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Palindromic Width of Wreath Products

A word is a palindrome if it reads the same forwards and backwards. We show that the wreath product $G \wr \mathbb{Z}^n$ of any finitely generated group G with $\mathbb{Z}^n$ has finite palindromic width. We also show that $C \wr A$ has finite palindromic width if C has finite commutator width and A is a finitely generated infinite abelian group. Further we prove that if H is a non-abelian group with finite palindromic width and G any finitely generated group, then every element of the subgroup $G' \wr H$ can be expressed as a product of uniformly boundedly many palindromes. From this we obtain that $P \wr H$ has finite palindromic width if P is a perfect group and further that $G \wr F$ has finite palindromic width for any finite, non-abelian group F.

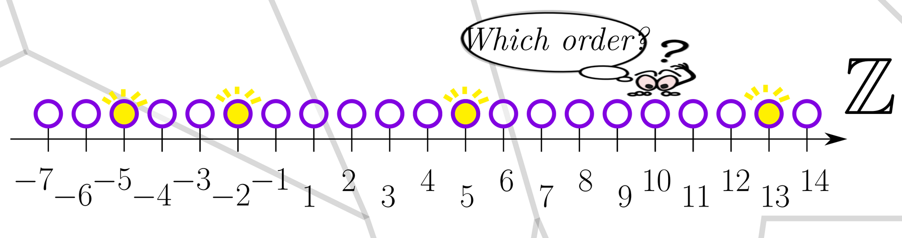
Wreath Products

A wreath product of two groups B and H is the semidirect product $\Pi_H B \rtimes H$ given by the multiplication

$$(g_\omega, h) \cdot (t_\kappa, f) = (g_{f(\omega)} t_\kappa, hf).$$

Despite this nice definition, it is best to think about it like this:

Example 1. *Lamplighter Group $\mathbb{Z}_2 \wr \mathbb{Z}$:*



We consider each copy of $\mathbb{Z}_2$ as a lamp on a string labelled by $\mathbb{Z}$. A Lamplighter runs up and down, lighting some lamps and then stops at a position. This represents an element of $\mathbb{Z}_2 \wr \mathbb{Z}$. It turns out the order in which he lit the lamps does not matter.

Words & Palindromes

A *group word* w in n letters is an element of the free group $F_n = \langle x_1, \dots, x_n \rangle$. So for example: $w(x_1, x_2) = x_1 x_2^2$ or $w(x_1, \dots, x_n) = x_1 \cdot \dots \cdot x_n$.

A *commutator word* is a word in two letters: $w(x, y) = x^{-1}y^{-1}xy$. A word is a *palindrome* if it reads the same forwards and backwards. For example:

$$w(x, y) = xyxyx,$$
$$w(x, y) = x^3y^2xy^2x^3,$$

or words like “level”, the German word “Reliefpfeiler” or the Finnish word “Saippuakivikauppias”.

Related Results

[RS14]: free metabelian groups and some wreath products

[BG14]: nilpotent-by-abelian groups, quotients and extensions thereof and certain soluble groups

Dependent on the generating set?

pw(BS(2,3))?

Word Width

We can now ask, how many words of a certain form does it take to express every element of a group as a product of such?

Surprisingly it turns out that this can sometimes be uniformly bounded over all group elements. It is easy to see that if A is a finitely generated abelian group with generating set $T = \langle t_1, t_2, t_3, t_4 \rangle$, then an element $a \in A$ is for example

$$a = t_1^5 t_2^6 t_3 t_4^{-1}.$$

Of course t_1^p is a palindrome by our definiton for every $p \in \mathbb{Z}$. Hence a is the product of 4 palindromes and we can do the same for every element of A. We then say that A has palindromic width $pw(A, T) = 4$ with respect to the generating set T.

$G \wr \mathbb{Z}^n$

Theorem 1 ([Fin14])**.** ***The wreath product a d-generated group G with $\mathbb{Z}^n$ has finite palindromic width at most $5d + 9n$ if n is even and at most $5d + 9n + 2$ if n is odd with respect to the natural generating set coming from $\mathbb{Z}^n$ and G.***

This sounds difficult to approach. Turns out though, you can just use this great result from [AM10]: If we let $B = \Pi_{\mathbb{Z}^n} F_d$, the base group of the wreath product $F_d \wr \mathbb{Z}^n$, then $w \in B'$ looks like this:

$$w = [a, t][b, t^2], \quad a, b \in B, t \in \mathbb{Z}^n.$$

Juggling around a bit with this lets us write with $t = t_1^{i_1} \cdot \dots \cdot t_n^{i_n}$

$$[a, t] = [a, t_1^{i_1} \cdot \dots \cdot t_n^{i_n}] = a^{-1}(t_1^{i_1} \cdot \dots \cdot t_n^{i_n})^{-1} a t_1^{i_1} \cdot \dots \cdot t_n^{i_n}$$
$$= a^{-1} t_n^{-i_n} \overline{a^{-1}} \cdot \overline{a} t_{n-1}^{-i_{n-1}} a \cdot a^{-1} t_{n-2}^{-i_{n-2}} \overline{a^{-1}} \cdot \dots \cdot \overline{a} t_1^{-i_1} a \cdot t_1^{i_1} \cdot \dots \cdot t_n^{i_n}$$

if n is even which is a product of $2n$ palindromes. Then we just do the same for the second commutator and use that a metabelian group has finite palindromic width as well by [BG14] and [RS14].

Commutator Width

If we assume that we have a wreath product $C \wr A$ where C has finite commutator width, then we can do something else. We take one commutator $[f, g]$ at position a_i and write it as
$a_i^{-1}(f^{-1}g^{-1}fg)\, a_i = a_i^{-1}(f^{-1}y^{-1}\overline{f^{-1}} \cdot y \cdot g^{-1}x^{-1}\overline{g^{-1}} \cdot x \cdot y^{-1} \cdot \overline{f} y f \cdot x^{-1}\overline{g} x g)\, a_i$
where x is some high power of one of the generators of A. Such powers are of course palindromes, so the above is a product of 7 palindromes describing one commutator. Put in a picture it looks like this:

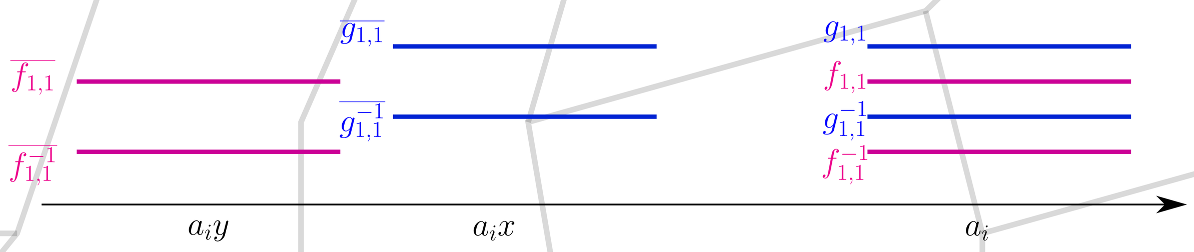


Figure 1: This is how we write one commutator on each position as a palindrome.

We can do this for every commutator and we get

Theorem 2 ([Fin14])**.** ***Let B be a d-generated group with finite commutator width $cw(B) = n$ and A an infinite abelian group generated by X with $|X| = r$. Then***

$$pw(B \wr A, X \cup S) \leq 5d + 6r + 7n$$

for any chosen generating set S of B.

Finite commutator width ⇔ Finite palindromic width?

$G \wr H$ - H non-abelian

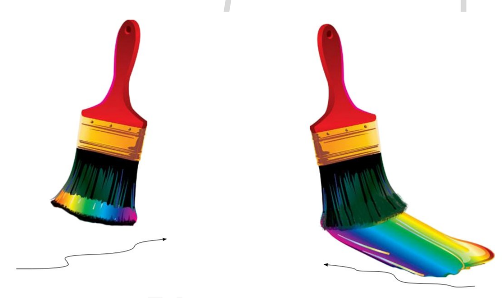
Theorem 3 ([Fin14])**.** ***Let H a non-abelian group with $pw(H, X) = k$. Then for every finitely generated group B with generating set Y, every element of the subgroup $B' \wr H$ is a product of at most $k+1$ palindromes with respect to $X \cup \{c\} \cup Y$, where $c \in H$, possibly trivial.***

This is a bit trickier because there’s no theorem about the commutator width of $G \wr H$ yet. It turns out though, all non-abelian groups without free quotients have a nice property:

Lemma 1. *If G is a finitely generated non-abelian group without free quotients with generating set X, then G has a presentation with generating set $X \cup \{c\}$, in which there exists a relation r in G such that $r \neq 1$ in G where $c \in G$ and possibly $c = 1$.*

Example 2. *Consider the group $BS(m, n) = \langle a, b \mid a^{-1}b^n a = b^m \rangle$. Clearly $a^{-1}b^n a b^{-m} = 1$, but $b^{-m}ab^n a^{-1} \neq 1$.*

That’s like drawing with a brush that has only paint on one side. Moving it right doesn’t do anything, but brushing reversely paints the canvas.



Now this is very practial in building commutators out of palindromes, using the same idea as above:

How do we prove this?

We again write one commutator a time and hope to shift the “reflected” parts a bit so that those will cancel. We write

$$f_1^{-1} r^{-1} g_1^{-1} r f_1 r^{-1} g_1 r f_2^{-1} r^{-1} g_2^{-1} r f_2 r^{-1} g_2 r \dots$$
$$\dots \overline{r} \cdot \overline{g_2} \cdot \overline{r^{-1}} \cdot \overline{f_2} \cdot \overline{r} \cdot \overline{g_2^{-1}} \cdot \overline{r^{-1}} \cdot \overline{f_2^{-1}} \cdot \overline{r} \cdot \overline{g_1} \cdot \overline{r^{-1}} \cdot \overline{f_1} \cdot \overline{r^{-1}} \cdot \overline{g_1^{-1}} \cdot \overline{r} \cdot \overline{f_1^{-1}}.$$

This looks complicated, but if we look closely, then we notice that the second line is just 1. This allows us to write arbitrarily long commutators whereever we want, because their mirrored parts will just always cancel.

References

[AM10] M. Akhavan-Malayeri. On commutator length and square length of the wreath product of a group by a finitely generated abelian group. *Algebra Colloq.*, 17(1):799–802, 2010.

[BG14] V. Bardakov and K. Gongopadhyay. Palindromic width of nilpotent groups. *Journal of Algebra*, 402:379–391, 2014.

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[RS14] T. Riley and A. Sale. Palindromic width of wreath products, metabelian groups and solvable max-n groups. *http://arxiv.org/abs/1307.4861*, 2014.