \( D_{tf} P \circ \Psi(\omega) \), combined with upper-bounds of its archimedian absolute values given by lemma 3.1, and to the product formula gives the lower bound announced. \( \square \)

3.3 Change of basis

Lemma 3.3 Let \( X = (x_1, \ldots, x_n) \) and \( Y = (y_1, \ldots, y_m) \) be two sets of vectors of \( T_0 G \) such that for all \( 1 \leq i \leq m \) we have \( y_i = \sum_{j=1}^{n} y_{i,j} x_j \), then

\[ \max_{|\pi|=T} |D_Y F(\pi)| \leq \left( \sum_{i,j} |y_{i,j}| \right)^T . \max_{|\pi|=T} |D_X F(\pi)| \]

Proof: This is lemma 6.15 of [8]. \( \square \)

4 Linear algebra: preparing for the auxiliary function

4.1 Rank of a linear system

We want to build a polynomial \( P \) in \( t + 1 + \sum (\nu_i + 1) \) variables, homogenous of degree \( N_i \) in the \( \nu_i + 1 \) variables attached to \( A_i \), homogenous of degree \( N_a \) in the ones attached to \( G_i^a \), and such that its value (and the one of its derivatives) in 0 are not too big.
Lemma 4.1  Let $G$ be an algebraic group, and $G'$ an algebraic connected subgroup of $G$, $B = (u_1, \ldots, u_m)$ a basis of a subspace $W$ of $T_0G$ such that the $m-\sigma$ last vectors of $B$ form a basis of $W \cap T_0G'$ (where $\sigma = \text{codim}_W W \cap T_0G'$). Let $T_1, \ldots, T_m$ be strictly positive integers, and $N'_i = \max(1, N_i)$. Then the rank of the system

$$S = \left\{ D^\omega_{\mathcal{B}}(P \circ \Phi)(\omega) = 0, \right.$$ where $t = (t_1, \ldots, t_m) \in \mathbb{N}^m$ and $t_i < T_i$ ($0 \leq i \leq m$)

$$\text{with unknown } P \in \mathbb{C}[X_1, \ldots, X_N, X_a](N_1, \ldots, N_n, N_a), \text{ is at most}$$

$$8g^{g+T_1 \ldots T_{\sigma}}H(G', N'_1, \ldots, N'_n, N'_a)$$

Proof: This is lemma 6.7 from [16] \hfill \Box

4.2  Thue-Siegel lemma

Lemma 4.2  Let $(a_{i,j})_{1 \leq i \leq L, 1 \leq j \leq C}$ be a matrix of rank at most $\rho$ ($a_{i,j} \in \mathbb{C}$). Let $(\Delta, M, m) \in (\mathbb{R}^*_+)^3$ be such that

$$\max_{1 \leq i \leq L} \sum_{j=1}^C |a_{i,j}| \leq m \quad \text{and} \quad (2\sqrt{2}Lm\Delta/M + 1)^{2\rho} \leq \Delta^C$$

then there is $(x_1, \ldots, x_C) \in \mathbb{Z}^C \setminus \{0\}$ such that

$$\max_{1 \leq j \leq C} |x_j| < \Delta \quad \text{et} \quad \max_{1 \leq i \leq L} \left| \sum_{j=1}^C a_{i,j}x_j \right| \leq M$$

Proof: it is the lemma 6.1 of [16]. \hfill \Box

4.3  Parameters

Let $C_0, C_T, C_{N_1}, \ldots, C_{N_n}, C_N$ be $n+3$ real positive numbers, and $\lambda$ a real in $[0; 1]$. We define:

$$U_0 = C_0^{2+g/\gamma}C_N \left( \prod_{i=1}^n \left( \frac{C_{N_i}}{\gamma} \right)^{g_i} \right)^1/2;$$

$$T^# = \frac{C_0^1/2U_0}{C_T};$$

$$N^#_a = N^#_a(\lambda) = \frac{\lambda U_0}{C_T^{\gamma}C_{N_a}}$$

and for $i \in \{1, \ldots, n\}$, $N^#_i = N^#_i(\lambda) = \frac{\lambda U_0}{C_T^{\gamma}C_{N_i}}$
4.4 Elimination of obstructive subgroups

Lemma 4.3 There exists $\lambda \in [0; 1]$ such that for all connected algebraic subgroup $G'$ of $G$, such that $T_0 G' + W \neq T_0 G$, we have, with $\sigma = \dim(W/W \cap T_0 G')$,

$$(T^\#)^\sigma H(G', N_1^\#, \ldots, N_n^\#) \geq C_0 H(G, N_1^\#, \ldots, N_n^\#, N_a^\#)$$

and there is a subgroup $\tilde{G}$ for which this is an equality.

Proof: Let $\mathfrak{G}$ be the set of all the connected subgroups $G'$ of $G$ such that $T_0 G' + W \neq T_0 G$.

Let $G'$ be an arbitrary element of $\mathfrak{G}$, $\sigma = \dim(W/W \cap T_0 G')$ and $r' = \dim(G/G')$. For any $\lambda \in [0; 1]$, denote $N_a = N_a^\#/\lambda$ and $N_i = N_i^\#/\lambda$: these are positive numbers, independents of $\lambda$. Consider then

$$f(G', \lambda) = \frac{(T^\#)^\sigma H(G', N_1^\#, \ldots, N_n^\#)}{C_0 H(G, N_1^\#, \ldots, N_n^\#, N_a^\#)} \lambda^{\dim G' - \dim G}$$

Since $r' < 0$, this is a decreasing function in $\lambda$.

If $f(G', 1) > 1$, we take $\lambda(G') = 1$, and otherwise $\lambda(G') = (f(G', 1))^{1/\lambda} \leq 1$.

$H(G', \cdot)$ is a polynomial with integers coefficients between 0 and $\deg_{\mathfrak{G}} G'$: for a fixed degree of $G'$, $H(G', N_1, \ldots, N_n, N_a)$ can only take a finite number of values.

In addition,

$$\deg_{\mathfrak{G}} G'.(\min_{1 \leq i \leq n} (N_i; N_a))^\dim G' \leq H(G', N_1, \ldots, N_n, N_a) \leq \deg_{\mathfrak{G}} G'.(\max_{1 \leq i \leq n} (N_i; N_a))^\dim G',$$

so $H(G', N_1, \ldots, N_n, N_a)$ (as well as $f(G', 1)$) grows proportionally with $\deg G'$.

We can deduce (we just need to look at a finite number of subgroups) that there exists a subgroup $\tilde{G} \in \mathfrak{G}$, such that

$$\tilde{\lambda} = \lambda(\tilde{G}) = \inf_{G' \in \mathfrak{G}} \lambda(G')$$

Finally, $(\tilde{\lambda}, \tilde{G})$ is the solution to the problem :

- Since for any subgroup $G'$, $\lambda \mapsto f(G', \lambda)$ is decreasing, we have

$$f(G', \tilde{\lambda}) \geq f(G', \lambda(G')) \geq 1,$$

which shows the first point of the lemma.
• by construction, \( \tilde{\lambda} \leq 1 \), since the particular case \( G' = 0 \) shows that \( f(G', 1) \leq 1 \) happens (and hence \( \tilde{\lambda} \leq \lambda(0) \leq 1 \):

\[
f(0, 1) = \frac{T^g}{C_0 \deg \phi \ G N_1^{g_1} \ldots N_n^{g_n} N^g_A} = \frac{C_0 \deg \phi \ G (U_0^g)^{1/(\log N)^r} \prod_{i=1}^n \left( \frac{U_0^g}{(C_0 \deg \phi \ G N_1^{g_1} \ldots N_n^{g_n} N^g_A)^{d_i}} \right)}{C_0 \deg \phi \ G} \leq 1
\]

This assures that \( f(G, \tilde{\lambda}) = 1 \) and gives the second point. □

We now suppose that \( \lambda \) (used in definition of the \( N_i^# \)) is fixed and equals to \( \tilde{\lambda} \).

Remark: since \( G \) is the product of an abelian variety \( A \) and of a power of \( C \), the connected subgroup \( \tilde{G} \) can be written as \( \tilde{A} \times V \), where \( V \) is a subspace of \( C' \), and \( \tilde{A} \) a subvariety of \( A \). For the conclusion of our main theorem, only \( \tilde{A} \) will matter. The following lemma gives an upper bound for its degree.

**Lemma 4.4** The degree of \( \tilde{A} \) is at most

\[
(C_0)^{g+1} \deg \phi \ G \max_{\sum d_i = \dim(W \cap T_0 G)} \prod_{i=1}^n \left( \frac{Dh^+(A_i) + a^2 \|\omega_i\|^2_R}{a} \right)^{d_i}
\]

where \( a = D \max h^+(A_i) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i\|_R \)

and

\[
T_0 \tilde{A} + \mathcal{W} \neq T_0 A
\]

**Proof:** By definition of \( \tilde{G} = \tilde{A} \times V \), we have that

\[
H(\tilde{G}, N_1^#, \ldots, N_n^#, N_a^#) = (N_a^#)^{\dim V} H(\tilde{A}, N_1^#, \ldots, N_n^#)
\]

We also have the easy lower bound:

\[
H(\tilde{A}, N_1^#, \ldots, N_n^#) \geq \deg \phi(\tilde{A}) \min_{\sum d_i = \dim A \atop 0 \leq d_i \leq g_i} \left\{ \prod_{i=1}^n \left( N_i^# \right)^{d_i} \right\}
\]

Hence

\[
\deg \phi(\tilde{A}) \leq \frac{H(\tilde{G}, N_1^#, \ldots, N_n^#, N_a^#)}{(N_a^#)^{\dim V} \min_{\sum d_i = \dim A \atop 0 \leq d_i \leq g_i} \left\{ \prod_{i=1}^n \left( N_i^# \right)^{d_i} \right\}}
\]  (4)
From definition of $\tilde{G}$ in lemma 4.3, we know
\[
H(\tilde{G}, N_1^#, \ldots, N_n^#, N_a^#) = \frac{C_0 H(G, N_1^#, \ldots, N_n^#, N_a^#)}{(T^#)^a}
\] (5)

Combining both equations (4) and (5), we get:
\[
\deg_{\Phi}(\tilde{A}) \leq \frac{C_0 H(G, N_1^#, \ldots, N_n^#, N_a^#)}{(T^#)^a \dim V \min_{\sum d_i = \dim \tilde{A}} \left\{ \prod_{i=1}^{n} \left( N_i^# \right)^{d_i} \right\}}
\]
\[
\leq \frac{C_0 \deg_{\Phi} G(N_a^#)^t \prod_{i=1}^{n} \left( N_i^# \right)^{g_i}}{(T^#)^a \dim V \min_{\sum d_i = \dim \tilde{A}} \left\{ \prod_{i=1}^{n} \left( N_i^# \right)^{d_i} \right\}}
\] (6)

Let $p$ be the projection on $T_0 \tilde{A}$. $p$ is an isomorphism from $W$ to $T_0 \tilde{A}$ ($W$ is a supplement of $\ker p$).

Since
\[
\dim(W \cap T_0 \tilde{G}) = \dim \left( p(W \cap T_0 \tilde{G}) \right)
\]
\[
\leq \dim \left( p(T_0 \tilde{G}) \right) = \dim \tilde{A},
\]
we can finally write
\[
\deg \tilde{A} \leq \frac{C_0 \deg_{\Phi} G(N_a^#)^t (N_1^#)^{g_1} \ldots (N_n^#)^{g_n}}{(T^#)^a \min_{\sum d_i = \dim(W \cap T_0 \tilde{G})} \prod_{i=1}^{n} N_i^#^{d_i}}
\]
\[
\leq \frac{C_0 \deg_{\Phi} G(N_a^#)^t \max_{\sum d_i = \dim(W \cap T_0 \tilde{G})} \prod_{i=1}^{n} (N_i^#)^{g_i - d_i}}{(T^#)^a}
\]

We replace now the values by our parameters, and after simplification,
\[
\deg \tilde{A} \leq (C_0)^{5a+1} \deg_{\Phi} \frac{\max_{\sum d_i = \dim(W \cap T_0 \tilde{G})} \prod_{i=1}^{n} \left( Dh(A_i) + a^2 \| \omega_i \|_2^2 \right)}{a} d_i
\]

To show the second part of the lemma, suppose that $T_0 \tilde{A} + W = T_0 \tilde{A}$.

By definition of $W$, that would imply that $T_0 \tilde{A}$ contains a free family of $t$ vectors $(\vec{v}_1, \ldots, \vec{v}_t)$ generating a supplementary subspace $V'$ of $W$ in $T_0 \tilde{A}$.

This $V'$ would also be a supplementary of $W$ in $T_0 \tilde{G}$:

- $\dim V' + \dim W = \dim V' + \dim W + t = \dim T_0 \tilde{A} + t = \dim T_0 \tilde{G}$
\[ \text{let } v \in W \cap V' : v \in W, \text{ then for all } 1 \leq k \leq t, L_k(v) = 0. \text{ Since } v \in V', v \in T_0A \text{ and in fact } L_k(v) = L_k(v), \text{ so for all } k, L_k(v) = 0, \text{ and } v \in W. \text{ Since } V' \cap W = \{0\}, \text{ we deduce dim } W \cap V' = 0. \]

Finally, we would have \( T_0G = V' \oplus W \subset T_0A + W \subset T_0\tilde{G} + W \not\subset T_0G \), which gives a contradiction. \( \square \)

5 Proof of theorem 1.2

We use the notations introduced in theorem 1.2: \( t \) and \( n \) are positive integers, \( A_1, A_2, \ldots, A_n \) are abelian varieties of dimension \( g_1, \ldots, g_n \), defined over \( \mathbb{Q} \), and principally polarized. We associate to each \( A_i \) a point \( \tau_i \) of Siegel-space \( H_{g_i} \).

We work on \( A = A_1 \times \ldots \times A_n \). We have \( t \) linear forms on \( T_0A \), whose coefficients in the dual basis of Shimura basis are the \( \beta_{k,j} \in \mathbb{Q} \).

We call \( K \) the common definition field of the \( A_i \) and \( \beta_{i,j} \).

We denote \( G = A_1 \times \ldots \times A_n \times G_t \), and extends the \( L_k \) to \( T_0G \):

\[ L_k(z_A, z') = L_k(z_A) - z_k, \]

and denote by \( W = \bigcap_{k=1}^t \ker L_k, \Lambda_k = L_k(\omega) \) and \( \Lambda = \max_{1 \leq k \leq t} |\Lambda_k| \).

Before the proof really starts in section 5.2, we still need a few more notations.

5.1 Basis

\( T_0G \) has a naturally defined basis, the Siegel basis \( \tilde{E} = (\tilde{e}_1, \ldots, \tilde{e}_{g+t}) \), where

- \((\tilde{e}_{d_i+1}, \ldots, \tilde{e}_{d_i+g_i})\) is the Siegel basis of \( T_0A_i \simeq \mathbb{C}^{g_i} \) (with \( d_i = \sum_{1 \leq j < i} g_j \))

- \((\tilde{e}_{g+1}, \ldots, \tilde{e}_{g+t})\) is the canonical basis of \( T_0G_t \)

In the paragraph 1.2, we introduced a matrix \( \Omega_1(\tau) \) and Shimura basis of \( A_i(\tau) \). Using the same notation, we define now the \((g+t) \times (g+t)\) block diagonal matrix

\[ \Omega_1 = \begin{bmatrix} \Omega_1^{(1)}(\tau_1) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \Omega_1^{(n)}(\tau_n) & 0 \\ I_t & \cdots & \cdots & I_t \end{bmatrix}, \]

which is the matrix of change from Siegel basis to ”Shimura basis” of \( T_0G \) noted \( E = (e_1, \ldots, e_{g+t}) \), where

- \((e_{d_i+1}, \ldots, e_{d_i+g_i})\) is Shimura basis of \( T_0A_i \simeq \mathbb{C}^{g_i} \) (with \( d_i = \sum_{1 \leq j < i} g_j \))

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• \((e_{g+1}, \ldots, e_{g+t})\) is the canonical basis of \(T_0G^t_a\) \((\tilde{e}_{g+k} = e_{g+k})\)

We finally introduced a particular basis of \(W\), by taking, for \(j \in \{1, \ldots, g\}\)

\[ f_j = e_j - \sum_{k=1}^{t} \beta_{k,j} e_{g+k} \]

(they all cancel the \(L_k\), they form a free family, their cardinal is \(g\), and \(W\) a subspace of codimension \(t\) in a space of dimension \(t + g\)). We can complete the family \(\{f_1, \ldots, f_g\}\) to form a basis \(F\) of \(T_0G\) by adding \(f_{g+j} = e_{g+j}\) for \(1 \leq j \leq t\).

We will use two norms on \(T_0G\):

• \(\|\cdot\|_E\) the norm for which \(E\) is orthonormal, and \(d\) the distance associated to this norm.

• The norm associated to Riemann-form : \(\|\vec{z}\|_R\) (attached to the principal polarization of \(\prod A_i\) : \(\|\vec{z}\|_R = \sqrt{\langle \vec{z} | \Im \tau \vec{z} \rangle} = \langle \vec{z} | E \vec{z} \rangle\), where \(\vec{z}_E\) is the column vector representing \(\vec{z}\) in Siegel basis)

We now introduce the point \(w\) defined by

\[ w = \omega + \sum_{k=1}^{t} \Lambda_k e_{g+k} \]

which may loosely be seen as a projection of \(\omega\) on \(W\). It is easy to see that \(L_k(w) = 0\) for all \(1 \leq k \leq t\), so \(w \in W\) and

\[ d(\omega, W) \leq \|\omega - w\|_E = \left( \sum_{k=1}^{t} |\Lambda_k|^2 \right)^{\frac{1}{2}} \leq t \Lambda \]

We can remark that \(w \neq 0\), otherwise \(\omega\) would be a linear combination of \(\{f_k\}_{k \in \{g+1, \ldots, g+t\}}\), \(i.e.\) of \(\{e_k\}_{k \in \{g+1, \ldots, g+t\}}\), which is obviously false.

### Estimates on \(w\)

If \(\omega \in T_0\tilde{G}\), then \(\omega \in T_0\tilde{A}\), and we will deduce from our estimates on the degree of \(\tilde{A}\) (lemma 4.4) the second conclusion of the theorem, so we can suppose that from now on

\[ \omega \notin T_0\tilde{G} \]

Following the main theorem from [1] (see also [8] proposition 6.5),

\[ d(\omega, T_0\tilde{G}) \geq \frac{1}{\deg(\tilde{G})} = \frac{1}{\deg(\tilde{A})}. \]
If $d(\omega, T_0 \tilde{G}) \leq 2t\Lambda$, then $\Lambda \geq \frac{1}{2y_{\omega \in \mathcal{A}}}$, using once again the estimates from lemma 4.4, we get easily the first the first conclusion of the theorem, so we can also suppose that

$$d(\omega, T_0 \tilde{G}) > 2t\Lambda$$

From this, we can assure that the point $\omega$ is not an element of $T_0 \tilde{G} \cap W$ : otherwise we would have $d(\omega, T_0 \tilde{G}) \leq \|\omega - w\|_R \leq t\Lambda$.

### 5.2 Building the auxiliary function

In all this paper appear absolute constants $(c_1, c_2, \ldots c_{23})$ depending only of $g$. They are in particular independent of the following parameters. We denote $C_0$, depending only of $g$, an upper bound of $(c_1, c_2, \ldots c_{23})$

$$C_0 = 1 + \max_{1 \leq i \leq 23} \{c_i\}.$$  

We now define:

$$U_0 = C_0^{2+5g/4}(D \log B + D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R \times$$

$$\prod_{i=1}^n \left( \frac{D h^+(A_i) + (D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R^2 \|\omega_i^+\|_R^2}{D h^+(A_i) + (D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R} \right)^{\frac{\lambda}{2}}$$

$\lambda$ being a real in $[0; 1]$, that we’ll precise using lemma 4.3. We define:

$$S^\# = C_0^{3/4}(D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R \text{ and } S = [S^\#]$$

$$T^\# = \frac{C_0^{1/2}U_0}{D \max h^+(A_i) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R} \text{ and } T = [T^\#]$$

$$T_0^\# = \frac{U_0}{C_0^{3/2}(D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R}$$

$$\lambda \frac{U_0}{C_0^{3/4}U_0/S^\#} \text{ and } T_0 = [T_0^\#]$$

$$N^\# = \frac{\lambda U_0}{C_0^2(D \log B + D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R}$$

and for $i \in \{1, \ldots, n\}$,

$$N_i^\# = \frac{\lambda U_0}{C_0^{3/2}D h^+(A_i) + (S^\#)^2\|\omega_i^+\|_R^2}$$

$$\lambda \frac{U_0}{C_0^{3/2}(D h^+(A_i) + (D \max h^+(A_i)) + \log^+ \max_{i \in \{1, \ldots, n\}} \|\omega_i^+\|_R^2 \|\omega_i^+\|_R)}$$
and $N_i = [N_i^\#]$.

In this part, we will build a function on $T_0G$, small enough (as well as its derivatives in directions of $W$ until a certain order) in $\omega$. This function will be searched as $F = P \circ \Phi$, where $P$ is a polynomial.

Our problem can be reformulate as a linear system with the coefficients of $P$ as unknowns.

We introduce here a basis $F' = (f'_{j})_{1 \leq j \leq g}$ of $W$, built as follow:

- we use an intermediate basis $\tilde{F} = (\tilde{f}_{j})_{1 \leq j \leq g}$ such that the matrix of basis change from $(\tilde{f}_{j})_{1 \leq j \leq g}$ to $(f_{j})_{1 \leq j \leq g}$ is unitary and the last vectors of $\tilde{F}$ form a basis $T_0G \cap W$ ($j$ from $d + 1$ to $g$).
- since $w$ is not in $T_0\tilde{G}$, it has a non-zero component on one of the first $\tilde{f}_{j}$. Up to a reordering, we can suppose that it is on $\tilde{f}_{1}$ and that this component is maximal.
- We now define $f'_{1} = w$ and $f'_{j} = \tilde{f}_{j}$ for $2 \leq j \leq g$.

With this notations, we have:

**Lemma 5.1** The coordinates $(w_1, w_2, \ldots, w_g)$ of $w$ in the basis $\tilde{F}$ verify:

$$|w_1| \geq \frac{d(\omega, T_0\tilde{G})}{2g} \quad \text{and} \quad 1 + |w_2| + \ldots + |w_g| \leq 1 + (g - 1)(\|\omega\|_E + t\Lambda)$$

**Proof:** (cf. Proposition 6.13 and Remark 6.17 from [8]) Let $\| \cdot \|_F$ be the norm associated to the scalar product such that $F$ (as well as $\tilde{F}$) is orthonormal.

Writing a vector $x$ in basis $E$ and $F$, we easily see that (modules $\beta_{i,j}$ are smaller than 1):

$$\|x\|_F \leq \|x\|_E \leq \sqrt{g}\|x\|_F.$$ 

We then write $w = \sum_{i=1}^{d} w_i \tilde{f}_i + \sum_{i=d+1}^{g} w_i \tilde{f}_i$, and we denote $x = \sum_{i=d+1}^{g} w_i \tilde{f}_i \in T_0G$.

$$|w_1| \sqrt{g} \geq \left( \sum_{i=1}^{d} |w_i|^2 \right)^{1/2} \geq \|w-x\|_F$$

$$\geq \frac{1}{\sqrt{g}} \|w-x\|_E$$

$$\geq \frac{1}{\sqrt{g}} (\|\omega-x\|_E - \|w-\omega\|_E)$$

$$\geq \frac{1}{\sqrt{g}} (d(\omega,T_0\tilde{G}) - t\Lambda) \geq \frac{1}{\sqrt{g}} \left( d(\omega,T_0\tilde{G}) - \frac{1}{2}d(\omega,T_0\tilde{G}) \right),$$
which prove the first inequality.

\[ 1 + |w_2| + \ldots + |w_g| \leq 1 + (g - 1)\|w\|_F \leq 1 + (g - 1)\|w\|_E \leq 1 + (g - 1)(\|w - \omega\|_E + \|\omega\|_E) \leq 1 + (g - 1)(t \Lambda + \|\omega\|_E) \]

5.2.1 Study of the system

We want to differentiate in the directions \((f'_j)\) as follow :

- a global order (i.e. in all directions) at most \(T\).
- a lesser order in the direction \(f'_1 = w\) : at most \(T_0\).

We denote

\[ T = \{\tau = (\tau_j)_{j=1...g}, \tau_j \leq T\} \quad \text{and} \quad T_0 = \{\tau \in T, \tau_1 \leq T_0\} \]  

(7)

The system we want to solve is :

\[ \{\|D\tau f'_F(u)\| \leq M, \quad \tau \in T_0\} \]

Number of unknowns:

we search \(P\) homogenous of degree \(N_a\) in \(X_a = (X_{a,0}, \ldots, X_{a,t})\), and homogenous of degree \(N_i\) in each \(X_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,\nu_i})\).

The dimension of the space of such polynomials is the value taken by the Hilbert-Samuel polynomial of \(G\) in \((N_1, \ldots, N_n, N_a)\),

\[ \text{Card}(I) = \dim(\mathbb{C}[P]/I(G))_{(N_1, \ldots, N_n, N_a)} \]

\[ = \dim(\mathbb{C}[P_{\nu_i}]/I(G^i_{\nu_i}))_{N_a} \prod_{i=1}^n \dim(\mathbb{C}[P_{\nu_i}]/I(A_i))_{N_i} \]

\[ \geq c_0 N_a^{\tau} \prod_{i=1}^n N_i^{\nu_i} (\text{if } N_i \neq 0, \text{ it is trivial}) \]

and if not, there is at least \(1 = N'_i\) polynomial

\[ \geq c_7 H(G, N'_1, \ldots, N'_n, N'_a) \]

System rank:

we apply lemma 4.1 with the basis \(F'\), the sub-group \(\tilde{G}\), \(T_1 = T_0\), and \(T_j = T\) for \(2 \leq j \leq g\). Since \(f'_1 = w \not\in T_0\tilde{G}\), a factor \(T_0\) will appear.

\[ \rho \leq (2^{g+t} + 1)T_0T^{\sigma-1}H(\tilde{G}, N'_1, \ldots, N'_n, N'_a) \]

\[ \leq 2^{2g+1}H(G, N'_1, \ldots, N'_n, N'_a) \frac{(T^\#)^{\ast}H(\tilde{G}, N'_1, \ldots, N'_n, N'_a)}{c_0 H(G, N'_1, \ldots, N'_n, N'_a)} \]
Since \( \frac{H(G_1 \ldots)}{H(G_2 \ldots)} \) is decreasing, and (following the idea from [16]) \( N_i' \geq N_i^# / 2 \) for all \( 1 \leq i \leq n \) and \( N_a' \geq N_a^# / 2 \), we deduce that

\[
\rho \leq 2^{2r+1} \tilde{C}_0^{-1} H(G, N_1', \ldots, N_n', N_a')
\]
\[
\times (T^#)^r \tilde{H}(G, N_1^#, \ldots, N_n^#, N_a^# / 2)\]
\[
C_0 H(G, N_1^#, \ldots, N_n^#, N_a^# / 2, N_a^# / 2)
\]
\[
\leq 2^{3 \dim G + 1 - \dim G} \tilde{C}_0^{-1} H(G, N_1', \ldots, N_n', N_a')
\]
\[
\times (T^#)^r \tilde{H}(G, N_1^#, \ldots, N_n^#, N_a^#)
\]
\[
C_0 H(G, N_1^#, \ldots, N_n^#, N_a^#)
\]

Finally, by definition of \( \tilde{G} \) (see lemma 4.3), we can simplify:

\[
\rho \leq 2^{3 \dim G + 1 - \dim G} \tilde{C}_0^{-1} H(G, N_1', \ldots, N_n', N_a')
\]
\[
\leq c_8 \tilde{C}_0^{-1} H(G, N_1', \ldots, N_n', N_a').
\]

**Lemma 5.2** None of the integers \( N_1, \ldots, N_n \) and \( N_a \) is zero.

**Proof:** Suppose that there is \( i_0 \) between 1 and \( n \) such that \( N_{i_0} = 0 \) (the case \( N_a = 0 \) should be treated similarly). Consider the following system:

\[
\{ D_i^\tau P \circ \Phi(\omega) = 0, \quad \tau \in T_0 \}
\]

with unknown the polynomial \( P \), with coefficients in \( \mathbb{C} \), homogenous of degree \( N_a \) in \( \overline{X}_0 \) and homogenous of degree \( N_i \) in \( \overline{X}_i \) (1 \( \leq i \leq n \)). We just show that the number of unknown is bigger than the system rank, therefore the system has a non-trivial solution.

Moreover, since \( N_{i_0} = 0 \), \( P \circ \Phi \) does not depend of the variables from \( T_0 A_{i_0} \); all derivatives of any order in directions of \( T_0 A_{i_0} \) are also zero.

Hence, if\(^2\) \( x \in T_0 A_{i_0} \setminus W \) (resp. \( x \in C^i \setminus W \)),

\( P \circ \Phi \) has a zero of order \( T_0 \) in \( \omega \) in the directions of \( \overline{W} = W \oplus \mathbb{C} x \). We now apply Philippon zero-lemma [15]: there is a proper algebraic connected sub-group \( G' \subset G \) such that:

\[
T_0^\sigma H(G', N_1', \ldots, N_n', N_a') \leq c_9 H(G, N_1', \ldots, N_n', N_a')
\]  
(8)

with \( \sigma' = codim_{\overline{W}}(\overline{W} \cap T_0 G') \)

- If \( \sigma' = codim_{T_0 G}(T_0 G') \), then the inequality (8) contradicts the definition of \( T_0 \): indeed, we get from this inequality

\[
T_0^{\sigma'} \leq c_9 \max\{N_1', N_a'^{\sigma'} \}
\]

which is similar to write \( C_0 \leq c_9 \) and contradicts the definition of \( C_0 \).

\(^2\) We can assume that such an \( x \) exists. Otherwise, \( T_0 A_{i_0} \subset W \subset W \), and we do all the theorem proof, without taking \( A_{i_0} \) into account, with the same number of linear forms. We would obtain estimates with \( (g - g_{i_0}) \) in exponents instead of \( g \), and they are better than the one we are looking for.
• Otherwise, we are in the case where \( \bar{W} + T_0 G' \neq T_0 G \).

We also have that \( W + T_0 G' \neq T_0 G \), so the subgroup \( G' \) is suitable for the lemma 4.3. The inequality (8) contradicts the conclusion of this lemma, since \( \sigma' = \sigma \) or \( \sigma + 1 \) and \( T_0 \simeq T/C_0^2 \). □

5.2.2 Polynomial choice

Let us recall that the set \( T_0 \) (as well as the set \( T \)) is defined by formula 7.

Lemma 5.3 There is a non-zero element \( P \) of \( K[\bar{X}_1, \ldots, \bar{X}_m, \bar{X}_a](N_1, \ldots, N_n, N_a) \) verifying the following properties :

1. for any \( \tau \in T_0 \), \( |D_\tau F(\omega)| \leq e^{-C_0 U_0} \), with \( F = \Phi \circ P \)

2. the height of the coefficients of \( P \) is at most \( C_1^{1/2} U_0 / D \)

Proof: We write \( P(X) = \sum p_\lambda X^\lambda \), with \( p_\lambda \in K \). Let \( (\xi_1, \ldots, \xi_D) \in K \) be a basis of \( K \) as a \( \mathbb{Q} \) vector space.

\( K \) is generated over \( \mathbb{Q} \) by \( \beta_{i,j} \) and \( \theta(i)_j(0) \). If we denote by \( \varsigma_1, \ldots, \varsigma_M \) these numbers, we can take for \( \xi_1, \ldots, \xi_D \) numbers of the form:

\[
\varsigma_1^{a_1} \cdots \varsigma_M^{a_M}
\]

with \( 0 \leq a_i \leq [\mathbb{Q}(\varsigma_i) : \mathbb{Q}] \) and \( \sum a_i \leq D \).

With this choice, we have \( (M_K \) denotes the places of \( K \)

\[
\begin{align*}
    h(\xi_1 : \ldots : \xi_D) &= \sum_{\nu \in M_K} \frac{D_\nu}{D} \log \max \{ |\xi_i|_\nu \} \\
    &\leq \sum_{\nu \in M_K} \frac{D_\nu}{D} D \log \max \{ |s_j|_\nu \} \\
    &\leq D \sum_{\nu \in M_K} \frac{D_\nu}{D} \left( \sum_{i=1}^t \log \max \{ |\beta_{i,j}|_\nu \} + \sum_{i=1}^n \log \max_j \left\{ \left| \frac{\theta(i)_j(0)}{\theta(0)} \right|_\nu \right\} \right) \\
    &\leq D \left( t \log B + \sum_{i=1}^n h(A_i) \right)
\end{align*}
\]

We will search the polynomial \( P \) as \( P(X) = \sum n_{\lambda,i} \xi_i X^\lambda \), where \( n_{\lambda,i} \in \mathbb{Z} \).

Denote \( a_{\lambda,\tau} = \frac{1}{D^r} D^r (\Phi^\lambda)(\omega) \): this way, the number we want to minimize are

\[
\sum_{\lambda} \sum_{i=1}^D (\xi_i a_{\lambda,\tau}) n_{\lambda,i}
\]

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Following lemma 3.1, we know that

$$|a_{\lambda, \tau}| \leq \exp \left( N_a \log B + T \log g + \sum_{i=1}^{n} N_i c_1 (1 + \|\omega_i\|_R)^2 \right) \leq e^{c_{10} C_{1/2}^{1/2} U_0}$$

We use now lemma 4.2, with unknowns $n_{\lambda, i}$, with coefficients of the system matrix the $\xi_i u_{\lambda, \tau}$, and:

$$\rho \leq c_8 C_0^{-1} H(G, N_1, \ldots, N_n, N_a)$$
$$C = D \# I \geq c_7 D H(G, N_1, \ldots, N_n, N_a)$$
$$L = \# T_0 = T^{g-1}, T_0 \leq C_0^{g/2-2} U_0^{g/2} \leq e^{c_{11} C_{1/2}^{1/2} U_0}$$
$$m = C. \max |\xi_i|, e^{c_{10} C_{1/2}^{1/2} U_0} \leq e^{c_{12} C_{1/2}^{1/2} U_0}$$
$$\Delta = e^{C_{1/4} U_0/D}$$
$$M = e^{-C_0 U_0}$$

since

$$\frac{Lm \Delta}{M} \leq e^{c_{11} C_{1/2}^{1/2} U_0 + c_{12} C_{1/2}^{1/2} U_0 + C_{1/4} U_0/D + C_0 U_0} \leq e^{c_{13} C_0 U_0}$$

and

$$\Delta^{C/2 \rho} \geq \left( e^{C_{1/4} U_0/D} \right)^{\frac{c_7 D H(G, N_1, \ldots, N_n, N_a)}{c_8 C_0^{-1} H(G, N_1, \ldots, N_n, N_a)}}$$
$$\geq e^{c_{14} C_0^{1/4} U_0}$$
$$\geq 2\sqrt{2} \frac{Lm \Delta}{M} + 1,$$

the hypothesis of Siegel lemma is verified, and we get the existence of $P$. Moreover,

$$p_{\lambda} = \sum_{i=1}^{D} n_{\lambda, i} \xi_i$$

So that,

$$v \not\to \infty \quad |p_{\lambda}|_v \leq \max |\xi_i|_v$$
$$v \to \infty \quad |p_{\lambda}|_v \leq D e^{C_{1/4} U_0/D} \max |\xi_i|_v$$

and finally

$$h(p_{\lambda}) \leq \log D + \frac{C_{1/4} U_0}{D} + h(\xi_1 : \ldots : \xi_D)$$

The term $C_{1/4} U_0$ being bigger than $D \log D$ and $Dh(\xi_1 : \ldots : \xi_D)$, we can simplify, and finally obtain the announced result. □
5.3 Extrapolation

We suppose in this section that

$$\log \Lambda \leq -C_0 U_0$$

(9)

Under this hypothesis, the points $\omega$ and $w$ are close, so $F$ will behave similarly, and we will be able to transform our estimates of $F$ in $\omega$ into estimates of $F$ in $w$, and reciprocally.

By doing this, we will show that the upper bound from lemma 5.3 is in fact true for a higher order of derivation, since we can replace the set $T_0$ by the set $T$, defined in (7), and this will finally yield to a contradiction.

**Lemma 5.4** For $\tau \in T$, $s \leq S$, we have

$$\left| \frac{1}{\tau!} D_{\tau}^2 F(s \omega) - \frac{1}{\tau!} D_{\tau}^2 F(s w) \right| \leq \exp(-C_0 U_0)$$

**Proof:** Let $\tau \in T$, and $s \leq S$. We consider the function

$$f: \mathbb{R} \to \mathbb{C} \quad x \mapsto \frac{1}{\tau!} D_{\tau}^2 F(s \omega + x(s w - s \omega))$$

By using lemma 3.1 we get an upper bound for $|f'(x)|$ on $[0, 1]$.

$$f'(x) = \sum_{i=1}^{g+t} x(w - \omega) \frac{1}{\tau!} D_{\tau} e_i \cdot D_{\tau} F(s \omega + x(s w - s \omega))$$

Recall that

$$w = \omega + \sum_{i=1}^{t} L_i(\omega)e_{g+i}$$

Hence for $i \leq g$, $(w - \omega)_i = 0$, and for $i \geq g + 1$, $(w - \omega)_i = L_{i-g}(\omega)$, so by hypothesis (9),

$$|w - \omega|_i \leq \Lambda \leq \exp(-C_0 U_0)$$

To estimate $\frac{1}{\tau!} D_{\tau} e_i \cdot D_{\tau} F(s \omega + z(s w - s \omega))$, we use lemma 3.1, with $u = s \omega + z(s w - s \omega)$, $(x_1, \ldots, x_{g+1}) = (f'_1, \ldots, f'_{g+1}, e_1)$, and $\overline{t} = (t_1, \ldots, t_{g+1}) = \overline{(t_1, \ldots, t_{g+1})}$. (Note that $\overline{t}! = \tau!$). Hence,

$$\log \left| \frac{1}{\tau!} D_{\tau} e_i \cdot D_{\tau} F(s \omega + x(s w - s \omega)) \right| \leq N_a \log \xi_a + (T + 1) \log \xi + (T + 1) \log(g + 1) + \log H + t N_a \log \max_{1 \leq i \leq t} (1 + |u_{g+i}|)$$

$$+ \sum_{i=1}^{n} 2c_1 N_i (g_i + (1 + |u_i|)^2)$$
In this case, \( \xi_a = \max(1, |\beta_{i,j}|, |L_i(\omega)|) \) and \( \xi = \max(1, |\omega_i|) \).

Then we have the following upper bound for \( z \in [0; 1] \),

\[
\log |f'(x)| \leq \log(tS) + \log \Lambda + N_a \log \xi_a + (T + 1) \log \xi \\
+ (T + 1) \log(g + 1) + \log H \\
+ t.N_a \log \max_{i \in \{g+1, \ldots, g+t\}} (1 + |u_i|) \\
+ \sum_{i=1}^{n} 2c_1 N_i \left( g_i + (1 + |u_i|)^2 \right)
\]

Finally, we use the mean value theorem between 0 and 1 with \( f \):

\[
\left| \frac{1}{\tau!} D_\tau^r F(s\omega) - \frac{1}{\tau!} D_\tau^r F(s\omega) \right| = \left| f(0) - f(1) \right| \\
\leq \max_{x \in [0;1]} |f'(x)|
\]

which, with (10), gives the lemma.

**Lemma 5.5** For \( \tau \in T, s \leq S/2 \), we have

\[
\frac{1}{\tau!} D_\tau^r F(s\omega) \leq \exp(-c_19 C_0^{3/4} U_0)
\]

**Proof:** Let \( \tau \in T \), we define \( \tau' = (0, \tau_2, \ldots, \tau_g) \).

That means we wish to avoid derivatives in direction \( f'_1 = w \). We study now the function:

\[
f(z) = \frac{1}{\tau!} D_\tau^r F(zw)
\]

From lemma 5.4, we know that for \( m \leq T_0, s \in 0, \ldots, S \)

\[
\left| \frac{1}{m!} f^{(m)}(s) \right| \leq \left| \frac{1}{\tau!} D_\tau^r F(s\omega) - \frac{1}{\tau!} D_\tau^r F(s\omega) \right| + \left| \frac{1}{\tau!} D_\tau^r F(s\omega) \right| \\
\leq \exp(-c_15 C_0 U_0)
\]

Then by mean of Schwarz-lemma (recalled below) with \( T_1 = T_0, r = S = S_r, R = 4e_S \), we get

\[
|f|_{2S} \leq 2|f|_{4eS} \left( \frac{1}{e} \right)^{T_0S} + 5.(18)^{T_0S} \max_{m \leq T_0} \left\{ \frac{1}{m!} |f^{(m)}(h)| \right\}
\]

From lemma 3.1, we have

\[
|f|_{4eS} \leq \exp(N_a (\log B) + T \log(N^2) + \log H + t.N_a (\log B) \log(4eS) \\
+ \sum_{i=1}^{n} 2c_1 N_i \left( g_i + (1 + 4eS)^2 \right)) \\
\leq \exp(C_0^{1/2} U_0)
\]
Since $T_0$ and $S$ were chosen such that $T_0 S \simeq C_0^{3/4} U_0$, we can deduce first, that

$$2|f|_{4eS} \left( \frac{1}{e} \right)^{T_0 S} \leq \exp(-c_{16} C_0^{3/4} U_0),$$

and second, that

$$5. (18) \max_{m \leq T_0, 0 \leq h \leq S} \left\{ \frac{1}{m!} |f^{(m)}(h)| \right\} \leq \exp(-c_{17} C_0^{3/4} U_0)$$

Finally, we use Cauchy inequalities to conclude:

$$\left| \frac{1}{\tau!} D^\tau F(s \omega) \right| = \left| \frac{1}{\tau!} \frac{\partial^{\tau_1}}{\partial z^{\tau_1}} f(s) \right| \leq |f|_{4S}$$

$$\leq \exp(-c_{18} C_0^{3/4} U_0) \tag{11}$$

By applying one more time lemma 5.4

$$\left| \frac{1}{\tau!} D^\tau F(s \omega) \right| \leq \left| \frac{1}{\tau!} D^\tau F(s \omega) - \frac{1}{\tau!} D^\tau F(s \omega) \right| + \left| \frac{1}{\tau!} D^\tau F(s \omega) \right|$$

$$\leq \exp(-C_0 U_0) + \exp(-c_{18} C_0^{3/4} U_0)$$

$$\leq \exp(-c_{19} C_0^{3/4} U_0).$$

\[ \square \]

**Lemma 5.6 (Schwarz lemma)** Let $f$ be an analytic function over the disc centered at the origin, of radius $R \geq 4$, $S_1 \geq 2$ an integer, $T_1$ an integer and $r \in [S_1, R/4]$. Then

$$|f|_{2r} \leq 2|f|_R \left( \frac{4r}{R} \right)^{T_1 S_1} + 5 \left( \frac{18r}{S_1} \right)^{T_1 S_1} \max_{0 \leq m \leq T_1} \left\{ \frac{1}{m!} |f^{(m)}(h)| \right\}$$

**Proof:** this is lemma 2.3 of [20] \[ \square \]

**Lemma 5.7** For $\tau \in T$, we have

$$D^\tau F(\omega) = 0$$

**Proof:** From lemma 5.5, for $\tau \in T$,

$$\left| \frac{1}{\tau!} D^\tau F(\omega) \right| \leq \exp(-c_{19} C_0^{3/4} U)$$

With lemma 3.3 we can deduce a similar higher-bound in basis $f$: the matrix of basis change is:

$$\begin{bmatrix}
\frac{1}{w_1} & 0 & 0 \\
-w_{2}/w_1 & I_{g-1} & 0 \\
\vdots & \vdots & \vdots \\
-w_{g}/w_1 & 0 & I_t \\
0 & 0 & I_t
\end{bmatrix}$$

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So we have
\[
\left| D^T F(\omega) \right| \leq \left( g + t - 1 + \frac{1 + |w_2| + \ldots + |w_g|}{|w_1|} \right)^T \exp(-c_{19}C_0^{3/4} U)
\]

We evaluate the first parenthesis with lemma 5.1.
\[
\left| D^T F(\omega) \right| \leq c_{20} \exp(-c_{19}C_0^{3/4} U) \quad (12)
\]

If \( |D^T F(\omega)| \) was not null for every \( \tau \in T \), we choose \( \tau \) minimal (for \(|\tau|\)) for which \( |D^T F(\omega)| \) is not zero.

From lemma 3.2 we can write:
\[
\left| D^T F(\omega) \right| \geq \exp(-c_0^{1/2} U) \quad (13)
\]

For \( C_0 \) big enough, (12) and (13) can not be true simultaneously, and the lemma follows.

\[ \square \]

5.4 Conclusion

We have built a non-trivial function which vanish along \( W \) at an order at least \( T \) in \( \omega \). From zero-lemma from [15], there is a connected proper algebraic subgroup \( G' \) of \( G \) such that
\[
T^\sigma H(G', N_1, \ldots, N_n, N_a) \leq c_{21} H(G, N_1, \ldots, N_n, N_a), \quad (14)
\]

where \( \sigma = \text{codim}_W (T_0 G' \cap W) \).

Two cases are possible:

- \( T_0 G' + W \neq T_0 G \), and lemma 4.3 contradicts the inequality (14).
- \( T_0 G' + W = T_0 G \), and necessarily \( \sigma = \dim(G/G') \):

\[
\dim(T_0 G) = \dim(T_0 G' + W) = \dim(W) + \dim(T_0 G') - \dim(T_0 G' \cap W),
\]

so that
\[
\dim(G/G') = \dim(T_0 G) - \dim(T_0 G') = \dim(W) - \dim(T_0 G' \cap W).
\]

We can now rewrite (14) as:
\[
C_0 U^\sigma \leq c_{21} U^\sigma,
\]

from what we deduce \( C_0 \leq c_{21} \), and that contradicts \( C_0 \) definition.

In both case, we obtained a contradiction. Our hypothesis must be wrong, which ends the demonstration.
References


